ELEMENTS OF K_1 VIA THE TRIPLE G-CONSTRUCTION

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Abstract. We represent the elements of K_1 of an exact category by means of two- and three-edge loops in the triple G-construction. The corresponding algebraic data are double short exact sequences and non-commutative 3×3-diagrams of short exact sequences. We state conjectural relations for the elements of K_1 in terms of 3×3-diagrams.

1. Review of the G- and the triple G-construction

In [GG] Gillet and Grayson attached a simplicial set $G.\mathcal{A}$ to any exact category $\mathcal{A}$ and proved that $|G.\mathcal{A}|$ is homotopy equivalent to $\Omega|Q\,\mathcal{A}| \sim \Omega|S.\mathcal{A}|$, the equivalence being natural in $\mathcal{A}$. Thus one can take the formula

$$K_m(\mathcal{A}) = \pi_m(G.\mathcal{A})\,, \quad m \geq 0\,,$$

for a definition of the higher K-groups of $\mathcal{A}$.

An n-simplex in $G.\mathcal{A}$ is a pair of triangular diagrams in $\mathcal{A}$ of the form

$$\begin{array}{ccccccccc}
 & & & & & & & & P_{n/n-1} \\
 & & & & & & & & \uparrow \\
 & & & & & & & & \cdots \\
 & & & & P_{2/1} & \to & \cdots & \to & P_{n/1} \\
 & & & & \uparrow & & & & \uparrow \\
 & & P_{1/0} & \to & P_{2/0} & \to & \cdots & \to & P_{n/0} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
P_0 & \to & P_1 & \to & P_2 & \to & \cdots & \to & P_n
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & & & & & P_{n/n-1} \\
 & & & & & & & & \uparrow \\
 & & & & & & & & \cdots \\
 & & & & P_{2/1} & \to & \cdots & \to & P_{n/1} \\
 & & & & \uparrow & & & & \uparrow \\
 & & P_{1/0} & \to & P_{2/0} & \to & \cdots & \to & P_{n/0} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
P'_0 & \to & P'_1 & \to & P'_2 & \to & \cdots & \to & P'_n
\end{array}
\tag{1.1}$$

subject to the conditions:
(i) the quotient index subtriangles in both diagrams coincide;
(ii) all the squares commute;
(iii) all the sequences of the form $P_j \to P_k \to P_{k/j}$, $P'_j \to P'_k \to P_{k/j}$, and $P_{j/i} \to P_{k/i} \to P_{k/j}$ with $i \leq j \leq k$ are s.e.s's in $\mathcal{A}$. (For short, we will write *s.e.s.* for *short exact sequence*).

In particular, a vertex in $G.\mathcal{A}$ is a pair of objects (P, P'), and an edge connecting (P_0, P'_0) to (P_1, P'_1) is a pair of s.e.s's $(P_0 \to P_1 \to P_{1/0}$, $P'_0 \to P'_1 \to P_{1/0})$ with equal cokernels.

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The i-th face of (1.1) amounts to deleting all the objects whose indices contain i. For instance, the faces of a generic 2-simplex

$$t = \left[\begin{array}{ccccccc} & & P_{2/1} & & & & P_{2/1} \\ & & \uparrow & & & & \uparrow \\ & P_{1/0} & \longrightarrow P_{2/0} & & & P_{1/0} & \longrightarrow P_{2/0} \\ & \uparrow & \uparrow & & & \uparrow & \uparrow \\ P_0 \longrightarrow & P_1 & \longrightarrow P_2 & & P'_0 \longrightarrow & P'_1 & \longrightarrow P'_2 \end{array}\right]$$

are given by

$$\begin{aligned} d_0 t &= (P_1 \to P_2 \to P_{2/1}, \quad P'_1 \to P'_2 \to P_{2/1}) \\ d_1 t &= (P_0 \to P_2 \to P_{2/0}, \quad P'_0 \to P'_2 \to P_{2/0}) \\ d_2 t &= (P_0 \to P_1 \to P_{1/0}, \quad P'_0 \to P'_1 \to P_{1/0}). \end{aligned}$$

The reader is referred to [Ne1-3] for a better introduction to $G.\mathcal{A}$. In the present paper we will also need another simplicial set $G^{III}_{.}\mathcal{A}$ with the properties:
(i) a vertex in it is a triple of objects $(P', P, P'') \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$, the upper index III referring to this fact;
(ii) an edge from (P'_0, P_0, P''_0) to (P'_1, P_1, P''_1) is a diagram of the form

$$\begin{array}{ccccc} P'_0 & \longrightarrow & P'_1 & \longrightarrow & P'_{1/0} \\ & & & & \downarrow \\ P_0 & \longrightarrow & P_1 & \longrightarrow & P_{1/0} \\ & & & & \downarrow \\ P''_0 & \longrightarrow & P''_1 & \longrightarrow & P''_{1/0} \end{array} \tag{1.2}$$

consisting of four s.e.s's;
(iii) an arbitrary n-simplex is a triple of admissible filtrations

$$\begin{aligned} &P'_0 \to P'_1 \to \ldots \to P'_n \\ &P_0 \to P_1 \to \ldots \to P_n \\ &P''_0 \to P''_1 \to \ldots \to P''_n \end{aligned}$$

endowed with

a) a choice of quotients $P'_{j/i}$, $P_{j/i}$, $P''_{j/i}$ for all possible composed maps $P'_i \to P'_j$, $P_i \to P_j$, and $P''_i \to P''_j$;

b) short exact sequences

$$0 \to P'_{j/i} \to P_{j/i} \to P''_{j/i} \to 0 \tag{1.3}$$