A note on the best tridiagonal approximation for the inverse of a totally nonnegative Matrix

Matthias Pott

*

Abstract

Let $\mathcal{T}$ be the set of the square totally nonnegative, $\mathcal{T}_{\mathcal{R}}$ of the tridiagonal, and $\mathcal{M}$ of the M-matrices. A result of M.Lewin in Ref.[5] shows that any two of the following three statements for a square regular matrix T imply the third one. $T^{-1} \in \mathcal{M}$, $T^{-1} \in \mathcal{T}_{\mathcal{R}}$, and $T \in \mathcal{T}$. An old presumption of a lot of researchers was: Given a matrix $T \in \mathcal{T}$. Is the solution of the problem $min_{X \in \mathcal{T}_{\mathcal{R}}} \|I - XT\|_F$ an M-matrix? Many numerical tests and theoretical considerations counted for a positive answer. We will prove in this article that this conjecture was wrong. But the numerical and theoretical observations leave a question open: Which further, probably *weak*, conditions do we need such that X is in $\mathcal{M}$?

Key words: Totally nonnegative matrices, minimizatition problems, tridiagonal M-matrices.

*Research supported by the Deutsche Forschungsgemeinschaft. My adress: Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany, e-mail: pott@mathematik.uni-bielefeld.de.

1 Introduction

Given a nonsingular matrix $A \in L(\mathbb{R}^n)$ and a $b \in \mathbb{R}^n$. For a practical computation it is often useful or even necessary first to determine a proper preconditioner $X \in L(\mathbb{R}^n)$, such that one can reduce the error for the numerical solution of the system

$$A\,x = b\,.$$

If the condition of XA is close to one the chances increase to get a good approximation for the exact solution, if one uses a direct numerical method for the computation of x. In advantageous cases one can even use iterative methods to solve $XA\,x = Xb$. To get an effective preconditioner, X should be sparse, so that the multiplications with A and b are cheap, second X should be easy to compute, and at last, to get a good condition number, XA should be close to the identity. So, one proper method could it be, to minimize under all tridiagonal matrices X the Frobenius-norm of $I - XA$. This leads to an interesting question of practical computing and matrix theory. Which structures has $X \in \mathcal{T}_{\mathcal{R}}$, if we know some structures of A? Here, if A is totally nonnegative, does it follow that X is an M-matrix?

Why do we ask this question? In Ref.[5] it is shown that if T is a regular totally nonnegative matrix, and T^{-1} is an M-matrix, then T^{-1} is tridiagonal. Also, if T^{-1} is a tridiagonal M-matrix, then T is totally nonnegative. Third, if T^{-1} is tridiagonal, and T totally nonnegative, then T^{-1} is an M-matrix. Further, it is easy to see, that if $T \in \mathcal{T}$ and regular, then T^{-1} is a checkerboard-matrix. But: Is the best tridiagonal approximation to the inverse of a totally nonnegative matrix, in the Frobenius-norm, an M-matrix?

This question was asked me by Mr. Michael Neumann during my stay at the University of Connecticut. Also some other researchers worked at this question. Generally, the inverse M-,Z-, and tridiadagonal matrix problems, are a main topic in matrix theory.

In this paper we will show that the abovementioned conjecture is wrong, but the practical experiments, we will show later on some examples, led *mostly* to X's which are M-matrices. So, the question which additional, *probably weak*, conditions for A we need, to get an M-matrix X, remains open.

Our main tool, to show that the conditions above are not sufficient, is a result of A.Ben-Tal and M.Teboulle in Ref.[1], which can also derived from more general theorem in L.Berg [2]. They show, that the least square solutions of rectangular systems lie in a