Note on Topological Hochschild Homology

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1 Introduction

In the appendix to [16] there is discussed a trace map $tr : K(R) \longrightarrow HH(R)$ from the algebraic K-theory of a simplicial ring to its Hochschild homology. In the main body of this paper a similar map from the algebraic K-theory $A(X)$ of a topological space X, as defined in [15], to stable K-theory is constructed, and one would like to interpret this in "algebraic" terms.

Informally one can view the K-theory of spaces as sort of a Quillen K - theory for rings up to homotopy. For an algebraic version of stable K - theory one would substitute the tensor products (over the integers) in the usual simplicial definition of Hochschild homology with a "tensor product" over the more basic ring up to homotopy $\Omega^\infty S^\infty$, i.e. with a suitable definition of the smash product of spectra.

Even for honest simplicial rings this view results in a nontrivial factorization of the trace map tr through a functor which – thanks to formal similarity – inherits the name "Topological Hochschild Homology" THH. There was also good reason to hope that $THH(R)$ would provide more information about $K(R)$ for classical rings than $HH(R)$.

A rigid construction of THH for FSP's – functors with smash product – was first described in Bökstedt's paper [1]. It involves explicit space level constructions for smash products of spectra. Since this smash product does not have the same formal properties as the tensor product on the nose, the definition of THH is not a verbatim translation of the one of HH.

In the meantime there have been constructed a number of symmetric monoidal smash product functors on suitable categories of spectra with the same formal properties as the classical tensor product [3], [5], [7], which allow for definitions of THH – for arbitrary A_∞ - rings – completely analogous to the one of HH. In the present paper we work in the setting of [3] and exploit this spectrum level view point of THH. We show that it leads to simpler, more conceptual proofs and extensions of early calculations of THH by Lindenstrauss [6], Pirashvili [11] and others who primarily used spectral sequence arguments.

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2 Categories of modules and algebras

Unless stated otherwise we use the terminology of [3].

2.1 Let R be a commutative $\mathbf{S}$-algebra and $\mathcal{M}od_R$ the category of R-module spectra. There is a commutative, associative and unital smash product functor $\wedge_R$ on $\mathcal{M}od_R$ making it a symmetric monoidal category [3, III. 3]. One has control over the homotopy type of $M \wedge_R N$ provided M or N is a cell R-module: there is a strongly convergent natural right half-plane spectral sequence of differential $\pi_*(R)$-modules

$$E^2_{p,q} = \mathrm{Tor}^{\pi_*(R)}_{p,q}(\pi_* M,\ \pi_* N) \Longrightarrow \pi_{p+q}(M \wedge_R N).$$

One can always resolve an R-module M: there is a cell R-module $\Gamma_R M$ and a weak equivalence $\Gamma_R M \longrightarrow M$. We say that $M \wedge_R N$ has the *correct homotopy type* if $\Gamma_R M \wedge_R N \to M \wedge_R N$ is a weak equivalence.

If R is a classical ring and M a right, N a left R-module, then the Eilenberg-MacLane spectra HM and HN can be represented as cell HR-module spectra and [3, IV. 2.1]

$$\pi_*(HM \wedge_{HR} HN) \cong \mathrm{Tor}^R_*(M, N).$$

More generally, if R is commutative and M is a CW HR-module spectrum there exists an associated cellular chain complex $C_*(M)$ of free R-modules such that $\pi_*(M) \cong H_*(C_*(M))$ [3, IV.2.3]. Moreover, if M and N are CW HR-module spectra there is a natural isomorphism

$$C_*(M \wedge_{HR} N) \cong C_*(M) \otimes_R C_*(N)$$

which is associative, commutative, and unital [3, IV.2.5] and [8, (2.4)].

2.2 The categories ${}_R\mathcal{M}od$, $\mathcal{M}od_R$, $\mathcal{A}lg_R$, and $\mathcal{CA}lg_R$ of left and right R-modules, R-algebras, and commutative R-algebras admit topological closed model category structures. Their weak equivalences are the weak equivalences of underlying spectra, their cofibrations, called q-cofibrations, are retractions of relative cell objects in their categories. Since retractions preserve weak equivalences it suffices to resolve an R-module M by a q-cofibrant one to obtain homotopical control of the smash product. For smash products of (commutative) algebras the argument is different because the forgetful functors

$$\mathcal{A}lg_R \longrightarrow \mathcal{M}od_R \qquad \mathcal{CA}lg_R \longrightarrow \mathcal{M}od_R$$

do not preserve q-cofibrant objects. Mandell's work [8] provides a remedy: if R is a classical commutative algebra and A and B are q-cofibrant commutative HR-algebras, then the isomorphism above induces a natural isomorphism of graded algebras

$$\pi_*(A \wedge_{HR} B) \cong H_*(C_*(A) \otimes_R C_*(B))$$

We will also need the following results.