ENDOFINITENESS IN STABLE HOMOTOPY THEORY

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Abstract. We study endofinite objects in a compactly generated triangulated category in terms of ideals in the category of compact objects. Our results apply in particular to the stable homotopy category. This leads, for example, to a new interpretation of stable splittings for classifying spaces of finite groups.

Introduction

A CW-complex is said to be *endofinite* if all stable homotopy groups $\pi_n^s(X) = \{S^n, X\}$ are finite length modules over the ring $\{X, X\}$ of stable self-maps of X. In this paper we study CW-complexes having this finiteness property. One motivation for this work is to understand stable splittings of certain spaces, for instance the classifying space BG of a finite group G. We shall give an algebraic interpretation of such splittings in terms of certain ideals in the category of finite spectra.

A natural framework for the study of endofinite objects are compactly generated triangulated categories. Thus we consider a triangulated category $\mathcal{C}$, for example the stable homotopy category, which has arbitrary coproducts and is generated by a set of compact objects (an object X in $\mathcal{C}$ is *compact* if the representable functor $\operatorname{Hom}(X, -)$ preserves coproducts). An object X in $\mathcal{C}$ is said to be *endofinite* if the group $\operatorname{Hom}(C, X)$ is a finite length module over the endomorphism ring $\operatorname{End}(X)$ for all compact objects C in $\mathcal{C}$. This finiteness condition has a number of interesting consequences. For example, in [15] it is shown that every endofinite object decomposes essentially uniquely into indecomposable objects with local endomorphism rings. Here, we shall study endofinite objects in terms of certain ideals in the full subcategory $\mathcal{C}_0$ of compact objects in $\mathcal{C}$. Viewing $\mathcal{C}_0$ as a ring with several objects, we consider for every X in $\mathcal{C}$ the *annihilator* $\operatorname{Ann} X$ of X which is the ideal of maps ϕ in $\mathcal{C}_0$ such that $\operatorname{Hom}(\phi, X) = 0$. Two types of ideals $\mathfrak{I}$ in $\mathcal{C}_0$ are of particular importance:

- $\mathfrak{I}$ is *cohomological* if for every sequence of maps in $\mathcal{C}_0$

$$X' \xrightarrow{\phi'} X \xrightarrow{\phi} Y \xrightarrow{\phi''} Y''$$

such that ϕ' is a fibre of $\phi''\phi$, the map ϕ belongs to $\mathfrak{I}$ if $\phi\phi'$ and $\phi''\phi$ belong to $\mathfrak{I}$. We shall see that the cohomological ideals in $\mathcal{C}_0$ are precisely the annihilator ideals of objects in $\mathcal{C}$. In fact, for every cohomological ideal $\mathfrak{I}$ there exists a canonical object $E_{\mathfrak{I}}$ such that $\mathfrak{I} = \operatorname{Ann} E_{\mathfrak{I}}$.

- $\mathfrak{I}$ is *cofinite* if for every X in $\mathcal{C}_0$ there exists $n \in \mathbb{N}$ such that for every sequence of maps in $\mathcal{C}_0$

$$X \xrightarrow{\phi_0} X_0 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_n} X_n$$

the composition of $\phi_{i-1} \dots \phi_0$ with a fibre of $\phi_i \dots \phi_0$ belongs to $\mathfrak{I}$ for some $i \leq n$. Let us denote by $\operatorname{Max} \mathcal{C}_0$ the set of maximal elements among the cofinite cohomological ideals $\mathfrak{I} \neq \mathcal{C}_0$ in $\mathcal{C}_0$.

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Theorem A. *An object X in $\mathcal{C}$ is endofinite if and only if the annihilator of X is cofinite. Moreover the maps $X \mapsto \operatorname{Ann} X$ and $\mathfrak{I} \mapsto E_{\mathfrak{I}}$ induce mutually inverse bijections between the set of isomorphism classes of indecomposable endofinite objects in $\mathcal{C}$ and* $\operatorname{Max} \mathcal{C}_0$.

Any endofinite object X has an essentially unique decomposition $X \cong \coprod_{i \in I} X_i$ into indecomposable endofinite objects, and the annihilator of X controls this decomposition. More precisely, an indecomposable object Y arises as a direct factor of an endofinite object X if and only if $\operatorname{Ann} X \subseteq \operatorname{Ann} Y$. This is a particular consequence of the following result:

Theorem B. *Let X be an endofinite object in $\mathcal{C}$ with $\mathfrak{I} = \operatorname{Ann} X$. Then*

$$\mathfrak{I} = \bigcap_{\mathfrak{I} \subseteq \mathfrak{J} \in \operatorname{Max} \mathcal{C}_0} \mathfrak{J}$$

and X has a unique decomposition

$$X \cong \coprod_{\mathfrak{I} \subseteq \mathfrak{J} \in \operatorname{Max} \mathcal{C}_0} X_{\mathfrak{J}}$$

such that each $X_{\mathfrak{J}}$ is a non-empty coproduct of copies of $E_{\mathfrak{J}}$. Moreover, $\operatorname{Ann} X \subseteq \operatorname{Ann} Y$ *holds for any object Y if and only if Y is isomorphic to a direct factor of a* (*co*)*product of copies of X.*

Let us now explain some special results for the stable homotopy category of spectra. A fairly immediate consequence of Theorem B is the fact that we can work one prime at a time.

Corollary. *Let X be an endofinite spectrum. Then the $\mathfrak{p}$-localization map $X \to X_{\mathfrak{p}}$ is a split epimorphism for every prime ideal $\mathfrak{p}$ in $\mathbb{Z}$. If X_{tor} denotes the fibre of the rational localization $X \to X_{(0)}$, then $X \cong X_{(0)} \vee X_{\mathrm{tor}}$ and $X_{\mathrm{tor}} \cong \bigvee_{\mathfrak{p}} (X_{\mathrm{tor}})_{\mathfrak{p}}$ where $\mathfrak{p}$ runs through all non-zero prime ideals in $\mathbb{Z}$.*

Having developed a general theory of endofinite spectra, one may ask for conditions on a CW-complex X such that its suspension spectrum is endofinite. It turns out that this happens if and only if all stable homotopy groups of X are finite length modules over the ring of stable self-maps of X, hence if the CW-complex X is endofinite. Here is a useful criterion:

Theorem C. *Let X be a connected CW-complex such that the singular homology groups $H_n(X; \mathbb{Z})$ are finite for all $n > 0$. Then X is an endofinite CW-complex, and therefore its suspension spectrum is also endofinite.*

Important examples of CW-complexes with finite homology are the classifying spaces of finite groups.

Corollary. *The classifying space BG of a finite group G is endofinite.*

It is a well-known consequence of Carlsson's solution of the Segal conjecture [2] that BG splits stably into indecomposable spectra; see [9] for an alternative proof. We recover this result, and our ideal theoretic approach reduces the problem of finding such splittings to the category of finite spectra.

The concept of endofiniteness has its origin in representation theory of finite dimensional algebras. Following Crawley-Boevey, a module X over some associative ring Λ is *endofinite* if it has finite length when regarded as a module over its endomorphism ring