DEGENERATION FOR PARABOLIC GROUP ACTIONS IN GENERAL LINEAR GROUPS

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Dedicated to Helmut Lenzing on the occasion of his sixtieth birthday

ABSTRACT. Let k be an algebraically closed field and V a finite dimensional k-space. Let $\mathrm{GL}(V)$ be the general linear group of V and P a parabolic subgroup of $\mathrm{GL}(V)$. Now P acts on its unipotent radical P_u and on $\mathfrak{p}_u = \mathrm{Lie}\, P_u$, the Lie algebra of P_u, via the adjoint action. More generally, we consider the action of P on the l-th member of the descending central series of $\mathfrak{p}_u$ denoted by $\mathfrak{p}_u^{(l)}$. All instances when P acts on $\mathfrak{p}_u^{(l)}$ for $l \geq 0$ with a finite number of orbits are known. In this note we give a complete description of the closure relations among the P-orbits on $\mathfrak{p}_u^{(l)}$ in all these finite cases.

There is a canonical bijection between the set of P-orbits on $\mathfrak{p}_u^{(l)}$ and the set $\mathcal{F}(\Delta)(\mathbf{e})$ of isomorphism classes of Δ–filtered modules of a particular dimension vector $\mathbf{e}$ of a certain quasi-hereditary algebra $\mathcal{A}(t,l)$. These isomorphism classes in turn are given by the orbits of the reductive group $G(\mathbf{e}) = \prod \mathrm{GL}(e_i)$ on the variety $\mathcal{R}(\Delta)(\mathbf{e})$ of all representations of the quasi-hereditary algebra $\mathcal{A}(t,l)$ with Δ–filtration and dimension vector $\mathbf{e}$.

In our chief result, Theorem 1.1, we show that provided there is only a finite number of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$ the following three posets coincide: (1) the Bruhat-Chevalley order on the set of P-orbits on $\mathfrak{p}_u^{(l)}$; (2) the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$; (3) the poset opposite to the so called hom–order on the set of isoclasses of $\mathcal{F}(\Delta)(\mathbf{e})$.

The advantage of this hom–order is that it is given purely by discrete invariants and that it can be computed explicitly for any given finite case. Theorem 1.1 then in turn allows us to explicitly determine the closure relations for the P-orbits on $\mathfrak{p}_u^{(l)}$ with the aid of this hom–order. We present some examples of Hasse diagrams in an Appendix.

We introduce the notion of a global minimal degeneration in the category $\mathcal{F}(\Delta)$ as one which cannot be refined by passing to higher dimension vectors by taking direct sums of modules, and show that these particular degenerations can be obtained directly from the Auslander-Reiten quiver of $\mathcal{F}(\Delta)$.

1. INTRODUCTION

Throughout, k is an algebraically closed field and V a finite-dimensional k-vector space and P is a parabolic subgroup of $\mathrm{GL}(V)$ with unipotent radical P_u. The Lie algebra of P_u is denoted by $\mathfrak{p}_u$. The descending central series of $\mathfrak{p}_u$ is defined as usual by $\mathfrak{p}_u^{(0)} := \mathfrak{p}_u$ and $\mathfrak{p}_u^{(l)} := [\mathfrak{p}_u, \mathfrak{p}_u^{(l-1)}]$, for $l \geq 1$. The group P acts on each $\mathfrak{p}_u^{(l)}$ via the adjoint action. The *Bruhat-Chevalley order* on the set of P-orbits on $\mathfrak{p}_u^{(l)}$ is given by the orbit closures: for two orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ we write $\mathcal{O}_1 \leq \mathcal{O}_2$ whenever $\mathcal{O}_1$ is contained in the (Zariski) closure of $\mathcal{O}_2$. The aim of this note is to

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give a complete description of this partial order provided the set of P-orbits on $\mathfrak{p}_u^{(l)}$ is finite. In [5], [6], and [2] precisely all those instances are determined.

A Levi subgroup L of P is (isomorphic to) a product of general linear groups, say $\mathrm{GL}(d_i)$, for $1 \leq i \leq t$ for some $t \in \mathbb{N}$, with $\dim V = \sum d_i$. The ordered tuple $\mathbf{d} = (d_1, \ldots, d_t)$ determines the conjugacy class of P in $\mathrm{GL}(V)$. To indicate this, we often write $P = P(\mathbf{d})$.

Given $\mathbf{d} \in \mathbb{N}^t$ we define the t-tuple $\mathbf{e} := \Sigma\mathbf{d}$ by $e_j = (\Sigma\mathbf{d})_j := \sum_{i=1}^{j} d_i$ for $1 \leq j \leq t$. For $P = P(\mathbf{d})$ there is a canonical bijection between the set of P-orbits on $\mathfrak{p}_u^{(l)}$ and the set of isomorphism classes of Δ–filtered modules $\mathcal{F}(\Delta)(\mathbf{e})$ of dimension vector $\mathbf{e}$ of a certain quasi-hereditary algebra $\mathcal{A}(t,l)$ ([2], [5]). These isomorphism classes in turn are given by the orbits of the reductive group $G(\mathbf{e}) := \prod \mathrm{GL}(e_i)$ on the variety $\mathcal{R}(\Delta)(\mathbf{e})$ of all representations of the quasi-hereditary algebra $\mathcal{A}(t,l)$ with Δ–filtration and dimension vector $\mathbf{e}$. In this context we may also consider the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$.

Let X and Y be in $\mathcal{F}(\Delta)(\mathbf{e})$. We write $X \leq_{\mathrm{hom}} Y$ provided $\dim \mathrm{Hom}(X, I) \leq \dim \mathrm{Hom}(Y, I)$ for every indecomposable module I in $\mathcal{F}(\Delta)$. As in the case of Artin algebras it turns out that this induces a partial order on the set of isomorphism classes of $\mathcal{F}(\Delta)(\mathbf{e})$ to which we refer to as the hom–*order* on $\mathcal{F}(\Delta)$.

Our principal result concerning the action of P on $\mathfrak{p}_u^{(l)}$ is

Theorem 1.1. *Let $t, l \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}^t$ and let V be a finite-dimensional k-vector space such that $\dim V = \sum d_i$. Let $P = P(\mathbf{d})$ be the standard parabolic subgroup of $\mathrm{GL}(V)$ associated to $\mathbf{d}$. Set $\mathbf{e} = \Sigma\mathbf{d}$. Suppose that the number of P-orbits on $\mathfrak{p}_u^{(l)}$ is finite. Then the following posets coincide:*

(i) *the Bruhat-Chevalley order on the set of P-orbits on $\mathfrak{p}_u^{(l)}$;*
(ii) *the Bruhat-Chevalley order on the set of $G(\mathbf{e})$-orbits on $\mathcal{R}(\Delta)(\mathbf{e})$;*
(iii) *the poset opposite to the* hom*–order on the set of isoclasses of $\mathcal{F}(\Delta)(\mathbf{e})$.*

For our setting the advantage of working with the hom–order is due to the fact that it is given purely by discrete invariants, more specifically, by the ordered tuples $\underline{\mathrm{hdim}}(X)$ whose j-th entry consists of $\dim \mathrm{Hom}(X, I_j)$, where I_j runs through a complete set of representatives of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$. Consequently, once the square matrix $(\dim \mathrm{Hom}(I_i, I_j))_{i,j}$ is computed, the hom–order on $\mathcal{F}(\Delta)(\mathbf{e})$ can be computed explicitly in any finite instance. Thus Theorem 1.1 allows us to explicitly determine the closure relations for the P-orbits on $\mathfrak{p}_u^{(l)}$ with the aid of this hom–order by computational means. We illustrate some examples in Appendix A.

While this method solves our initial problem of computing the Hasse diagram for any fixed choice of the parameters t, l and $\mathbf{d}$, in Section 5 we take a more conceptual approach to degenerations which is independent of these parameters. The crucial idea in this context is the notion of a *global minimal degeneration*: by that we mean a pair of modules X and Y in $\mathcal{F}(\Delta)$ of the same dimension vector without common direct summand such that the difference $\underline{\mathrm{hdim}}(X) - \underline{\mathrm{hdim}}(Y)$ is a standard basis vector in $\mathbb{Q}^r$, where r is the number of isomorphism classes of indecomposable modules in $\mathcal{F}(\Delta)$. In our principal result in Section 5 we show that such pairs correspond bijectively to almost split sequences in the Auslander-Reiten quiver of $\mathcal{F}(\Delta)$. The AR-quivers in turn can be computed explicitly in each of the relevant cases; for $l \geq 3$ these quivers can even be constructed in a uniform fashion. Ultimately, we obtain from these AR-quivers explicit formulae for the number of