Comparison of systems of stochastic partial differential equations

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Abstract

We provide a general comparison theorem for systems of stochastic partial differential equations, which is a powerful tool to study stability as well as stochastic invariance of the corresponding solution processes. An application showing stability properties of a predator-prey system is given.

KEY WORDS stochastic partial differential equation, comparison theorem

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1 Introduction

In contrast to the theory of ordinary stochastic differential equations, the theory of stochastic partial differential equations lacks of an important tool, namely, we have not a well-applicable Itô formula. One technique to handle the difficulties arising from the missing Itô formula is the comparison technique.

So, several authors, [3], [4], [5], [6], [8], proved comparison theorems for stochastic partial differential equations of the form

$$\frac{\partial}{\partial t}u(t,y) \;=\; Au(t,y) + f(t,y,u(t,y)) + g(t,y,u(t,y)) \cdot \dot{\eta}(t,y) \tag{1}$$

which are of special interest. Here A denotes a differential operator, f, g are real functions on $[0,T]\times\Omega\times\mathcal{O}\times\mathbb{R}$, $\mathcal{O}\subseteq\mathbb{R}^d$, and the gaussian noise $\dot{\eta}(t,y)$ is white in time resp. white or coloured in space. We mention that P. Kotelenez, [4], was the first one who considered a wide class of such equations with nontrivial diffusion coefficients. But in their recent paper [9], G. Tessitore and J. Zabczyk especially show that some arguments of the proof given in [4] are not satisfactory. The authors in [3], [6], [8] only investigate the case $d=1$ under the additional assumption that A is the one-dimensional Laplacian with certain boundary conditions. T. Shiga resp. C. Mueller (cf. [8],[6]) approximate the Laplacian using a discretization procedure and we do not see how to apply this method in the case of more general differential operators. Differently the method used by C. Donati-Martin and E. Pardoux (cf. [3]), we feel to be applicable also in the case of an operator A which is the generator of a positivity preserving semigroup on $L^2(\mathcal{O})$, $\mathcal{O}\subseteq\mathbb{R}^d$, even in dimensions $d\geq 2$. However, this extension having not yet been done in [3] is now accomplished in [5] using a different method.

What we are doing in this paper is less to prove a better version of the comparison theorems given in [3] resp. [5] but more to apply them to <u>systems</u> of stochastic partial differential equations,

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that is f, g in (1) are not longer supposed to be real functions but, for example, n-dimensional vector fields. In this sense, our results extend the results in [1] for systems of ordinary stochastic differential equations to the more complicated case of stochastic partial differential equations, partly. We give remarks concerning the generality of the conditions we need.

Finally, in order to demonstrate the force of our comparison theorem, we study a hypothetical predator-prey system describing the phenomenon that the growth rate of the predator population depends on the ratio between prey and predator. At first, we discuss aspects of the stability of the basic deterministic system in a geometrical way which is interesting in itself and second, applying the same method, we show this stability properties also for the stochastic system obtained from the deterministic one by disturbing the coefficients, randomly.

Remark that we only deal with a more realistic model in a forthcoming paper because the difficulties arising from coefficients with singularities would not fit into the frame of this paper.

2 Notations

On a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ let be given m independent cylindrical Wiener processes $W_1(t), ..., W_m(t)$ in $\mathfrak{D}'(\mathcal{O})$ where $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with sufficiently smooth boundary. Denote by $\mathbf{L}^2(\mathcal{O})$ the Hilbert space of vector fields $\mathbf{u} : \mathcal{O} \to \mathbb{R}^n$, $n \geq 2$, satisfying

$$\|\mathbf{u}\|_2^2 := \int_{\mathcal{O}} \|\mathbf{u}(y)\|_{\mathbb{R}^n}^2 \, \mathrm{d}y \; < \; \infty.$$

Here and in what follows, we place emphasis on n-dimensional objects by printing them as bold face letters. Clearly, n is the number of equations our system of stochastic partial differential equations consists of.

Define the multiplication operator

$$\boldsymbol{\mathcal{M}} : \mathbf{L}^2(\mathcal{O}) \to \mathfrak{L}(L^\infty(\mathcal{O}), \mathbf{L}^2(\mathcal{O}))$$

by

$$[\boldsymbol{\mathcal{M}}(\mathbf{u})\phi](y) \; = \; \phi(x)\mathbf{u}(y), \; x \in \mathcal{O},$$

and fix m bounded linear operators $B_1, ..., B_m : L^2(\mathcal{O}) \to L^\infty(\mathcal{O})$ with the property that

$$\|\boldsymbol{\mathcal{M}}(\mathbf{u}) \circ B_j\|_{\mathfrak{L}^2}^2 \; \leq \; C(B)\|\mathbf{u}\|_2^2, \; \mathbf{u} \in \mathbf{L}^2(\mathcal{O}), \; j = 1, ..., m, \tag{2}$$

where $\| \cdot \|_{\mathfrak{L}^2}$ denotes the Hilbert-Schmidt norm on $\mathfrak{L}(L^2(\mathcal{O}), \mathbf{L}^2(\mathcal{O}))$.

On $\mathbf{L}^2(\mathcal{O})$ we consider a separate operator $(\mathbf{A}, D(\mathbf{A}))$, that is

$$\mathbf{A}\boldsymbol{\phi} \; = \; \begin{pmatrix} A_1\phi_1 \\ \vdots \\ A_n\phi_n \end{pmatrix}, \quad \boldsymbol{\phi} \in D(\mathbf{A}),$$

which is assumed to be the generator of a strongly continuous semigroup $(\mathbf{S}(t))_{t \geq 0}$. Obviously, $\mathbf{S}(t)$ is also separated into n strongly continuous semigroups on $L^2(\mathcal{O})$:

$$\mathbf{S}(t)\boldsymbol{\phi} \; = \; \begin{pmatrix} S_1(t)\phi_1 \\ \vdots \\ S_n(t)\phi_n \end{pmatrix}, \quad \boldsymbol{\phi} \in \mathbf{L}^2(\mathcal{O}).$$

Using the Banach-Steinhaus theorem, for a fixed time horizon $T > 0$, we have

$$\sup_{t \leq T} \|\mathbf{S}(t)\|_{\mathfrak{L}(\mathbf{L}^2(\mathcal{O}))} \; \leq \; C \; = \; C(T) \; < \; \infty.$$