Three Symmetries Groups.

Felikson A. A. & Tumarkin P. V.

Introduction

Let $\Gamma = \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ be a free product of three involutions s_1, s_2 and s_3. Denote by r the composition of these involutions $s_1 s_2 s_3$. Consider the discrete embedding $\Gamma \to PSL_2(\mathbf{R})$, that is injective homomorphism whose image G is a discrete group acting on hyperbolic plane L^2. We are interested in the embeddings such that G has a non-compact fundamental domain of finite volume. The image of s_i is the symmetry with respect to a point z_i. Since the volume is finite, the image of r is parabolic. In this paper we study Teihmuller and moduli spaces of these groups.We also find the modular group.

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Three symmetries group

Let z_1, z_2 and z_3 be three distinct points on the hyperbolic plane; let G be a group generated by symmetries τ_1, τ_2, τ_3 with respect to these points, and let ρ be the composition of these symmetries $s_1 s_2 s_3$. In [1] it is proved that

1) if ρ is elliptic, then either G is non-discrete or its fundamental domain is compact.

2) if ρ is hyperbolic, then G is discrete, but its fundamental domain has infinite volume.

3) if ρ is parabolic, then G is a discrete group with non-compact fundamental domain of finite volume.

In the last two cases ρ has at least one fixed point at infinity. Hence, in both cases we have an ideal triangle whose sides m_1, m_2 and m_3 contain the points z_1, z_2 and z_3 respectively (see Fig. 1). We find an easy geometric way to understand if ρ is parabolic.

Theorem 1. *Let h_i, $(i = 1, 2, 3)$ be a line perpendicular to m_i and $z_i \in h_i$. The composition ρ is parabolic if and only if the lines h_i either have a common point or have a common perpendicular.*

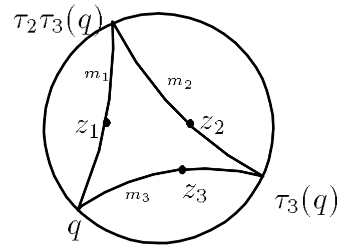

Figure 1.

Plan of the proof

By **altitude** of the triangle ABC we mean a line AH perpendicular to the side BC.

First, we prove that for any pair of points z_1, z_2 ($z_1 \in m_1, z_2 \in m_2$) there exists a unique point $z_3 \in m_3$ such that the composition ρ is parabolic (see lemma 1).

Then we prove that the line h_3 is orthogonal to a line $\overline{z_1 z_2}$ containing z_1 and z_2 (see lemma 2). Therefore, h_3 is an altitude of the triangle $z_1 z_2 z_3$. Similarly, h_1 and h_2 are altitudes of this triangle. Three altitudes of a hyperbolic triangle either have a common point (may be at infinity) or have a common perpendicular. In the Klein model, it means that the altitudes have a common point (though it may be outside the disc).

Thus, we have a necessary condition. The sufficiency is proved in lemma 3.

Remark. Let z_1, z_2 and z_3 be three distinct points on the hyperbolic plane. The proof of the theorem involves a way to understand if the composition of the symmetries $\tau_1\tau_2\tau_3$ is parabolic. Let h_i ($i = 1, 2, 3$) be a line containing z_i, $h_i \perp \overline{z_j z_k}$ (where i, j, k are different). Let s_i be a line through z_i and orthogonal to h_i. Then $\tau_1\tau_2\tau_3$ is parabolic if and only if the lines s_i compose an ideal triangle (moreover, it is easy to prove that $\tau_1\tau_2\tau_3$ is parabolic if s_1 intersects s_2 at infinity).

Equidistant line

Consider points z_1, z_2 and z_3 in the upper halfplane model.
Assume that $z_1 = i$, $z_2 = ki$, where $\ln k$ is the distance between z_1 and z_2. Let $z_3 = a + bi$. Then the symmetries τ_i are represented by matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sim \tau_1, \qquad \frac{1}{k}\begin{pmatrix} 0 & -k^2 \\ 1 & 0 \end{pmatrix} \sim \tau_2, \qquad \frac{1}{b}\begin{pmatrix} a & -a^2 - b^2 \\ 1 & -a \end{pmatrix} \sim \tau_3$$

(up to the multiple $-E$).
We obtain $|tr\ \rho| = |\frac{a}{b}|\frac{k^2-1}{k}$.

It is well known that ρ is parabolic if and only if $|tr\ \rho| = 2$. This equation determines two rays on the halfplane; these rays compose the equidistant to the line $\overline{z_1 z_2}$. The distance d between this equidistant and $\overline{z_1 z_2}$ satisfies the following condition: $\cosh d = \frac{k^2+1}{k^2-1}$. If z_3 is located above the equidistant, then $|tr\ \rho| < 2$ and ρ is elliptic. Analogously, ρ is hyperbolic if z_3 lies under the equidistant.

In the unit disc model the equidistant is an arc (part of euclidean circle) containing "the ends" of the line.