On condensed forms for partially commuting matrices

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Abstract

Two complex $n \times n$ matrices A and B are said to be partially commuting if A and B have a common eigenvector. We propose a condensed form for such matrices that can be obtained from A and B by a finite rational computation. The condensed form is a pair of block triangular matrices, with the sizes of the blocks being uniquely defined by the original matrices. We then show how to obtain additional zeros inside the diagonal blocks of a condensed form by using the generalized Lanczos procedure as given by Elsner and Ikramov. This procedure can also be considered as a finite rational process. We point out several applications of the constructions above. It turns out that for Laffey pairs of matrices, i.e., for matrices (A, B) such that rank $[A, B] = 1$, the condensed form is a pair of 2×2 block triangular matrices. Using this fact, we show an economical way to find a spanning set for the matrix algebra generated by Laffey matrices A and B. Another application concerns so-called k-selfadjoint matrices. We examine such matrices in the unitary space as well as in a Krein space of defect 1. As an addendum, we give a new description of the Shemesh subspace of matrices A and B. This is the maximal common invariant subspace of A and B, on which these matrices commute.

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1 Introduction

There exist many variations of the familiar notion of a commuting pair of matrices: nearly commuting matrices, quasi-commuting matrices, and so on. We wish to extend this list by a new term.

Definition Two complex $n \times n$ matrices A and B are said to be *partially commuting* if A and B have a common eigenvector.

One good reason for coining this term is obvious: if x is a nonzero vector such that

$$Ax = \lambda x, \qquad Bx = \mu x,$$

then there exist nontrivial common invariant subspaces of A and B, on which these matrices commute. The span of the vector x above is one of such subspaces.

A criterion for two complex matrices to have a common eigenvector was given by D. Shemesh in [7].

Theorem 1 Let $A, B \in M_n(\mathbf{C})$. Then A and B are partially commuting if and only if the subspace $\mathcal{N}$ defined by the formula

$$\mathcal{N} = \bigcap_{k,\ell=1}^{n-1} \ker\, [A^k, B^\ell] \tag{1}$$

is nontrivial.

Here and henceforth, the symbol $M_n(\mathbf{C})$ stands for the space of complex $n \times n$ matrices, and $[F, G]$ denotes the commutator of F and G: $[F, G] = FG - GF$.

In fact, the proof of Theorem 1 in [7] contains more than the statement above reveals. We state this extra as a separate theorem.

Theorem 2 The subspace (1) is invariant with respect to both matrices A and B, and A and B commute on $\mathcal{N}$. Moreover, $\mathcal{N}$ contains any other subspace of $\mathbf{C}^n$ with these properties.

It is our intention in this paper to point out some implications of Theorem 2 relating to the following problem.

Problem For given partially commuting matrices A and B, find matrices R and S with as many zeros as possible that can be obtained from the original matrices by a simultaneous similarity transformation

$$A \to R = P^{-1}AP, \qquad B \to S = P^{-1}BP. \tag{2}$$

Moreover, it is required that the similarity matrix P could be found by a finite computation employing only arithmetical operations. (For brevity, expressions like "P admits a finite rational computation" or "P can be computed rationally" are used in the paper.)

To make the point clearer, consider matrices A and B that commute on the whole space $\mathbf{C}^n$. It is almost universally known that such matrices are reducible