On Gelfand-Zetlin modules over orthogonal Lie algebras

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Descriptive title: Gelfand-Zetlin modules
1991 Mathematics Subject Classification: 17B10, 17B35

Abstract

We construct a new family of simple modules over orthogonal complex Lie algebra associated with Gelfand-Zetlin formulae for simple finite dimensional modules and study the corresponding Gelfand-Zetlin subalgebra.

1 Introduction

Explicit formulae which effectively define all simple finite dimensional modules over the groups of unimodular and orthogonal matrices were obtained by Gelfand and Zetlin in their famous papers [GZ1, GZ2]. Using these formulae it is possible to define and investigate big families of modules over the corresponding Lie algebras, as was done (for special real forms on the Lie algebras) in [O1, O2]. The analogous formulae are also known for quantum groups (see [J, NT, GK] and references therein). In the same way, using the formulae for the unimodular group, Drozd, Futorny and Ovsienko constructed a large family of simple modules over the reductive Lie algebra $\mathfrak{G} = \mathfrak{g}l(n, \mathbb{C})$ in [DFO1, DFO2]. Roughly speaking this is an $n(n+1)/2$-parameter family of simple $\mathfrak{G}$-modules and each module is presented in a very convenient base and hence is quite simple for computations. It seems that this is the biggest known family of simple weight modules for $\mathfrak{G}$.

Gelfand-Zetlin modules are defined as $U(\mathfrak{G})$-modules which can be decomposed into a direct sum of finite-dimensional modules with respect to the so-called Gelfand-Zetlin subalgebra, which is a big commutative subalgebra in $U(\mathfrak{G})$. In fact, in [O] it was shown that Gelfand-Zetlin subalgebra is a maximal commutative subalgebra in $U(\mathfrak{G})$. Roughly speaking, Gelfand and Zeltlin show that Gelfand-Zetlin subalgebra has a simple spectrum on all finite-dimensinal modules. Simple modules constructed in [DFO1, DFO2] inherit this property and have a natural base parametrized by the so-called (Gelfand-Zetlin) tableaux (or patterns), associated with the points of the spectrum, making them especially easy to handle.

Further properties of Gelfand-Zetlin modules were obtained in [M1, M2, MO, O]. For example a huge family of Verma and generalized Verma modules were realized as Gelfand-Zetlin modules, which allows one to describe the structure of these modules. It was also proved that any simple module over the Gelfand-Zetlin algebra possesses a non-trivial extension to an $U(\mathfrak{G})$-module. Regretfully, all the above results were obtained only for the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$.

The aim of this paper is to construct the Gelfand-Zetlin modules for orthogonal complex Lie algebras and to investigate their properties. Our technique is based on the Gelfand-Zetlin formulae for orthogonal group, see also [GZ1, BR]. In particular, we construct a new family of simple modules over orthogonal complex Lie algebras, investigate the structure of some modules having a tableaux realization and construct the orthogonal operator algebras associated with Gelfand-Zetlin formulae.

The paper is organized as follows: in Section 2 we collect all necessary preliminaries and notations. In Section 3 we recall the Gelfand-Zetlin formulae for the orthogonal group. In Sections 4,5 we introduce Gelfand-Zetlin subalgebra in an orthogonal Lie algebra and construct the family of simple gelfand-Zetlin modules. In Section 6 we propose a simplification for the Gelfand-Zetlin formulae. In Section 7 using the simplified Gelfand-Zetlin formulae we construct two families of modules having a tableaux realization (in he sense of [M1]). Finally, in Section 8 we construct the corresponding orthogonal algebras.

2 Notations and preliminaries

Throughout the paper we fix a base field $\mathbb{C}$ of complex numbers. As usual, we will denote by $\mathbb{Z}$ and $\mathbb{N}$ the sets of integers and positive integers respectively. For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra and by $Z(\mathfrak{A})$ the center of $U(\mathfrak{A})$. By module we will mean left module.

Suppose that A is an associative algebra and B is its commutative subalgebra. An A-module V is said to be a B-weight (root) module (or simply a weight or root module, if B is fixed) provided it can be decomposed into a direct sum of one dimensional (finite dimensional) B-modules or, equivalently,

$$V = \bigoplus_{\lambda\in B^*} V_\lambda \quad \left(V = \bigoplus_{\lambda\in B^*} V^\lambda\right),$$

where V_λ (V^λ) is a weight (root) subspace corresponding to the weight (root) $\lambda \in B^*$. The set of all non-zero weights of the given B-weight (root) module V is usually called the B-support of V.

We will also denote by e_{ij} the matrix units, i.e. matrices in $Mat_{n\times n}(\mathbb{C})$ whose ij-component is one and all others are zero.