Hereditary Triangulated Categories

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ABSTRACT. We call a triangulated category *hereditary,* in case it is triangle equivalent to the bounded derived category $D^b(\mathcal{A})$ of a hereditary abelian category $\mathcal{A}$. Such a hereditary triangulated category has a very lucid structure: it is the additive closure of the shifted copies $\mathcal{A}[t]$ of $\mathcal{A}$, and there are non-zero homomorphisms from objects in $\mathcal{A}[t]$ only to objects in $\mathcal{A}[t]$ and $\mathcal{A}[t+1]$. We are going to show that these properties furnish a characterization of hereditary triangulated categories.

Our main interest lies in triangulated categories $\mathcal{C}$ where every object is a finite direct sum of indecomposable objects and where every idempotent splits. We will consider paths in such categories: they connect indecomposable objects and are given by a sequence of non-zero maps and of jumps from an object X to $X[1]$. In case $\mathcal{C}$ is hereditary, there does not exist a path from $X[1]$ to X, for any indecomposable object X. Conversely, we are going to show that if $\mathcal{C}$ is path-connected and if there exists at least one indecomposable object X with no path from $X[1]$ to X, then $\mathcal{C}$ is hereditary.

These characterizations of hereditary triangulated categories can be used in the study of piecewise hereditary (and thus of quasi-tilted) k-algebras, where k is a field. It follows from our considerations that a finite dimensional connected k-algebra A with a directing object in $D^b(\operatorname{mod} A)$ is derived equivalent to a finite dimensional hereditary algebra, in particular A is piecewise hereditary.

1. Triangles

We assume that $\mathcal{C}$ is a triangulated category. The triangles which we consider will always be distinguished ones. Let us recall from [H], 1.4, the following assertion:

Lemma 1. *Let $X \to Y \to Z \to$ be a triangle with maps u, v, w. The map u is split epi if and only if $v = 0$, if and only if w is split mono.*

Lemma 2. *Let $X \to Y \to Z \to$ be a triangle with maps u, v, w. Assume that there is given a decomposition $Y = Y_1 \oplus Y_2$, write $u = (u_1, u_2)$ and $v = (v_1, v_2)$, where $u_i \colon X \to Y_i$ and $v_i \colon Y_i \to Z$. If $u_2 = 0$, then v_2 is a split monomorphism.*

Proof: Consider the triangle $0 \to Y_2 \to Y_2 \to$ with maps $0, 1, 0$. There are

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commutative squares

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Y_2 \arrow[d, "\iota"] \\
X \arrow[r, "u"] \arrow[d] & Y_1 \oplus Y_2 \arrow[d, "\pi"] \\
0 \arrow[r] & Y_2
\end{tikzcd}$$

where the right hand maps are just the canonical inclusion ι and the canonical projection π (thus the composition is the identity). We may complete these maps to maps of triangles:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Y_2 \arrow[r, "1"] \arrow[d, "\iota"] & Y_2 \arrow[r] \arrow[d, "h"] & {} \\
X \arrow[r, "u"] \arrow[d] & Y_1 \oplus Y_2 \arrow[r, "v"] \arrow[d, "\pi"] & Z \arrow[r] \arrow[d, "h'"] & {} \\
0 \arrow[r] & Y_2 \arrow[r, "1"] & Y_2 \arrow[r] & {} .
\end{tikzcd}$$

Since the compositions on the left are isomorphisms, also $h'h$ is an isomorphism, thus h is split mono. On the other hand, $v_2 = v\iota = h$, thus v_2 is split mono.

Lemma 2*. *Let $X \to Y \to Z \to$ be a triangle with maps u, v, w. Assume that there is given a decomposition $Y = Y_1 \oplus Y_2$, write $u = (u_1, u_2)$ and $v = (v_1, v_2)$, where $u_i \colon X \to Y_i$ and $v_i \colon Y_i \to Z$. If $v_2 = 0$, then u_2 is a split epimorphism.*

Proof. The assertion is just the dual one to Lemma 2.

Lemma 3. *Let $X \to Y \to Z \to$ be a triangle with maps u, v, w. Assume that X, Y are indecomposable and that u is non-zero and not invertible. Also assume that there is given a decomposition $Z = \bigoplus_{i=1}^{t} Z_i$ with indecomposable objects Z_i and write $v = (v_1, \dots, v_t)$, $w = (w_1, \dots, w_t)$, with maps $v_i \colon Y \to Z_i$ and $w_i \colon Z_i \to X[1]$. Then $t \geq 1$ and all the maps v_i, w_i are non-zero and non-invertible.*

Proof: First of all, $t = 0$ would mean $Z = 0$, but then u has to be invertible. Assume that some $v_i = 0$. Then by Lemma 2, w_i is split mono. Since both Z_i and $X[1]$ are indecomposable, it follows that w_i is an isomorphism, thus w is split epi. But this implies that $u = 0$, by Lemma 1, a contradiction. This shows that all v_i are non-zero. By duality, also all w_i are non-zero. On the other hand, if some w_i is invertible, then w is split epi, and by Lemma 1, it follows that $u = 0$, a contradiction. Again, we use duality in order to see that also no map v_i can be invertible.

2. Hereditary triangulated categories

Given an additive category $\mathcal{A}$ and classes $\mathcal{U}, \mathcal{V}$ of objects in $\mathcal{A}$, then we write $\operatorname{Hom}(\mathcal{U}, \mathcal{V}) = 0$ provided $\operatorname{Hom}(U, V) = 0$ for all $U \in \mathcal{U}$, and $V \in \mathcal{V}$.