Triangulated Categories and Kac-Moody Algebras*

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Dedicated to our teacher Shaoxue Liu for his 70th birthday

Abstract

By using the Ringel-Hall algebra approach, we find a Lie algebra arising in each triangulated category with $T^2 = 1$, where T is the translation functor. In particular, the generic form of the Lie algebras determined by the root categories, the 2-period orbit categories of the derived categories of finite dimensional hereditary associative algebras, gives a realization of all symmetrizable Kac-Moody Lie algebras.

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1 Introduction

1.1 A Cartan datum is a pair $(I, (,))$ consisting of a finite set I and a symmetric bilinear form $(-,-)$ on the free abelian group $\mathbf{Z}[I]$ with values in $\mathbf{Z}$ such that the following conditions are satisfied.

(a) $(i,i) \in \{2,4,6,\cdots\}$ for any $i \in I$.

(b) $2(i,j)/(i,i) \in \{0,-1,-2,\cdots\}$ for any $i \neq j$ in I.

Then $C = (a_{ij})_{i,j\in I}$ with $a_{ij} = \frac{2(i,j)}{(i,i)}$ is just a symmetrizable generalized Cartan matrix. Without loss of generality, we may identify a Cartan datum $(I, (,))$ with the symmetrizable generalized Cartan matrix $C = (a_{ij})_{i,j\in I}$.

There are many well-developed mathematical disciplines corresponding to a given Cartan datum. In this article, we will consider the relations between the Kac-Moody algebras and the derived categories of representations of finite dimensional hereditary associative algebras.

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1.2 Let $C=(a_{ij})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix. The associated Kac-Moody algebra $\mathbf{g}(C)$ is a complex Lie algebra on $3n$ generators e_i, f_i, h_i $(i\in I, n=|I|)$ and the following defining relations:

$$\begin{cases} [h_i,h_j]=0\ ,\quad [e_i,f_i]=h_i\ ,\quad [e_i,f_j]=0 \quad \text{if } i\neq j\ , \\ [h_i,e_j]=a_{ij}e_j\ ,\quad [h_i,f_j]=-a_{ij}f_j\ , \\ (\mathrm{ad}e_i)^{1-a_{ij}}e_j=0\ ,\quad (\mathrm{ad}f_i)^{1-a_{ij}}f_j=0 \quad \text{if } i\neq j\ , \end{cases}$$

where $\mathrm{ad}e_i=[e_i,-]$. Note that the Lie algebra $\mathbf{g}(C)$ defined here is the derived algebra of the Kac-Moody algebra given in the book [K1] and the two Lie algebras share a common structure and representation theory.

1.3 Given a field k, let A be a finite dimensional hereditary associative k-algebra, where "hereditary" means $\mathrm{Ext}^2_A(-,-)=0$ everywhere. Let $\mathrm{mod}A$ be the category of finitely generated A-modules. And let $G(\mathrm{mod}A)$ be the Grothendieck group of $\mathrm{mod}A$, the factor group of the free abelian group generated by isomorphism classes $[X]$ of all A-modules X by the subgroup generated by the elements $[X]-[Y]+[Z]$ provided there exist exact sequences of form $0\to X\to Y\to Z\to 0$ in $\mathrm{mod}A$. It is easy to see $G(\mathrm{mod}A)=\mathbf{Z}[I]$, here I is the set of isomorphism classes of simple A-modules. For $M\in\mathrm{mod}A$, by $\underline{\dim}M$ we denote the canonical image of M in $\mathbf{Z}[I]$, called the dimension vector of M. We have the symmetric Euler form defined on $\mathbf{Z}[I]$: for any $M,N\in\mathrm{mod}A$

$$\begin{aligned}(\underline{\dim}M,\underline{\dim}N) &= \dim_k\mathrm{Hom}_A(M,N)-\dim_k\mathrm{Ext}_A(M,N)\\ &+ \dim_k\mathrm{Hom}_A(N,M)-\dim_k\mathrm{Ext}_A(N,M).\end{aligned}$$

This is well-defined since $\mathrm{Ext}^2_A(-,-)=0$.

We have the following observation (see [R8]).

If $(-,-)$ is the symmetric Euler form defined on the Grothendieck group $\mathbf{Z}[I]$ of $\mathrm{mod}A$, then the pair $(I,(-,-))$ is a Cartan datum. Conversely, any Cartan datum can be realized by a symmetric Euler form defined on the Grothendieck group of $\mathrm{mod}A$ for some finite dimensional hereditary k-algebra A.

1.4 According to the Gabriel theorem [G1] on representations of quivers and an extended version given by Dlab-Ringel[DR1], we know that finite dimensional hereditary k-algebras of finite representation type are given by Dynkin diagrams of type $A_n, D_n, B_n, C_n, E_6, E_7, E_8, F_4$ and G_2, and the map $\underline{\dim}$ induces a bijection between the isomorphism classes of all indecomposable modules over a hereditary algebra A of Dynkin type and the positive roots of the corresponding semisimple complex Lie algebra $\mathbf{g}$.

It is natural to ask whether it is possible to recover the Lie algebra structure of $\mathbf{g}$ in terms of the representations of A. This was succesfully attained by