A THIRD LOOK AT WEIGHT DIAGRAMS

Nikolai Vavilov

Department of Mathematics and Mechanics
University of Sanct-Petersburg
Petrodvorets, 198904, Russia

Abstract. In this paper, which is a sequel of [PSV], we develop a completely elementary approach to calculations in Chevalley groups $G = G(\Phi, R)$ of types $\Phi = \mathrm{E}_6$ and E_7 over a commutative ring R using only the weight diagrams (alias, crystal graphs) of their minimal modules. After an elementary construction of a crystal base we explicitly describe action of root subgroups and of the extended Weyl group, multilinear invariants, equations defining the orbit of the highest weight vector and Freudenthal tranvections. As an illustration of our methods we give the first complete a priori proof of the central step in the method of decomposition of unipotents (see [VPS], [V2], [VP], [SV], [VPe]) for these cases. Namely we prove that any singular column v is stabilised by a non-trivial Freudenthal transvection of a certain type ('fake root unipotent') and that there are in fact enough of those to generate the whole elementary group of type Φ over R as the v ranges over the columns of a matrix $g \in G$. It is known that this result immediately implies the main structure theorems for G (description of normal subgroups, standard commutator formulae and the like). The results of the present paper provide complete proofs for the algebraic part of [V2] in the cases of E_6 and E_7, complete proofs for the geometric part of the above paper are given in [V7].

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Introduction

Let Φ be a reduced irreducible root system, $W = W(\Phi)$ be the corresponding Weyl group. We fix an order on Φ and denote by $\Pi = \{\alpha_1, \dots, \alpha_l\}$, Φ^+ and Φ^- the corresponding sets of fundamental, positive and negative roots respectively. Then the Weyl group is generated by the set S of fundamental reflections $s_1 = w_{\alpha_1}, \dots, s_l = w_{\alpha_l}$. As usual $P(\Phi)$ denotes the lattice of integral weights of Φ and $P(\Phi)_{++}$ is the cone of dominant integral weights. Recall that any weight $\omega \in P(\Phi)_{++}$ is a non-negative integral linear combination of the fundamental weights $\overline{\omega}_1, \dots, \overline{\omega}_l$.

Further, let $G = G(\Phi, R)$ be the *simply connected* Chevalley group of type Φ over a commutative ring R with 1. One can find all the relevant notions in [A], [B], [C], [Hé], [H], [M], [St1], [S] (see [V2], [V4], [VP] for many additional references). Fix a dominant weight $\omega \in P(\Phi)_{++}$ and let $V = V(\omega)$ be the Weyl module of G with the highest weight ω. The corresponding representation $G \longrightarrow \mathrm{GL}(V)$ will be denoted by π. By $\Lambda(\pi)$ we denote the set of weights of the representation π *with multiplicities*. An *admissible base* v^λ, $\lambda \in \Lambda(\pi)$, of V consists of weight vectors and has the property that the action of the root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, is described by matrices whose entries are polynomials in ξ with integral coefficients.

The *weight diagram* of π is a marked graph, whose nodes correspond to the weights of π (usually with multiplicities) and two nodes λ and μ are joined by a bond marked i if their difference $\lambda - \mu = \alpha_i$ is the i-th fundamental root. Eventually, one should indicate the positive direction as well. Sometimes this is done by drawing arrows instead of lines. We draw the diagrams in such a way that a larger weight stands to the left of and/or higher than a smaller one, with the landscape orientation usually being primary. Another convention is that we omit at least one of the two equal labels at the opposite sides of a parallelogramm.

Below we reproduce two typical weight diagrams, the one of the representation of the Chevalley group $G(\mathrm{E}_6, R)$ with the highest weight $\overline{\omega}_1$,

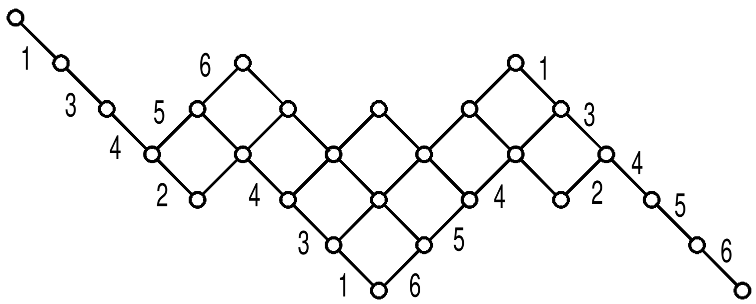

Figure 1. $(\mathrm{E}_6, \overline{\omega}_1)$

and the one of the representation of the Chevalley group $G(\mathrm{E}_7, R)$ with the highest weight $\overline{\omega}_7$.