SUBGROUPS OF SL_n OVER A SEMILOCAL RING

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Let R be a commutative semilocal ring, $G = \mathrm{SL}(n,R)$ be the special linear group of degree $n \geq 3$ over R and $T = \mathrm{SD}(n,R)$ be its subgroup of diagonal matrices. In the present paper we describe subgroups of G containing T under some mild restrictions on the residue fields of R. More precisely we prove the following result (we refer to Sections 2^o and 3^o for an explanation of all definitions and notation used in this statement).

Theorem 1. *Let $n \geq 3$ be a natural number and R be a semilocal ring, all of whose residue fields contain at least $3n+2$ elements. Then the standard description of subgroups of $\Gamma = \mathrm{SL}(n,R)$ containing the diagonal subgroup $T = \mathrm{SD}(n,R)$ holds. In other words, for any such subgroup H there exists a unique D-net of ideals σ such that $\Gamma(\sigma) \leq H \leq N_\Gamma(\sigma)$.*

This result generalises my earlier results, published in [V3], where R was supposed to be a field. However the proof here does not repeat the proofs presented in [V3]. To the contrary, it is based on a reduction to the results of [V3]. From a general viewpoint Theorem 1 is a description of overgroups of a split maximal torus in a Chevalley group of type A_l. It is worth to stress the fact that unlike the case of *extended* Chevalley groups description of such overgroups in ordinary Chevalley groups was known only for fields.

The paper is organised as follows. In 1^o we discuss previously known related results and in 2^o and 3^o we introduce the necessary notation and

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state some lemmas used in the proof. In 4^o we reduce the proof of Theorem 1 to the special case of subradical subgroups, embodied in Theorem 2 and in 5^o and 6^o finish the proof of Theorem 2. Finally, in 7^o we mention some further related problems.

1^o. **Introduction.** A brief historical survey is in order to put Theorem 1 in context. We refer the reader to the surveys [V6], [V7], [V9] and [Z] for a much broader picture of the field and many further related references. First of all, observe that for the general linear group analogous results are substantially easier. Subgroups of the general linear group $\mathrm{GL}(n, R)$ over a semilocal ring R containing the diagonal subgroup $\mathrm{D}(n, R)$ were described some two decades ago in a series of papers by Z.I.Borewicz and the present author [B3], [BV], [V1], [V2]. The ring R was not even assumed to be commutative. The methods of the above papers based on the use of pseudoreflections turned out to be extremely productive and were generalized in several directions. In particular, they carry over also to other extended classical groups over rings, like the general symplectic group $\mathrm{GSp}(2l, R)$ or the special orthogonal group $\mathrm{SO}(n, R)$, see [DV], [V6], [V7], [V9] or [HV] for a detailed survey. At least for the case of skew fields these methods found definitive explanations in [V4] and [V10].

The case of the special linear group SL_n turned out to be much harder even over fields. For finite fields K, $|K| \geq 13$, $\operatorname{char} K \neq 2$, it was solved by G.M.Seitz [S]. For infinite fields one should distinguish between the cases $n \geq 3$ and $n = 2$, the group SL_2 behaving like a symplectic group rather than a linear one. The case $n \geq 3$ was treated by the author, first in [V3, I] for infinite fields and then in [V3, II – IV] for all fields K such that $|K| \geq 7$. While the methods of [V3, I] still relied on the presence of a non-trivial 'homology' (i.e. a scalar multiple of a pseudoreflection $d \neq e$) in $\mathrm{SL}(n, K)$, they introduced calculations with the conjugates of the elements $d_{rs}(\varepsilon) = e + (\varepsilon - 1)e_{rr} + (\varepsilon^{-1} - 1)e_{ss}$, which are, from the point of view of algebraic groups, nothing else than semisimple root elements in SL_n. These methods too were later generalised to arbitrary Chevalley groups, see [V7] – [V9] and references there.

For the group SL_n itself in the matrix language the basic trick consisted in considering a conjugate $y(\varepsilon) = gd_{rs}(\varepsilon)g^{-1}$ of the semisimple root element by an element g from the subgroup $T \leq H \leq G$. Then for any pair $i \neq j$ the condition $y(\varepsilon)_{ij} = 0$ is an equation in ε of degree ≤ 2 one of whose roots equals 1 and we are almost done since one usually can solve *linear* equations over fields (maybe after varying g, since we want our root to be distinct from 0 and 1). When $|K|$ or n (or both) are large enough the proof goes smoothly, but the remaining small cases required substantial extra-work.

As it turned out, the case of SL_2 over an infinite field is incredibly tough and was actually the last one to be treated. Partial results were obtained