STRUCTURE OF HYPERBOLIC UNITARY GROUPS
I. ELEMENTARY SUBGROUPS

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ABSTRACT. This is the first in the series of papers dedicated to the structure of hyperbolic unitary groups over form rings and their subgroups. In this part we recall foundations of the theory and study elementary subgroup of the hyperbolic unitary group over a ring. In particular, using a variant of Suslin's patching method, we prove standard commutator formulae for relative elementary subgroups.

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Introduction

In this series of papers we intend to systematically study subgroups of hyperbolic unitary groups $\mathrm{U}(2n, R, \Lambda)$ over a form ring (R, Λ), corresponding to a symmetry λ (see [B1] – [B3], [Ba3], [HO'M]). Our primary objectives are two-fold. On one hand we plan to update the foundational results from [B1], [Ba3], removing, whenever possible, stability conditions, or replacing them by weaker commutativity or finiteness conditions. The excellent recent exposition [HO'M] concentrates on the Steinberg groups, structure of unitary groups over division rings and isomorphism theory and does not contain proofs of structure results in full generality. We fill this gap. In partivular, we carry over to unitary groups over form rings the main structure theorems over general rings and their generalizations obtained for the usual classical groups during the last 20 years: [A3], [A4], [B4], [B5], [BV3], [CK], [Go1] – [Go5], [GM1], [GM2], [Hb1], [Kh1] – [Kh3], [Ko1], [Ko3], [L1], [L2], [LL], [Md3], [Su2], [SK], [Ta1] – [Ta3], [Tu], [V2] – [V8], [Va1], [Va3] – [Va6], [VPS], [Wi][1]. On the other hand we generalize to unitary groups over form rings some more specialized structure results, related to the description of various classes of subgroups. In particular, we carry over results of [BV1] – [BV3], [Va1], [Va4], [Go4] (see [Va6], [Va7], [Za] for many further references).

In the first half of this first part we fix the general framework for the rest of the series. The second half studies elementary subgroups $\mathrm{EU}(2n, R, \Lambda)$ of hyperbolic unitary groups $\mathrm{U}(2n, R, \Lambda)$. We study also relative elementary groups $\mathrm{EU}(2n, I, \Gamma)$ corresponding to form ideals (I, Γ) in (R, Λ). In particular we prove the following result (recall that a ring R is called *almost commutative* if it is finitely generated as a module over its centre).

Theorem. *Let (R, Λ) be an almost commutative form ring, $n \geq 3$. Then for any form ideal (I, Γ) the corresponding elementary subgroup $\mathrm{EU}(2n, I, \Gamma)$ is normal in the hyperbolic unitary group $\mathrm{U}(2n, R, \Lambda)$. Moreover,*

$$\mathrm{EU}(2n, I, \Gamma) = [\mathrm{EU}(2n, R, \Lambda), \mathrm{CU}(2n, I, \Gamma)],$$

where $\mathrm{CU}(2n, I, \Gamma)$ is the full congruence subgroup of level (I, Γ).

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[1]We refer to [Di], [HO'M], [Md2], [Md3] for a description of the earlier stages of development. For the linear groups the old era ended with [Ba1], [BMS], [Ba2]. The thesis [B1] was not published and was not easily available, especially in Russia and China. This did a lot of harm. In fact, many works were appearing *up to late 80-ies*, proving structure theorems for classical groups over such rings, as, say, zero-dimensional ones. We do not cite these publications in our bibliography, see, for example, references in [HO'M] and also in [CK], [V5], [V6], [Va6].