Series Logo
Volume **00**, Number 00, Xxxx 19xx

# CLASSIFICATION OF 2-REFLECTIVE HYPERBOLIC LATTICES OF RANK 4

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Abstract. A hyperbolic lattice is called 2-reflective if its automorphism group contains a finite index subgroup generated by 2-reflections. We find all 2-reflective hyperbolic lattices of rank 4. (For all other ranks this was done by V. V. Nikulin.)

In this work we classify 2-reflective hyperbolic lattices of rank 4 (see the definitions below). The classification was done by the author in 1981 and mentioned in [N3] but the proof was not published up to the present. A classification of 2-reflective hyperbolic lattices is the key step in the classification of algebraic $K3$-surfaces whose automorphism group is finite [N1], [N2].

A *quadratic lattice* $L$ is a free abelian group equipped with a non-degenerate integral symmetric bilinear form, which is called the *scalar product.* The lattice $L$ is called *Euclidean*, if the scalar product in it is positive definite, and *hyperbolic*, if the scalar product is of signature $(n, 1)$.

One can think of $L$ as of a lattice in the (pseudo)Euclidean vector space $V = L \otimes \mathbb{R}$. The automorphism group $\mathrm{O}(L)$ of $L$ is a lattice in the (pseudo)orthogonal group $\mathrm{O}(V)$.

In the hyperbolic case, one of two connected components of the hyperboloid

$$(x, x) = -1 \tag{1}$$

can be considered as a model of $n$-dimensional Lobachevsky space $L^n$ in such a way that the group of motions of $L^n$ is the subgroup $\mathrm{O}'(V)$ of index 2 in $\mathrm{O}(V)$ consisting of transformations leaving invariant each connected component of the hyperboloid (1). The Lobachevsky planes in this model are non-empty intersections of $L^n$ with subspaces of $V$. The points at infinity of $L^n$ correspond to isotropic one-dimensional subspaces of $V$. The group

$$\mathrm{O}'(L) = \mathrm{O}(L) \cap \mathrm{O}'(V)$$

is a discrete group of motions of $L^n$ with a finite covolume.

A primitive vector $e$ of a quadratic lattice $L$ is called a *root* or, more exactly, a *$k$-root*, if $(e, e) = k > 0$ and

$$2(e, x) \in k\mathbb{Z} \quad \forall x \in L. \tag{2}$$

Any root $e$ gives rise to an orthogonal reflection

$$R_e : x \mapsto x - \frac{2(e,x)}{(e,e)} e,$$

which leaves $L$ invariant, i.e. belongs to $\mathrm{O}(L)$. In the hyperbolic case, $R_e$ defines a reflection in the the hyperplane

$$H_e = \{x \in L^n : (e,x) = 0\}$$

of the space $L^n$. The reflection defined by a $k$-root, is called a *$k$-reflection*. Obviously, a root is defined by the corresponding reflection up to sign.

We are especially interested in 2-reflections. Note that any vector $e \in L$ with $(e,e) = 2$ automatically satisfies (2), i.e. is a root.

Denote by $\mathrm{O}_r(L)$ (resp. by $\mathrm{O}_r^{(2)}(L)$) the (normal) subgroup of $\mathrm{O}(L)$ generated by all reflections (resp. by all 2-reflections) in $\mathrm{O}(L)$. A hyperbolic lattice $L$ is called *reflective* (resp. *2-reflective*) if $\mathrm{O}_r(L)$ (resp. $\mathrm{O}_r^{(2)}(L)$) has a finite index in $\mathrm{O}(L)$.

If $L'$ is a sublattice of $L$, then $\mathrm{O}_r^{(2)}(L') \subset \mathrm{O}_r^{(2)}(L)$, while $\mathrm{O}(L')$ and $\mathrm{O}(L)$ are commensurable. It follows that if $L'$ is 2-reflective, then $L$ is also 2-reflective.

A quadratic lattice $L$ is called *even*, if $(x,x) \in 2\mathbb{Z}$ for any $x \in L$, and *odd* otherwise. Any odd lattice $L$ contains a unique even sublattice $L^{even}$ of index 2, namely,

$$L^{even} = \{x \in L : (x,x) \in 2\mathbb{Z}\}.$$

Since $\mathrm{O}(L) \subset \mathrm{O}(L^{even})$ and any 2-root of $L^{even}$ is a 2-root of $L$, we have

$$\mathrm{O}_r^{(2)}(L) = \mathrm{O}_r^{(2)}(L^{even}).$$

It follows that an odd hyperbolic lattice $L$ is 2-reflective if and only if $L^{even}$ is 2-reflective. This allows us to cosider only even lattices.

Let $L$ be a hyperbolic lattice. Then the groups $\mathrm{O}_r(L)$ and $\mathrm{O}_r^{(2)}(L)$ are discrete reflection groups in $L^n$. Denote by $P$ and $P^{(2)}$ their fundamental polyhedra (in general infinite). The lattice $L$ is reflective (resp. 2-reflective) if and only if $\operatorname{vol} P < \infty$ (resp. $\operatorname{vol} P^{(2)} < \infty$) or, equivalently, if $P$ (resp. $P^{(2)}$) is finite [V2]. There is an algorithm allowing for any concrete $L$ to find recursively all the faces of $P$ or $P^{(2)}$ and to determine if there are only finitely many of them [V2], [V4], [B].

V. V. Nikulin [N1], [N2] classified all 2-reflective hyperbolic lattices of rank $\neq 4$. R. Scharlau [SW] published a tentative list of all reflective isotropic hyperbolic lattices $L$ of rank 4 with a maximal group $\mathrm{O}_r(L)$, but no proof has appeared as yet.

In order to formulate the result, let us introduce some notation.

[C] – the quadratic lattice, the scalar product in which is given by a (symmetric) matrix $C$ in some basis,

$U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ – the standard 2-dimensional hyperbolic lattice,

$A_n$ – the (Euclidean) root lattice of type $A_n$,