MOTIVIC EQUIVALENCE OF QUADRATIC FORMS

OLEG T. IZHBOLDIN*

ABSTRACT. Let X_ϕ and X_ψ be projective quadrics corresponding to quadratic forms ϕ and ψ over a field F. If X_ϕ is isomorphic to X_ψ in the category of Chow motives, we say that ϕ and ψ are motivic isomorphic and write $\phi \overset{m}{\sim} \psi$. We show that in the case of odd-dimensional forms the condition $\phi \overset{m}{\sim} \psi$ is equivalent to the similarity of ϕ and ψ. After this, we discuss the case of even-dimensional forms. In particular, we construct examples of generalized Albert forms q_1 and q_2 such that $q_1 \overset{m}{\sim} q_2$ and $q_1 \not\sim q_2$.

Let F be a field of characteristic $\neq 2$ and ϕ be a quadratic form of dimension ≥ 3 over F. By X_ϕ we denote the projective variety given by the equation $\phi = 0$. It is well known that the variety X_ϕ determines the form ϕ uniquely up to similarity. More precisely, the condition $X_\phi \simeq X_\psi$ holds if and only if $\phi \simeq k\psi$ for a suitable element $k \in F^*$. Now, let $\mathcal{M} : \mathcal{V}_F \to \mathcal{C}$ be an arbitrary functor from the category $\mathcal{V}_F$ of smooth projective F-varieties to a category $\mathcal{C}$. Is it possible to say anything specific about ϕ and ψ if we know that $\mathcal{M}(X_\phi) \simeq \mathcal{M}(X_\psi)$? Clearly, the answer depends on the category $\mathcal{C}$ and the functor $\mathcal{M}$. In the present paper, we mainly consider the example of the category $\mathcal{C} = \mathcal{M}\mathcal{V}_F$ of Chow motives. In this particular case, we set $\mathcal{M}(X) = M(X)$, where $M(X)$ denotes the motive of X in the category of Chow motives. If $M(X_\phi) \simeq M(X_\psi)$, we say that ϕ is motivic equivalent to ψ (and we write $\phi \overset{m}{\sim} \psi$).

Recently, Alexander Vishik has proved that $\phi \overset{m}{\sim} \psi$ iff $\dim\phi = \dim\psi$ and $i_W(\phi_L) = i_W(\psi_L)$ for all extensions L/F (see [23]). His proof uses deep results concerning the Voevodsky motivic category. In [8], Nikita Karpenko found a new, more elementary, proof that, in contrast to Vishik's proof, deals only with Chow motives. In §2, we give an elementary proof of Vishik's theorem in the case of odd-dimensional forms. In fact, we prove a more precise result. Namely, we show that, in the case of odd-dimensional forms, the condition $\phi \overset{m}{\sim} \psi$ is equivalent to the similarity of the forms ϕ and ψ. In other words, the condition $M(X_\phi) \simeq M(X_\psi)$ is equivalent to the conditions $X_\phi \simeq X_\psi$ for the odd-dimensional quadrics X_ϕ and X_ψ. In the proof we use some results of §1 concerning low dimensional forms belonging to $W(F(\phi)/F)$.

In §3, we show that the condition $\phi \overset{m}{\sim} \psi$ is equivalent to the condition $\phi \sim \psi$ for all forms of dimension ≤ 7. Besides, we discuss the case of even-dimensional forms of dimension ≥ 8. This case is much more complicated. For

1991 *Mathematics Subject Classification.* Primary 11E81; Secondary 19E15.
Key words and phrases. Quadratic form, quadric, Pfister form, Chow motives.
*Supported by TMR-Network Project ERB FMRX CT-97-0107.

instance, for all $n \geq 3$, there exist an example of anisotropic 2^n-dimensional forms ϕ and ψ such that $\phi \overset{m}{\sim} \psi$ but $\phi \not\sim \phi$. In §4, for any n and m such that $0 \leq m \leq n-3$, we construct generalized Albert forms q_1 and q_2 such that $\dim(q_1)_{an} = \dim(q_2)_{an} = 2(2^n - 2^m)$, $q_1 \overset{m}{\sim} q_2$ but $q_1 \not\sim q_2$. This example gives a negative answer to a question stated by T. Y. Lam [16].

Some words about terminology and notation. Mainly we use the same terminology and notation as in the book of T. Y. Lam [15], W. Scharlau [20], and the fundamental papers of M. Knebusch [9, 10]. However, there exist several differences. We use the notation $\langle\langle a_1, \ldots, a_n \rangle\rangle$ for the Pfister form $\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$ (in [15] and [20], $\langle\langle a_1, \ldots, a_n \rangle\rangle = \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$). We write $\phi \sim \psi$ if there exists an element $k \in F$ such that $k\phi \simeq \psi$ (i.e., if ϕ is similar to ψ). We say that ϕ and ψ are half-neighbors if $\dim \phi = \dim \psi$ and there exist $s, r \in F$ such that $\pi = s\phi \perp r\psi$ is a Pfister form (see, e.g., [4]). In this case, we will write $\phi \overset{hn}{\sim} \psi$ and we say that ϕ and ψ are half-neighbors of π. Our definition differs from the original definition of Knebusch [10]. However, we prefer to use the new definition since we want to regard any pair ϕ, ψ of 2^n-dimensional similar forms as half-neighbors. We denote by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to n-fold Pfister forms is denoted by $GP_n(F)$. We also use the notation $P_*(F) = \cup_n P_n(F)$ and $GP_*(F) = \cup_n GP_n(F)$.

Acknowledgments. This work was supported by TMR-Network Project ERB FMRX CT-97-0107. Also, the author would like to thank the *Universität Bielefeld*, and the *Université de Franche-Comté* for their hospitality and support. The author wishes to thank Nikita Karpenko for useful discussions.

1. Low dimensional forms in $W(F(\phi)/F)$

In this section, we give slight generalizations of some results of M. Knebusch. In fact, we modify some proofs of [10] by using Hoffmann's theorem [3]. We recall that Hoffmann's theorem asserts that for a pair of anisotropic quadratic forms ϕ and ψ satisfying the condition $\dim \phi \leq 2^n < \dim \psi$, the form ϕ remains anisotropic over $F(\psi)$.

Proposition 1.1. *Let ϕ and ψ be anisotropic quadratic forms over F such that $\dim \phi \geq \dim \psi$. Suppose that the form $\pi \overset{\text{Def}}{=} \phi \perp \psi$ belongs to the group $W(F(\phi)/F)$. Then*

(1) *if π is isotropic, then π is hyperbolic,*
(2) *if π is anisotropic, then π is similar to a Pfister form.*

Proof. (1) Assume that π is isotropic but not hyperbolic. This means that $0 < \dim \pi_{an} < \dim \pi$. In the Witt ring $W(F)$, we have $\pi - \phi = \psi$. Therefore,

$$\dim(\pi_{an} \perp -\phi)_{an} = \dim \psi \leq \dim \phi < \dim \pi_{an} + \dim \phi = \dim(\pi_{an} \perp -\phi).$$

Consequently, the form $\pi_{an} \perp -\phi$ is isotropic. Hence the set $D_F(\pi_{an}) \cap D_F(\phi)$ is nonempty.