# On the convergence in the variation norm for the images of measures under differentiable mappings

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**Abstract**

Let $F_j$, $F\colon \mathbf{R}^n \to \mathbf{R}^n$ be measurable mappings such that $F_j \to F$ and $\partial_{x_i} F_j \to \partial_{x_i} F$ in measure on a measurable set $E$. We give conditions ensuring that the images of Lebesgue measure $\lambda|_E$ on $E$ under the mappings $F_j$ converge in the variation norm to the image of $\lambda|_E$ under $F$. For example, a sufficient condition is the convergence of $F_j$ to $F$ in the Sobolev space $W^{p,1}(\mathbf{R}^n, \mathbf{R}^n)$ with $p > n$ and the inclusion $E \subset \{\det DF \neq 0\}$. Analogous results are obtained for mappings between Riemannian manifolds and mappings from infinite dimensional spaces.

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In this work we investigate the following problem. Let $\xi_j$ be a sequence of random vectors in $\mathbf{R}^n$ convergent in probability to a random vector $\xi$. When do the distributions of the vectors $\xi_j$ converge to the distribution of $\xi$ in the variation norm? This problem arises in the study of the limit theorems in probability theory (see, e.g., [7]); it is of independent interest as well. A simple and convenient sufficient condition for the convergence in variation has been obtained by Yu. A. Davydov in [8], where

$\{\xi_j\}$ are absolutely continuous functions on the interval $[0, 1]$, convergent uniformly to an absolutely continuous function $\xi$ such that $\|\xi_j' - \xi'\|_{L^1[0,1]} \to 0$. Under these assumptions, the restriction $\lambda|_E$ of Lebesgue measure to the set $E = \{\xi' \neq 0\}$ is transformed by the functions $\xi_j$ to measures that converge in the variation norm to the image of the measure $\lambda|_E$ under the mapping $\xi$. Below we give a simple proof of a considerably more general result. In particular, the above assertion is true if $\xi_j$ and $\xi$ are measurable mappings in $\mathbf{R}^n$ with Lusin's property (N) such that they have derivatives (or at least regular approximate derivatives) $D\xi_j$ and $D\xi$ almost everywhere, these deriavtives are locally uniformly integrable, the mappings $\xi_j$ converge uniformly on compact sets to $\xi$, and the mappings $D\xi_j$ converge in measure on a bounded measurable set $E$ to the mapping $D\xi$, which is nondegenerate on $E$. Thus, also in the one dimensional case the result of the work [8] is reinforced. Analogous results are obtained for mappings between finite dimensional Riemannian manifolds, where the nondegeneracy of the derivative $D\xi$ of the limiting mapping is replaced by the surjectivity of $D\xi$. In addition, we obtain corollaries of the main result for mappings from infinite dimensional spaces, applicable to wide classes of measures, in particular, to Gaussian measures and to Gibbs distributions.

# 1 Notation and terminology

Let us recall some definitions. For any open set $\Omega \subset \mathbf{R}^d$ and any number $p \geq 1$, the Sobolev space $W^{p,1}(\Omega)$ consists of the functions $f \in L^p(\Omega)$, whose generalized partial derivatives are also elements of $L^p(\Omega)$. The Sobolev space $W^{p,1}(\Omega, \mathbf{R}^n)$ consists of the mappings $f = (f_1, \dots, f_n)\colon \Omega \to \mathbf{R}^n$ such that $f_i \in W^{p,1}(\Omega)$. This space is equipped with the norm

$$\|f\|_{p,1} = \Big\| \Big( \sum_{i=1}^n f_i^2 \Big)^{1/2} \Big\|_{L^p(\Omega)} + \Big\| \Big( \sum_{i \le d,\, j \le n} (\partial_{x_i} f_j)^2 \Big)^{1/2} \Big\|_{L^p(\Omega)}.$$

The symbol $W^{p,1}_{loc}(\Omega, \mathbf{R}^n)$ stands for the class of all mappings $f\colon \Omega \to \mathbf{R}^n$ such that $\zeta f \in W^{p,1}(\Omega, \mathbf{R}^n)$ for all $\zeta \in C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ is the class of all smooth functions with compact supports in $\Omega$.

Let $A \subset \mathbf{R}^d$ and let $x \in A$. If

$$\lim_{r \to 0} \frac{\lambda\big(K(x, r) \cap A\big)}{\lambda\big(K(x, r)\big)} = 0,$$

where $K(x, r)$ is the closed ball of radius $r > 0$ centered at $x$, then the point $x$ is called a point of density 0 for the set $A$. A point $y \in Y$ is called the approximate limit of the mapping $f\colon A \to \mathbf{R}^n$ at the point $x$ if the density of the set $\mathbf{R}^d \setminus f^{-1}(U)$ at $x$ equals 0 for every neighborhood $U$ of the point $y \in Y$. Clearly, there might be