FIBRE SUM FUNCTORS AND THE BIMODULE EXT

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1. Introduction

It is a classical technique in the representation theory of finite-dimensional algebras A to consider the A-modules as extensions of modules over smaller subcategories $\mathcal{F}$ and $\mathcal{T}$ thus identifying the module category with the representation category of the bimodule $\mathrm{Ext}^1_A(-,-)$ acting on $\mathcal{F} \times \mathcal{T}$. If $\mathcal{T} = \mathrm{add}\, S$ for a simple module S the category of representations of this bimodule can be identified with the subspace category of a vector space category (see [Ri1]).

In [Dr1] the fibre sum functor with respect to a module P is introduced. It relates the module with the representations of a bimodule as well. In case the endomorphism algebra of P is a field or more specially if P is a simple module, this leads to a vector space category as well.

The aim of this note is to analyse the relation between these two reduction processes to vector space categories. After recalling the first reduction concept in the next section, in the final section we will clarify the relation completely.

For simplicity we consider only finite-dimensional algebras over an algebraically closed field k which we assume to be basic. We use the term algebra for this concept. For notation and background we refer to [GR] and [Ri2].

2. Reduction to the bimodule $\mathrm{Ext}_A(-,-)$

2.1. Let us start out by recalling the concept of a bimodule. Following [GR] an aggregate is a k-additive category with finite-dimensional morphism spaces such that each object is the direct sum of subobjects with local endomorphism algebras. A typical example for an aggregate is the category $A-\mathrm{mod}$ of finite-dimensional (left) modules over an algebra A.

A bimodule over two aggregates $\mathcal{F}$ and $\mathcal{T}$ is a k-linear bifunctor $H\colon \mathcal{F} \times \mathcal{T} \to k-\mathrm{mod}$ which is covariant in the second and contravariant in the first argument. The category $\mathrm{rep}(H)$ of representations of H has as objects the triples (X, h, Y) where $X \in \mathcal{F}$, $Y \in \mathcal{T}$ and $h \in H(X, Y)$. A morphism from (X, h, Y) to (X', h', Y') is a pair (s, t) of morphisms $s\colon X \to X'$ in $\mathcal{F}$ and $t\colon Y \to Y'$ in $\mathcal{T}$ such that $H(X, t)(h) = H(s, Y')(h')$. The category $\mathrm{rep}(H)$ is again an aggregate.

2.2. The classical example to study the category $A-\mathrm{mod}$ of an algebra A by representations of a bimodule is the following: Let $\mathcal{F}$ and $\mathcal{T}$ be two full subaggregates of $A-\mathrm{mod}$ and $G : A-\mathrm{mod} \to A-\mathrm{mod}$ a subfunctor of the identity functor such that $G(X) \in \mathcal{T}$ and $X/G(X) \in \mathcal{F}$ for all X in $A-\mathrm{mod}$. We consider the bimodule $H = \mathrm{Ext}^1_A(-,-)$ acting on $\mathcal{F} \times \mathcal{T}$ and obtain a full functor $R : A-\mathrm{mod} \to \mathrm{rep}(H)$ by mapping the module X to its canonical exact sequence:

$$0 \to G(X) \xrightarrow{\iota_X} X \xrightarrow{\pi_X} X/G(X) \to 0$$

In general the functor R is neither dense nor faithful. Its kernel consists of the morphisms $f : X \to Y$ which factorise as $f = \iota_Y g \pi_X$. Since this kernel is contained in the radical of the aggregate $A-$mod we see that $A-$mod is representation equivalent to its image category inside rep(H). Thus we have 'reduced' the study of $A-$mod to the study of its image. It turns out that in many cases representations of a bimodule are easier to handle than the module category itself.

Let us provide some examples for choices of $\mathcal{F}$, $\mathcal{T}$ and G. We first start with a full subaggregate $\mathcal{T}$ of $A-$mod which is closed under factor modules. For an A-module X we define $G = G_{\mathcal{T}}(X)$ as the sum over all images $f(Z)$ for all Z in $\mathcal{T}$ and $f \in \mathrm{Hom}_A(Z, X)$. For $\mathcal{F}$ we choose a subaggregate of $A-$mod containing all modules $X/G(X)$. Another class of examples arises by considering an ideal I of A. We take $\mathcal{T}$ as the subcategory of A/J-modules and $\mathcal{F}$ as the subcategory of A/I-modules of $A-$mod where J is the left annihilator of I in A. The functor $G = G_I$ is defined as $G_I(X) = IX$. Dually, we can take $\mathcal{T} = A/I-$mod, $\mathcal{F} = A/J-$mod and $G = G^I$ as the annihilator of I in X where J is the right annihilator of I in A (this means $J = G^I(I)$).

Note, that the lack of density of R is repaired if one assumes that the pair $(\mathcal{F}, \mathcal{T})$ is a torsion theory or in other words G is a radical. Moreover, under this assumption R will also be faithful. Being a torsion class forces $\mathcal{T}$ to be closed under extensions and factor modules. Therefore, if $\mathcal{T} = \mathrm{add}\, S$, then $\mathrm{Ext}^1_A(S, S) = 0$.

2.3. Let us continue to consider the situation $\mathcal{T} = \mathrm{add}\, S$ where additionally S satisfies $\mathcal{T}(S, S) \cong k$. In this case rep H can be rewritten as a subspace category. Let us recall the relevant notation.

A vector space category is a k-additive functor $M\colon \mathcal{F} \to k-$mod where $\mathcal{F}$ is an aggregate. Its subspace category $\check{\mathcal{U}}(M)$ is an aggregate which has as objects the triples $U = (U_\omega, \gamma_U, X_U)$ where $U_\omega \in k-$mod, $U_0 \in \mathcal{F}$ and $\gamma_U \in \mathrm{Hom}_k(U_\omega, M(U_0))$. Morphisms $U \to U'$ in this category are pairs $f = (f_\omega, f_0)$ such that $f_\omega\colon U_\omega \to U'_\omega$ is k-linear, $f_0\colon U_0 \to U'_0$ is a morphism in $\mathcal{F}$ and $M(f_0)\gamma_X = \gamma_{X'} f_\omega$.

If now H is a bimodule such that $\mathcal{T} = \mathrm{add}\, S$, then H is completely determined by the contravariant functor $M := H(-, S)\colon \mathcal{F} \to k-$mod which we consider as covariant functor $\mathcal{F}^{op} \to k-$mod. Moreover, any object of rep(H) lying in $H(X, S^n)$ may by the Yoneda lemma be identified with a morphism in $\mathrm{Hom}_{\mathcal{T}(S,S)}(\mathcal{T}(S^n, S), H(X, S))$. This identification yields an antiequivalence rep$(H) \to \check{\mathcal{U}}(M)$.

Dually, one can transform rep(H) into a subspace category if $\mathcal{F} = \mathrm{add}\, S$. This happens e.g. for G^I introduced above for an ideal I satisfying $A/J \cong k$. This is used in [GNRSV] where the considered functor R usually will not be dense, but nevertheless the image can be calculated precisely.

3. Fibre sum functors

3.1. Let $(\mathcal{F}, \mathcal{T})$ be a torsion theory in $A-$mod. We define $\mathcal{K}_{\mathcal{T}}$ to be the full subaggregate of $A-$mod whose objects are the modules V satisfying $\mathrm{Ext}^1_A(V, \mathcal{T}) = 0$. The bimodule L acts on $\mathcal{T} \times \mathcal{K}_{\mathcal{T}}$ as $\mathrm{Hom}_A(-,-)$. We put $\mathrm{rep}_{mon}(L)$ to be the full subcategory of rep(L) given by all monomorphisms $h : X \to V$. Then the definition of $\mathcal{K}_{\mathcal{T}}$ implies that the functor $\widetilde{F} : \mathrm{rep}_{mon}(L) \to A-$mod which sends h to its cokernel is full.

As a torsion class $\mathcal{T}$ is closed under factor modules. Let us assume that $\mathcal{T}$ has a cover P i.e. $\mathcal{T} = \mathrm{fac}\, P$ for some module P in $A-$mod. Then we can even assume that P is a minimal cover and therefore $\mathrm{Ext}_(P, \mathrm{fac}\, P) = 0$. It follows from [AR] that $\widetilde{F}$ is dense.