GRASSMANNIAN FIXED POINT RATIOS

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Abstract. We provide estimates for the fixed point ratios in the permutation representations of a finite classical group over a field of order q on k-subspaces of its natural n-dimensional module. For sufficiently large n each element must either have a negligible ratio or act linearly with a large eigenspace. We obtain an asymptotic estimate in the latter case, which in most cases is q^{-dk} where d is the codimension of the large eigenspace. We also obtain more specific results that are applicable to the monodromy conjecture of Guralnick and Thompson.

Introduction

If G is a finite group, $g \in G$, and M is a maximal subgroup of G, then the fixed point ratio of g in the associated primitive permutation action is denoted by $f(g, M)_G$, or simply $f(g, M)$. Our concern here is with the fixed point ratios $f(g, M)_G$ when G is a nearly simple finite classical group and $M \cap F^*(G)$ stabilizes a nontrivial proper subspace of the natural module for $F^*(G)$, the generalized Fitting subgroup of G. [Following, [11], we say that a group G is *nearly simple* if there is a nonabelian simple group S such that $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(S)$ containing $\mathrm{Inn}(S)$.] Apart from a few small exceptions, these permutation actions include for each classical group the permutation actions of lowest degree in which $F^*(G)$ acts non-trivially.

In [8], Guralnick and Thompson conjectured that for every non-negative integer g there is a finite set $\mathcal{E}(g)$ of finite simple groups such that if S is a nonabelian composition factor of $\mathrm{Gal}(N/\mathbb{C}(t))$ where N is the normal closure of a finite, genus g extension L of $\mathbb{C}(t)$ and t is transcendental over $\mathbb{C}$ then either S is an alternating group or $S \in \mathcal{E}(g)$. They also suggested that it would be desirable to know a minimal set $\mathcal{E}^*(0)$ explicitly. The Riemann-Hurwitz formula connects the genus of L to the permutation action of $\mathrm{Gal}(N/\mathbb{C}(t))$, and the work of several authors shows that to establish the conjecture it suffices to consider primitive permutation actions of nearly simple groups. (See [8] for a more detailed discussion.)

Several recent papers (see [3], [4], [5], [6], [10], [13], [14], [15], and [16]) have investigated fixed point ratios in nearly simple finite groups. The collective effect of such analyses (and the Classification of Finite Simple Groups) is that the Guralnick-Thompson Conjecture has been verified for all primitive permutation actions of all nearly simple finite groups apart from the ones considered here, and the non-classical members of $\mathcal{E}^*(0)$ have been determined. In a forthcoming paper [7], we use the results here to complete the proof of the Guralnick-Thompson Conjecture.

Permutation actions of sufficiently large degree in which all fixed point ratios are sufficiently small must have large genus. Liebeck and Saxl showed in [13] that,

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apart from a few exceptions, if $F^*(G)$ is a group of Lie type written over $\mathbf{F}_q$, $g \in G^\sharp$, and M is a maximal subgroup of G not containing $F^*(G)$ then $f(g, M) < 4/3q$.

We refine that here by producing element-wise bounds. We show that for every $\varepsilon > 0$ there is an integer n such that if G has Lie rank at least n, then $f(g, M) < 1/q + \varepsilon$ for every $g \in G \setminus Z(G)$ and every maximal subgroup M under consideration here. More generally, we show that if G is a classical group of large Lie rank defined over $\mathbf{F}_q$ then the only elements with non-negligible fixed point ratios are elements that act linearly on the natural module with a large eigenspace. We further show that the fixed point ratios for such elements are essentially q^{-dk} where d is the codimension of the eigenspace and k is the smaller of the dimension or codimension of the subspace stabilized by the maximal subgroup.

We show specifically that if G is a classical group with natural module V of dimension n over $\mathbf{F}_q$, M is a maximal subgroup of G such that $M \cap F^*(G)$ stabilizes a subspace of V of dimension or codimension k, where $k \leq n/2$, then either $f(g, M) \leq (4000 + 150n)q^{-(n-6)/2}$ or g acts linearly on V with an eigenspace of codimension d, with $dk < n/2$, and $f(g, M) \leq q^{-dk} + 2000q^{-(n-6)/2}$.

Contents

1. Preliminaries

1.1. **Nearly simple groups of classical type.** For the most part, we follow standard conventions, but the reader should note that the parameter q in the definition of unitary groups is the size of the field over which the associated vector space is