ON MAXIMAL SUBGROUPS OF FINITE CLASSICAL GROUPS

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Abstract. In [1] M. Aschbacher described the structure of the possible maximal subgroups of a classical group G with natural module M. Each such subgroup is either a member of a canonical class of subgroups, or it is the normalizer of a quasi-simple, absolutely irreducible subgroup H of G. In the latter case suppose that H is a classical group whose defining characteristic is coprime to that of G. The aim of this article is to help classify those intermediate groups $H \leq K \leq G$, where K is again a classical group whose defining characteristic is equal to the one of G. If H is linear, then under mild hypotheses on the rank of H and the size of the field $\mathbb{F}_q$ of definition of H we show that there are no such proper K.

1. Introduction

In [2] M. Aschbacher and L. Scott showed that the maximal subgroup problem for finite groups can be reduced to the following two problems:

1. Determine the maximal subgroups of the almost simple groups.
2. Determine $H^1(G, V)$, where G is almost simple and V is an irreducible G-module.

This paper is a contribution to part (1) of this project. For most non-classical almost simple groups reasonably complete information on maximal subgroups exists. For classical groups M. Aschbacher proved in [1]:

Theorem 1.1. *If G is a classical group defined over a finite field F of characteristic l acting on its minimal dimensional FG-module M and H is a subgroup of G, then one of the following holds:*

1. *$F^*(H)$ stabilizes one of the "natural" structures of M.*
2. *$F^*(H)$ is quasi-simple acting absolutely irreducibly and tensor indecomposably on M.*

This structure theorem reduces the study of the maximal subgroups of the classical groups to the following two problems:

1. Determine the stabilizers of the *natural* structures and the maximality among such stabilizers.
2. Determine maximality among subgroups H appearing in part (2) of Theorem 1.1 above.

Under the assumption that $\dim M \geq 13$ the first problem was solved by P. Kleidman and M. Liebeck in [9]. In connection with the second, one would like to prove the following:

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Conjecture 1.2. *Let G be a finite classical group with natural module M. Let H be a subgroup of G as in part (2) of Theorem 1.1 above. Then generically H is maximal, i.e., there is a small list of specified cases when H fails to be maximal.*

Using Theorem 1.1 one can show that if a subgroup H as in (2) of Theorem 1.1 is not maximal, then either K is a member of $\mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{C}_6 \cup \mathcal{C}_7$ (Aschbacher classes, cf. [1]) or there exists a quasi-simple subgroup K of G with H in K and both H and K act absolutely irreducibly and tensor indecomposably on M. Thus, to establish Conjecture 1.2, one has to classify the list of such triples (H, K, G). We shall refer to this as the *general problem*.

From the classification of finite simple groups we know that H and K are either alternating, sporadic, classical or of exceptional Lie type. In case K is alternating there are partial results by J. Saxl [16] in characteristic zero, P. Kleidman and D. Wales [10], J.C. Jantzen and G. Seitz [8], and B. Ford [3]. If K is sporadic or of exceptional Lie type, enough is known about the maximal subgroups of K, i.e., the possible choices of H, to put a solution to the general problem within reach. For example, when K is of exceptional Lie type it is easy to show that there are very few possible configurations; in each case $F^*(H)$ is the centralizer in K of an outer automorphism of order at most 3. The remaining and most difficult cases of the general problem are those where H is alternating or of Lie type and K is a classical group.

W. Husen studied the case where H is of alternating type [7]. For example, he has shown that there are at most twenty possible triples (H, K, G) provided K is not alternating or of Lie type and the characteristic of the field of definition of K is equal to l. Moreover, he obtained some partial results in case these two characteristics coincide.

In our work, we will consider the case where both H and K are of Lie type. For a classical group X we denote its natural module by N_X. Suppose that H and K are defined over fields of order $q = p^a$ and $s = r^b$, respectively.

If the defining characteristic l of K is distinct from those of G and H, then using [12] twice, we get that roughly $s^{(q^{\dim N_H/2})/2} \leq \dim N_G$. On the other hand $\dim N_G \leq r^{\frac{1}{2}(\dim N_H)^2} \leq |H|^{\frac{1}{2}}$, where H is defined over $\mathbb{F}_q$, i.e.,

$$\frac{1}{2} q^{\dim N_H/2} \leq \log_s \dim N_G \leq \frac{(\dim N_H)^2}{4} \log_s q.$$

Both inequalities can certainly not be satisfied for infinitely many values of q and r. Thus this case seems easy to handle. Work of M. Liebeck, G. Seitz, and D. Testerman ([18], [12], [22]) establishes the conjecture when H, K, and G are all defined over fields of the same characteristic subject to mild restrictions on the characteristic. Work in progress by Liebeck and Seitz aims at removing these restrictions. In [19] G. Seitz verified Conjecture 1.2 for the case where H and K are defined over fields of characteristic l, K is classical, and G is defined over a field of characteristic $p \neq l$. The situation when all three groups are classical, H is defined over a field of characteristic p, while K and G are defined over fields of characteristic $l \neq p$, is a last major open case. This is the one under consideration here.

Our basic strategy is to obtain upper and lower bounds for $\dim N_G$ such that generically, there are no such proper triples of groups, as these bounds contradict each other; thus leading to a small list of possible configurations. When H is linear we actually show that there are no such triples provided $q \geq 17$ and $n \geq 16$, see Corollary 1.6. To obtain a lower bound we use the Landazuri-Seitz-Zalesskii bound