STABLE EQUIVALENCE AND GENERIC MODULES

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Abstract. Let Λ and Γ be finite dimensional algebras. It is shown that any stable equivalence $f\colon \overline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{mod}}\,\Gamma$ between the categories of finitely generated modules induces a bijection $M \mapsto M_f$ between the sets of isomorphism classes of generic modules over Λ and Γ such that the endolength of M_f is bounded by the endolength of M up to a scalar which only depends on f. Using Crawley-Boevey's characterization of tame representation type in terms of generic modules, one obtains as a consequence a new proof for the fact that a stable equivalence preserves tameness. This proof also shows that polynomial growth is preserved.

Generic modules were introduced by Crawley-Boevey in order to give a conceptual description of the representation type of a finite dimensional algebra [1]. In this note we study the relation between the generic modules over two finite dimensional algebras Λ and Γ under the assumption that there exists an equivalence $\overline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{mod}}\,\Gamma$ between the stable categories of finitely generated modules.

Recall that a Λ-module is *generic* if it is indecomposable and endofinite but not of finite length over Λ. A Λ-module M is *endofinite* if the *endolength* $\operatorname{endol}(M)$ which is the length of M over its endomorphism ring $\operatorname{End}_\Lambda(M)$ is finite.

The endolength of a module is *not* a categorical invariant. In other words, a Morita equivalence $F\colon \operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma$ between two module categories does not necessarily preserve the endolength. However, it is not hard to see that there exists a constant c_F such that $\operatorname{endol}(FM) \leq c_F.\operatorname{endol}(M)$ for every Λ-module M. The purpose of this paper is to show that this fact remains true for a stable equivalence.

Using Crawley-Boevey's characterization of tame representation type in terms of generic modules, we obtain in this way a simple proof for the fact that a stable equivalence preserves tame representation type [2]. In fact, we improve the results obtained in [2] and show that a stable equivalence preserves polynomial growth.

In [2], it was shown that an equivalence $f\colon \overline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{mod}}\,\Gamma$ extends to an equivalence $F\colon \overline{\operatorname{Pinj}}\,\Lambda \to \overline{\operatorname{Pinj}}\,\Gamma$ between the full subcategories of (not necessarily finitely generated) pure-injective modules in $\overline{\operatorname{Mod}}\,\Lambda$ and $\overline{\operatorname{Mod}}\,\Gamma$. Any endofinite module is pure-injective, and our main result states that the functor F controls the endolength as follows.

Theorem. *Let Λ and Γ be artin algebras and suppose that $f\colon \overline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{mod}}\,\Gamma$ is an equivalence. Denote by $F\colon \overline{\operatorname{Pinj}}\,\Lambda \to \overline{\operatorname{Pinj}}\,\Gamma$ the equivalence between the full subcategories of pure-injective modules in $\overline{\operatorname{Mod}}\,\Lambda$ and $\overline{\operatorname{Mod}}\,\Gamma$ which extends f (see [2]). Then there exists a constant $c_f \in \mathbb{N}$ such that*

$$\operatorname{endol}(FM) \leq c_f.\operatorname{endol}(M)$$

for every endofinite non-zero Λ-module M.

Corollary 1. *An equivalence $f\colon \overline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{mod}}\,\Gamma$ induces a bijection $M \mapsto M_f$ between the sets of isomorphism classes of indecomposable endofinite and non-injective modules over Λ and Γ such that $\operatorname{endol}(M_f) \leq c_f.\operatorname{endol}(M)$ for all M.*

Before we can give the proof of the main theorem we need some preparations. Let us start with some notation. We fix an artin algebra Λ and denote by $\operatorname{Mod}\Lambda$ the category of all Λ-modules. The full subcategory of finitely generated modules is denoted by $\operatorname{mod}\Lambda$. The *stable category* $\overline{\operatorname{Mod}}\,\Lambda$ has the same objects as $\operatorname{Mod}\Lambda$ but $\overline{\operatorname{Hom}}_\Lambda(M,N)$ is the group $\operatorname{Hom}_\Lambda(M,N)$ of Λ-maps modulo the subgroup of maps which factor through some injective module. Analogously, the stable category $\overline{\operatorname{mod}}\,\Lambda$ of finitely generated modules, and the stable category $\overline{\operatorname{Pinj}}\,\Lambda$ of pure-injective modules are defined.

We denote by $S_1,\dots,S_p$ a representative set of simple Λ-modules, and $T_1,\dots,T_q$ denotes a representative set of simple Γ-modules. Keeping the equivalence $f\colon \overline{\operatorname{mod}}\,\Lambda \to \overline{\operatorname{mod}}\,\Gamma$ fixed, we choose Λ-modules $U_1,\dots,U_q$ sucht that $f(U_j) \cong T_j$ in $\overline{\operatorname{mod}}\,\Gamma$ for all j. Now let M and X be Λ-modules and suppose that X has finite length. We denote by $[X,M]$ the length of $\operatorname{Hom}_\Lambda(X,M)$ as $\operatorname{End}_\Lambda(M)$-module, and $[\![X,M]\!]$ denotes the length of $\overline{\operatorname{Hom}}_\Lambda(X,M)$ as $\overline{\operatorname{End}}_\Lambda(M)$-module. The multiplicity of S_i in a composition series of X is denoted by $m_i(X)$. We note the following elementary fact.

Lemma 1. *Let M and X be Λ-modules and suppose that X has finite length. Then*

$$[\![X,M]\!] \le [X,M] \le \sum_{i=1}^{p} m_i(X)[S_i,M].$$

Proof. The first inequality follows from the fact that the length of $\overline{\operatorname{Hom}}_\Lambda(X,M)$ as an $\overline{\operatorname{End}}_\Lambda(M)$-module equals the one as an $\operatorname{End}_\Lambda(M)$-module. In order to prove the second inequality, observe that any exact sequence $0 \to X' \to X \to X'' \to 0$ of Λ-modules gives an exact sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(X'',M) \longrightarrow \operatorname{Hom}_\Lambda(X,M) \longrightarrow \operatorname{Hom}_\Lambda(X',M)$$

and so $[X,M] \le [X',M] + [X'',M]$. By induction on the length of X we get the second inequality. □

The next lemma is simple but extremely useful.

Lemma 2. *Let M be a Λ-module without any non-zero injective submodule, and let S be a simple Λ-module. Then $\operatorname{Hom}_\Lambda(S,M) = \overline{\operatorname{Hom}}_\Lambda(S,M)$ and therefore $[S,M] = [\![S,M]\!]$.*

Proof. Any non-zero map $S \to M$ which factors through some injective module, also factors through the injective envelope $E(S)$ of S, and induces therefore a monomorphism $E(S) \to M$. This is impossible by our assumption on M. Thus $\operatorname{Hom}_\Lambda(S,M) = \overline{\operatorname{Hom}}_\Lambda(S,M)$. □

Proof of the Theorem. Let M be an endofinite non-zero Λ-module and let $N = FM$. We denote by I_N a maximal injective submodule of N which exists by Zorn's lemma since direct limits of injective modules are injective. This gives a direct decomposition $N = N_I \coprod I_N$. Using the preceding lemmas we obtain the following bound for the