ON REGULARITY OF TRANSITION PROBABILITIES AND INVARIANT MEASURES OF SINGULAR DIFFUSIONS UNDER MINIMAL CONDITIONS

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Abstract. Let $A = (a^{ij})$ be a matrix-valued Borel mapping on a domain $\Omega \subset \mathbb{R}^d$, let $b = (b^i)$ be a vector field on Ω, and let $L_{A,b}\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi$. We study Borel measures μ on Ω that satisfy the elliptic equation $L^*_{A,b}\mu = 0$ in the weak sense: $\int L_{A,b}\varphi\, d\mu = 0$ for all $\varphi \in C_0^\infty(\Omega)$. We prove that, under mild conditions, μ has a density. If A is locally uniformly nondegenerate, $A \in H^{p,1}_{loc}$ and $b \in L^p_{loc}$ for some $p > d$, then this density belongs to $H^{p,1}_{loc}$. Actually, we prove Sobolev regularity for solutions of certain generalized nonlinear elliptic inequalities. Analogous results are obtained in the parabolic case. These results are applied to transition probabilities and invariant measures of diffusion processes.

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1. Introduction and notation

Though the results and methods of this paper are purely analytic, we first want to describe our original motivation in the direction of probabilistic applications (which also explains the title of this paper). A

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fundamental problem in probability theory is the study of equilibrium distributions (i.e., invariant measures) of diffusion processes governed by the stochastic differential equation on $\mathbb{R}^d$

$$d\xi_t = \sigma(\xi_t)dw_t + b(\xi_t)dt \tag{1.1}$$

under minimal conditions on the coefficients σ and b. The main issues are, of course, existence and uniqueness of such invariant measures, but also their regularity properties. In order to avoid a priori conditions that ensure the existence of a solution to (1.1) or a corresponding semigroup describing its transition probabilities, we shall use the purely analytic reformulation of the notion of an invariant measure in terms of the underlying generator

$$L = \frac{1}{2}(\sigma\sigma^T)^{ij}\partial_i\partial_j + b^i\partial_i,$$

i.e., we consider solutions μ to the equation

$$L^*\mu = 0.$$

In the terminology of [4], we are thus concerned with the adjoint solutions for L (see also [13] where additional references can be found). Existence and uniqueness results for probability measures solving this equation have been proved in [2], [10], again mainly based on analytic tools. In particular, the corresponding uniqueness results depend on the third issue, namely, the regularity of solutions in an essential way. In the present paper, we, therefore, address the latter problem again. Our results widely generalize our previous ones in the elliptic case [8], since the assumptions on the diffusion coefficients are much more general and we also include the parabolic case.

To be more specific, let Ω be an open set in $\mathbb{R}^d$, let $A = (a^{ij})$ be a Borel mapping on Ω with values in the space of the nonnegative matrices on $\mathbb{R}^d$, and let $b = (b^i)\colon\ \Omega \to \mathbb{R}^d$ be a Borel vector field. Let us set

$$L_{A,b}\varphi = a^{ij}\partial_i\partial_j\varphi + b^i\partial_i\varphi, \qquad \varphi \in C_0^\infty(\Omega), \tag{1.2}$$

where we use the standard summation rule for repeated indices. Suppose that μ is a locally finite Borel measure on Ω such that a^{ij}, $|b| \in L^1_{loc}(\mu)$ and

$$L^*_{A,b}\mu = 0 \tag{1.3}$$

in the following sense:

$$\int_\Omega L_{A,b}\varphi\, d\mu = 0, \qquad \forall\, \varphi \in C_0^\infty(\Omega). \tag{1.4}$$