ON SET COVERINGS IN CARTESIAN PRODUCT SPACES

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Preface

This is a literal copy of a paper from the fall of 1971, which remained unpublished, because its main result had been obtained independently, but somewhat earlier in the paper "Hide and seek, data storage and entropy, Annals Math. Statistics 42, 1706–1716, 1971" by E.C. Posner and Mc Eliece.

Further developments have shown that our paper still has its merits, because of its methods. They immediately yield the Covering Theorem for the case of non–identical factors and a general class of infinite hypergraphs without any topological assumptions.

Furthermore there is a thorough discussion of connections to other solved and unsolved combinatorial problems. The paper under [5] appeared in the Journal of Applied Probability 10(4), 824–836, 1973. It deals with packings of controlled overlaps and led us to investigate coverings.

Only familiarity with a few elementary combinatorial and probabilistic concepts is demanded from the reader.

Abstract

Consider $(X, \mathcal{E})$, where X is a finite set and $\mathcal{E}$ is a system of subsets whose union equals X . For every natural number $n \in \mathbb{N}$ define the cartesian products $X_n = \prod_1^n X$ and $\mathcal{E}_n = \prod_1^n \mathcal{E}$. The following problem is investigated: how many sets of $\mathcal{E}_n$ are needed to cover X_n ? Let this number be denoted by $c(n)$. It is proved that for all $n \in \mathbb{N}$

$$\exp\{C \cdot n\} \leq c(n) \leq \exp\{Cn + \log n + \log\log |X|\} + 1.$$

A formula for C is given. The result generalizes to the case where X and $\mathcal{E}$ are not necessarily finite and also to the case of non–identical factors in the product. As applications one obtains estimates on the minimal size of an externally stable set in cartesian product graphs and also estimates on the minimal number of cliques needed to cover such graphs.