Cramér's Theorem in Banach Spaces

Peter Eichelsbacher
Institut für Angewandte Mathematik
Universität Zürich
Winterthurer-Str. 190
CH-8057 Zürich

Abstract

In this note we point out that an application of a special contraction principle combined with a general representation of large deviation rates provides a new proof of the classical result of Cramér for Banach–valued random variables.

AMS Subject Classification (1992): 60F10

Keywords: large deviations $*$ rate function – representation $*$ contraction principle

1 Introduction

The asymptotic computation of small probabilities on an exponential scale is what large deviation theory analyses: consider a normal topological space X endowed with its Borel σ-field $\mathcal{B}(X)$ and let $(P_n)_n$ be a sequence of probability measures on X. A large deviation principle (LDP) characterizes the limiting behavior as $n \to \infty$ of the sequence $(P_n)_n$ in terms of a rate function. $(P_n)_n$ satisfies a LDP on X with rate function $I : X \to [0, \infty)$ if for all $A \in \mathcal{B}(X)$:

$$- \inf_{x \in A^o} I(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P_n(A) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_n(A) \leq - \inf_{x \in \bar{A}} I(x), \tag{1}$$

where A^0 denotes the interior of A and $\bar{A}$ the closure of A. By definition a rate function has compact level sets $N_I(L) := \{x : I(x) \leq L\}$ for all $L \in [0, \infty)$. After establishing the existence of a LDP we try to use one of various approaches for the identification of the resulting rate function. A very general observation is the following: If $(P_n)_n$ satisfies a LDP the rate function I can be represented as

$$I(x) = \sup\{f(x) - \Phi(f), f \in C_b(X)\}, \tag{2}$$

where $C_b(X)$ denotes the Banach space of all bounded continuous functions $f : X \to \mathbb{R}$ with the supremum norm and

$$\Phi(f) := \lim_{n\to\infty} \frac{1}{n} \log \int_X \exp(nf(x))dP_n(x) \tag{3}$$

(see Dinwoodie [9]). We obtain that I takes a form analogous to a convex conjugate. Dinwoodie applied this identification to the empirical law of a Markov chain (see [10]).

The aim of this note is to improve a rate representation given in [12], where the class $C_b(X)$ is replaced by an arbitrary seperating class of the space of all continuous, real-valued functions. This provides a new proof of the classical result of Cramér for Banach–valued random variables.

2 Rate representation

First we describe two main tools in large deviation theory:
The first one is Varadhan's integral lemma which actually is one of the starting points for developing the large deviation theory. Suppose $(P_n)_n$ satisfies a LDP with rate I and let $f : X \to \mathbb{R}$ be continuous. Assume further the tail condition

$$\lim_{L\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log \int_{\{f(x)\geq L\}} \exp(nf(x))dP_n(x) = -\infty,$$

then

$$\Phi(f) = \sup\{f(x) - I(x), x \in X\}.$$

The second is Bryc's inverse of Varadhan's Laplace method (see [5]) which he told the asymptotic value approach: Let $\mathcal{F}$ be an arbitrary separating class of $C(X)$, the space of all continuous, real-valued functions: $\mathcal{F}$ contains the constant functions, is closed under finite pointwise minima and separates points of X, e.g. for all $x, y \in X$ and for all $a, b \in \mathbb{R}$