Questions about tame algebras

Collected by Peter Dräxler

Introduction

This is a collection of questions about tame algebras which emerged from (formal and informal) discussions with colleagues during the last 2 years after the workshop on *The Notion of Tameness* in Bielefeld, May of 1993. The presentation is an extended version of the one used by the author for a problem session at the Oberwolfach conference about *Darstellungstheorie endlich-dimensionaler Algebren* in July of 1995. In particular some sections are added which due to lack of time could not be presented in the session.

Since this paper is only meant as a hopefully helpful list of problems for interested people working close to the area, known results and definitions are only provided if necessary and no completeness is intended. In the references we include a list of survey papers on tameness or various related aspects which a reader interested in more details and literature should use.

Although the author is indebted and very grateful to many colleagues for communicating the listed problems to him, no problem will be related to a specific person. One reason is that the author does not always remember who posed which problem for the first time. The other is that the author may have misunderstood questions and does not want to embarrass others for his errors in the formulations arising in this way.

The questions were collected concentrating on representations of algebras. In particular, we looked for problems related to the definition of tameness and for parallels to the well-understood finite type case. Following this concept, we will only touch or even omit some important topics like selfinjective algebras (see [Sk3]), the infinite radical (see [Sk2]), modules over orders (see [Di]), representations of bocses and subspace categories of vector space categories (see [CB1], [Si]). Moreover, by restricting to an algebraically closed base field all problems related to a definition of tameness for arbitrary fields or even artin algebras are not touched at all.

1 The definition of tameness

Let us fix the algebraically closed field k and a finite-dimensional basic k-algebra A which we may suppose to be given as $A = k[\Delta]/\langle\rho\rangle$ were Δ is a finite quiver and ρ is a (finite) set of admissible relations.

For a given dimension vector $d = (d_x) \in \mathbb{N}_0^{\Delta_0}$ we denote by $(\Delta, R)-\mathrm{rep}_d$ the set $\prod_{\alpha:x\to y\in\Delta_1} R^{d_y\times d_x}$ where $R^{d_y\times d_x}$ is the set of matrices with d_y rows and d_x columns over another (possibly infinite-dimensional) k-algebra R. The subset of $(\Delta, R)-\mathrm{rep}_d$ given by all tuples of matrices M such that $M(r) = 0$ for each $r \in \rho$ will be called $(A, R)-\mathrm{rep}_d$. Obviously any element of $(A, R)-\mathrm{rep}$ can be considered as a A-R-bimodule which is free of rank $\sum d_x$ as right R-module.

Note that $(A, k)-\mathrm{rep}_d$ is an algebraic subset of the linear variety $(\Delta, k)-\mathrm{rep}_d$ on which the algebraic group $Gl(d) = \prod_{x\in\Delta_0} Gl(d_x)$ acts by conjugation. On the other hand A-k-bimodules are just left A-modules and consequently $(A, k)-\mathrm{rep}_d$ can be identified with the set $A-\mathrm{mod}_d$ of left A-modules of dimension vector d. Via this identification the isomorphism classes correspond to the $Gl(d)$-orbits.

The algebra A is said to be tame provided that for any dimension vector $d \in \mathbb{N}_0^{\Delta_0}$ there exists a finite sequence of non-zero polynomials $f_1^{(d)}, \ldots, f_\nu^{(d)}$ together with a finite sequence $B_1^{(d)}, \ldots, B_\nu^{(d)}$ of bimodules $B_i^{(d)} \in (A, k[X, (f_i^{(d)})^{-1}])-\mathrm{rep}_d$ with the following three properties:

(i) $B_i^{(d)} \otimes - \, : \, k[X, (f_i^{(d)})^{-1}]-\mathrm{mod} \to A-\mathrm{mod}$ preserves indecomposability and isoclasses.

(ii) For any dimension vector $d' \in \mathbb{N}_0^{\Delta_0}$ there are only finitely many isoclasses of indecomposable modules in $A-\mathrm{mod}_{d'}$ which have empty intersection with the image of each of the functors $B_i^{(d)} \otimes -$.

(iii) For $(d, i) \neq (d', i')$ the images of the functors $B_i^{(d)} \otimes -$ and $B_{i'}^{(d')} \otimes -$ do not contain non-zero isomorphic modules.

Note that the indecomposable modules in the image of the functor $B_i^{(d)} \otimes -$ are just the modules $B_i^{(d)} \otimes k[X]/(X-\lambda)^n \cong B_i^{(d)}(J(n, \lambda))$ where $\lambda \in k$ is not a zero of $f_i^{(d)}$ and $J(n, \lambda)$ is the Jordan block matrix of size $n \in \mathbb{N}$.

On the other hand A is said to be wild if there exists $B \in (A, k\langle X, Y\rangle)-\mathrm{rep}$ such that the functor $B \otimes - : k\langle X, Y\rangle-\mathrm{mod} \to A-\mathrm{mod}$ preserves indecomposability and isoclasses. A is called strictly wild if this functor is actually full.

The fundamental theorem on tameness proved first by Drozd [Dd] (see also [CB1]) says that any finite-dimensional algebra A over an algebraically closed field k is either tame or wild but not both. Note that the original definition of tameness is weaker. To get the above definition of tameness one also has to invoke [CB2].

The proofs of the tame and wild dichotomy in [Dd] and [CB1] rest on a reduction to bocses, a very general class of matrix problems. [G-V] presents an alternative access based on a reduction to vector space categories, a much smaller class of one-sided matrix problems. This also allows to characterize tameness and wildness by involving only linear bimodules. To be more precise: