Stable homotopy of algebraic theories

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0 Introduction

The original motivation for this paper came from the attempt to understand how a rational result about the stable homotopy theory of commutative rings might generalize to arbitrary characteristic. We showed in [Sch, 3.2] that for a commutative $\mathbb{Q}$-algebra B, the stable homotopy theory of augmented commutative simplicial B-algebras is equivalent to the homotopy theory of simplicial B-modules. This theorem does not hold, as it stands, for general B. However, the stable homotopy category of commutative simplicial B-algebras is still a category of modules if one allows brave new rings rather than simplicial rings, i.e., if one works with a suitable notion of homotopy ring.

It turned out that the right notion to consider here is that of a 'Functor with Smash Product' (FSP), introduced in [Bö]. There is currently no Quillen model category structure available for modules over an FSP, hence we chose to work with a variant of this notion, which we call DFSPs. DFSPs are based on a symmetric monoidal smash product for Γ-spaces with good homotopical properties ([Se],[BF],[Ly]). They are 'FSPs defined on finite sets' and they represent all homotopy types of connective FSPs. We give model category structures for modules over a fixed DFSP, for the whole category of DFSPs and more generally for algebras over a commutative DFSP. If $H\mathbb{Z}$ denotes the (commutative DFSP model of the) Eilenberg-MacLane ring spectrum of the integers, we show that the homotopy theory of $H\mathbb{Z}$-algebras is equivalent to that of simplicial rings.

The generalization of the rational result mentioned at the beginning then reads:

Theorem: Let B be a commutative ring, possibly simplicial. Then the stable homotopy theory of augmented commutative simplicial B-algebras is equivalent to the homotopy theory of modules over a certain DFSP DB. The graded ring of homotopy groups of DB is isomorphic to the ring of stable homotopy operations in the category of commutative augmented simplicial B-algebras. If B is a $\mathbb{Q}$-algebra, this DB is weakly equivalent to (the DFSP model of) B.

We give more information about the homotopy type of DB in section 3. The theorem is appropriately dealt with in a more general framework. We are led to consider pointed simplicial algebraic theories, or just simplicial theories for short. A simplicial theory T has a category of algebras; definitions will be given later, but thinking of a T-algebra as a simplicial set with certain algebraic operations and equational relations is a good guide line. Examples include simplicial sets, simplicial sets with an action of a simplicial group, (commutative) simplicial groups, modules over a simplicial ring, augmented (commutative) algebras over a (commutative) simplicial ring, and many more. The category of T-algebras is naturally a closed simplicial model category. There are further structural properties that the homotopy theory of T-algebras shares with the ordinary homotopy theory of spaces. These include for example Blakers-Massey, Hurewicz and Whitehead theorems. The category of spectra of T-algebras can be formed and also forms a closed simplicial model category. Its homotopy theory will be referred to as the stable homotopy theory of T. The above theorem then becomes a special case of

Theorem: To a simplicial theory T, there is functorially associated a DFSP T^s. The graded ring of homotopy groups of T^s is isomorphic to the ring of stable homotopy operations in the category of T-algebras. The stable homotopy theory of T-algebras is equivalent to the homotopy theory of modules over T^s.