NORMAL MATRICES: AN UPDATE

L. Elsner
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131

33501 Bielefeld
FRG

Kh. D. Ikramov[1]
Faculty of Computational Mathematics
and Cybernetics
Moscow State University

119899 Moscow
Russia

Abstract. A list of seventy conditions on an n-by-n complex matrix A, equivalent to its being normal, published nearly ten years ago by Grone, Johnson, Sa, and Wolkowicz has proved to be very useful. Hoping that, in an extended form, it will be even more helpful, we compile here another list of about twenty conditions. They either have been overlooked by the authors of the original list or have appeared during the last decade.

[1]Research supported by Sonderforschungsbereich 343 *Diskrete Strukturen in der Mathematik*

1. Introduction.

A square complex matrix A is called normal if

$$(1) \qquad A^*A = AA^*$$

where A^* is the adjoint of A. If this relation is taken as condition 0, then the seventy conditions 1–70 equivalent to (1) constitute Section II of [6].

At the end of the introduction to their list, the authors of [6] say: "Reflecting the fact that normality arises in many ways, it is hoped that not only will it be useful now, but its utility will grow over time as conditions are added."

Nearly a decade after [6] was published, one can say that it (called henceforth the GJSW list) has indeed proved to be very useful. It is perhaps the right time to increase its utility, as the authors suggest, by supplementing the original list with additional conditions equivalent to the normality of A.

Section 2 of this paper contains nearly twenty such conditions. They either have been overlooked by the authors of [6] or have appeared during the last decade. Since our list is meant to be a continuation of that in [6], our first condition is numbered 71. The outline of proofs and/or comments to most of the conditions are given in Section 3. In nearly all cases, we point to a publication where the corresponding condition has first appeared. Our proofs are, up to minor details, the original proofs in these publications. As in [6], proofs in an obvious direction are omitted.

We conclude this section by introducing the notation, which does not in all cases coincide with that of [6]. Also, those conditions in the original GJSW list are given explicitly which are used in the body of our paper.

Our matrices belong (almost) invariably to the set $\mathbb{C}^{n,n}$ of complex $n \times n$–matrices. It is sometimes beneficial to regard $\mathbb{C}^{n,n}$ as a unitary space equipped with the inner product

$$(2) \qquad < A, B >= \mathrm{tr}\ (B^*A)$$

where $\mathrm{tr}(\cdot)$ stands for the trace of a matrix.

$\mathbb{C}^n$ denotes the set of all complex column vectors with n components. If necessary, $\mathbb{C}^n$ is also considered as a unitary space with the usual inner product $(\ ,\ )$. The symbol $||\cdot||$ is used for the 2–norm of a vector and, if not stated otherwise, for the spectral norm of a matrix. For A nonsingular, the condition number

$$(3) \qquad \mathrm{cond}\ A = ||A||\ ||A^{-1}||$$

is always meant to be the spectral one. If λ is a simple eigenvalue of A, x and y corresponding unit right and left eigenvectors, then

$$(4) \qquad \kappa(\lambda) = |(x, y)|^{-1}$$