Hairdressing in groups: a survey of combings and formal languages

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Abstract

A group is combable if it can be represented by a language of words satisfying a fellow traveller property; an automatic group has a synchronous combing which is a regular language. This article surveys results for combable groups, in particular in the case where the combing is a formal language.

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1 Introduction

The aim of this article is to survey work generalising the notion of an automatic group, in particular to classes of groups associated with various classes of formal languages in the same way that automatic groups are associated with regular languages.

The family of automatic groups, originally defined by Thurston in an attempt to abstract certain finiteness properties of the fundamental groups of hyperbolic manifolds recognised by Cannon in [12], has been of interest for some time. The defining properties of the family give a geometrical viewpoint on the groups and facilitate computation with them; to such a group is associated a set of paths in the Cayley graph of the group (a 'language' for the group) which both satisfies a geometrical 'fellow traveller condition' and, when viewed as a set of words, lies in the formal language class of regular languages. (A formal definition is given in section 2.) Epstein et al.'s book [15] gives a full account; the papers

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[3] and [16] are also useful references (in particular, [16] is very readable and non-technical).

The axioms of an automatic group are satisfied by all finite groups, all finitely generated free and abelian groups, word hyperbolic groups, the fundamental groups of compact Euclidean manifolds, and of compact or geometrically finite hyperbolic manifolds ([15, 26]), Coxeter groups ([10]), braid groups, many Artin groups ([13, 14, 28, 24]), many mapping class groups ([27], and groups satisfying various small cancellation conditions ([18]). However some very interesting groups are not automatic; the family of groups fails to contain the fundamental groups of compact 3-manifolds based on the Nil and Sol geometries, and, more generally, fails to contain any nilpotent group (probably also any soluble group) which is not virtually abelian. This is maybe surprising since nilpotent groups have very natural languages, with which computation is very straightforward.

A family of groups which contains the fundamental groups of all compact, geometrisable 3-manifolds was defined by Bridson and Gilman in [9], through a weakening of both the fellow traveller condition and the formal language requirement of regularity for automatic groups. The fellow traveller condition was replaced by an asynchronous condition of the same type, and the regularity condition by a requirement that the language be in the wider class of 'indexed languages'. The class of groups they defined can easily be seen to contain a range of nilpotent and soluble groups.

Bridson and Gilman's work suggests that it is sensible to examine other families of groups, defined in a similar way to automatic groups with respect to other formal language classes. This paper surveys work on this theme. It attempts to be self contained, providing basic definitions and results, but referring the reader elsewhere for fuller details and proofs. Automatic groups are defined, and their basic properties described in section 2; the more general notion of combings is then explained in section 3. A basic introduction to formal languages is given in section 4 for the sake of the curious reader with limited experience in this area. (This section is included to set the results of the paper into context, but all or part of it could easily be omitted on a first reading.) Section 5 describes the closure properties of various classes of combable groups, and section 6 gives examples (and non-examples) of groups with combings in the classes of regular, context-free, indexed and real-time languages.

2 Automatic Groups

Let G be a finitely generated group, and X a finite generating set for G. We define a *language* for G over X to be a set of *words* over X (that is, products in the free group $F(X)$) which maps onto G under the natural homomorphism from $F(X)$; such a language is called *bijective* if the natural map is bijective.

G is defined to be automatic by the existence of a language satisfying two