Classification problems in the representation theory of finite-dimensional algebras

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Contents

1 Introduction

1.1 Combinatorics arising in the representation theory of algebras

The aim of the representation theory of algebras is to understand the category Mod$-A$ of modules over a given associative unital k-algebra A where k is a commutative ring. We will restrict ourselves to the case that k is a field and A is finite-dimensional over k. Familiar examples for algebras of this kind are the group algebras kG for a finite group G or the factor algebras $k[X]/I$ of the polynomial ring $k[X]$ by a non-zero ideal I.

Frequently, we will assume that the field k is algebraically closed. Moreover, we will mainly deal only with the full subcategory mod$-A$ of Mod$-A$ given by all right A-modules M which are finite-dimensional as vector spaces over k.

The key opening representation theory of finite-dimensional algebras to combinatorial methods is the observation that these algebras as well as their categories of finite-dimensional modules can be described to a large extent by locally finite quivers. Recall that 'quiver' is the short notation for 'oriented graph' which is used commonly in representation theory. The quiver describing the algebra is sometimes called the 'Gabriel quiver' whereas for the quiver of the category of finite-dimensional modules meanwhile 'Auslander-Reiten quiver' is the standard name.

In the Auslander-Reiten quiver an indecomposable module shrinks to a mere vertex. A compromise between this total loss of structure and the wish to work only with combinatorial data is furnished by looking at the corresponding element of the module in the Grothendieck group (with respect to short exact sequences). Note, that this Grothendieck group is always of the shape $\mathbb{Z}^n$ for some $n \in \mathbb{N}$.

For general algebras the Auslander-Reiten quiver together with the Grothendieck group does not lead to many effective combinatorial algorithms. The situation improves drastically if one considers strongly simply connected algebras. Associated with the representation theory of these algebras another combinatorial structure namely partially ordered sets arise. Quivers, Grothendieck groups, partially ordered sets together with various actions on these structures form the combinatorial environment of the classifications and algorithms which we will present.

1.2 Representation types, classifications and CREP

In the last 30 years a lot of effort was made to study and classify finite-dimensional algebras with respect to their representation type. An algebra is said to be of finite representation type if there are only finitely many isomorphism classes of indecomposable finite-dimensional modules. A generalization is given by the tame algebras which are characterized by having for any fixed dimension only finitely many 1-parameter families of isomorphism classes of indecomposable modules. By a theorem of Drozd algebras which are not tame have to be wild which means that the classification of their indecomposable modules is as difficult as the classification of pairs of square matrices under simultaneous conjugation.

In the study of finite representation type it turned out that it is better to pass to the universal covering of a suitable degeneration called the 'standard form' of a given algebra. Thus one arrives at a strongly simply connected algebra. For the study of tame algebras a similar approach seems to work although up to now there are no complete results in this case.

In order to understand finite representation type for strongly simply connected algebras, the aim was to classify the minimal representation infinite algebras and those algebras which have a faithful indecomposable module. This led to the classifications of the tame concealed algebras and the sincere representation directed algebras. At this place computers had to enter the business because it turned out that there were so many exceptional structures yielding huge amounts of combinatorial data that otherwise the explicit classifications would have been impossible. Our main aim is to review these classifications introducing the necessary material to understand the algorithms which were used.

The original classifications were performed by stand-alone programs. Meanwhile we have developed the program system CREP (Combinatorial REPresentation theory) which allows to deal with algorithms, examples and classifications in a more conceptual and systematic way (see [DN2]). Another aim of these notes is to illustrate this progress and to indicate new developments. In particular, we will complete a classification of certain tame incidence algebras associated with partially ordered sets using our algorithms