An Example of an Arithmetic 3–Manifold With Cubic Ground Field

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Introduction

In this note we work out some technical details which are typically faced in the context of hyperbolic lattices in dimension three. We take a specific example of a group given by an abstract presentation and construct a (cocompact) lattice representation in the group of isometries of hyperbolic three space.

The group is

$$\Pi_\infty = \langle \xi, \eta, \mu | \mu\xi\mu^{-1} = \eta\xi, \mu\eta\mu^{-1} = (\eta\xi)^{-5}\eta \rangle;$$

it is well known to be the fundamental group of a three dimensional manifold M_∞ which is hyperbolic (see [5]). This manifold fibers over the circle, the fiber being a once punctured torus; it may be discribed in the following way: Let T_0 a topological torus with a small open disc removed. We select a pair $\{\xi, \eta\}$ of generators of $\pi_1(T_0)$ such that

$$\lambda = \eta^{-1}\xi^{-1}\eta\xi$$

is represented by a loop around the hole. We take a homeomorphism

$$f : T_0 \longrightarrow T_0$$

which is the identity close to the hole and which, on $\pi_1(T_0)$, induces the isomorphism

$$f_* : (\xi, \eta) \longrightarrow (\eta\xi, (\eta\xi)^{-5}\eta).$$

Then M is defined to be

$$M = (T_0 \times [0, 1]) / \sim$$

where the equivalence relation is

$$(p, 0) \sim (f(p), 1) \qquad \text{for all } p \in T_0.$$

A peripheral system (see [3]) is composed by λ and μ. We define

$$\Pi = \langle \xi, \eta, \mu | \mu\xi\mu^{-1} = \eta\xi, \mu\eta\mu^{-1} = (\eta\xi)^{-5}\eta, \mu^2 = (\eta^{-1}\xi^{-1}\eta\xi)^{-3} \rangle$$

as a quotient of Π_∞. This group is the fundamental group of a closed oriented three-manifold which comes as result of a specific Dehn surgery performed on M_∞. It is hyperbolic, compare [5]. We explicitly exhibit a cocompact lattice $\tilde{\Gamma}$ in $Iso^+(\mathbf{H}^3)$ which as a group is isomorphic to $\Pi = \pi_1(M)$. This lattice turns out to be arithmetic. We are in the position to present a fundamental domain for Γ which allows to compute the volume of $M \cong \Gamma \setminus \mathbf{H}^3$.

It should be noted that actually a certain lattice $\tilde{\Gamma}$ in $Iso^+(\mathbf{H}^3)$ will be presented which is easier to handle than Γ. This $\tilde{\Gamma}$ will be an index two extension of Γ.

It turns out that when doing matrix calculations in $GL(2, \mathbf{C})$ it makes no sense to insist on determinant 1. So matrix identities are when checking group relators most frequently understood as identities modulo the center of $GL(2, \mathbf{C})$.

It is natural to try to apply the general formula for the volume of an arithmetic manifold which is proven in [2]. Also in that case it seems to be indispensable to have a fundamental domain at one's disposal. In addition one would have to estimate Dedekind's ζ-function for a certain cubic number field at the argument $s = 2$ with high accuracy. We use Lobačevskij's function, instead.

A quaternion algebra

The ground field of the quaternion algebra which will contain a lattice commensurable to Γ is cubic and defined over the rationals by

$$t^3 - t - 1 = 0.$$

We denote the abstract field $\mathbf{Q}(t)$ by K. K has one real and one pair of complex conjugate embeddings into $\mathbf{C}$. The discriminant of K is -23; any root of the above polynomial generates the absolute class field over $\mathbf{Q}(\sqrt{-23})$, see [4]. This field is Galois over $\mathbf{Q}$ with Galois group dihedral of order 6. If t is any root the two others are

$$-\frac{1}{2}t \pm \frac{4 + 9t - 6t^2}{2\sqrt{-23}}$$

If we choose t to be the real root this is the explicit decomposition of the nonreal ones into real and imaginary part. The ring $\mathbf{o}$ of integers in K is easily identified to $\mathbf{Z} + \mathbf{Z}t + \mathbf{Z}t^2$. Note that t is a unit in the ring of integers of K; it is easy to see that t is not a square in K.

For computations in geometrical context we prefer to work with

$$x = 1 - \frac{1}{t}.$$

Its minimal equation over $\mathbf{Q}$ is

(1)
$$x^3 - 4x^2 + 5x - 1 = 0.$$