HNN extension of cyclically presented groups *

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Abstract

It is shown that if the abelianization of a cyclically presented group is finite and the defining word is admissable then its natural HNN extension is the group of a high dimensional knot. As an example we define a family of cyclically presented groups which contains Sieradski groups, Fibonacci groups, and Gilbert-Howie groups. It is proven that HNN extensions of these groups are LOG groups and so, are fundamental groups of complements of codimension two closed orientable connected tamely embedded ℓ-dimensional manifolds ($\ell \geq 2$).
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1 Introduction

In the present paper we study the natural HNN extension of cyclically presented groups. We shall prove that, under some assumptions, they are the groups of a high dimensional knot, labelled oriented graph (LOG) groups and a fundamental groups of complement of codimension two submanifold.

In section 2 (Theorem 1) we give conditions when the natural HNN extension of a cyclically presented group is high dimensional knot group. This result improves on the main result of [15] and [16]. Next, we introduce a family of cyclically presented groups which contains well-known Fibonacci

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groups [10], Sieradski groups [2], Gilbert-Howie groups [6], and groups of some three-dimensional manifolds. In section 3 we shall prove (Corollary 2) that the natural HNN extensions of the groups from the above family are LOG groups. Hence, they are the fundamental groups of complements of codimension two submanifolds. This section was mainly motivated by the results of [6].

Recall some terminology which will be used below. A group G is said to be *cyclically presented* [10] if for some n and w it has the presentation

$$G = G_n(w) = \langle x_1, \dots, x_n \mid w, \eta(w), \dots, \eta^{n-1}(w)\rangle,$$

where $\eta : \mathbb{F}_n \to \mathbb{F}_n$ is an automorphism of the free group $\mathbb{F}_n = \langle x_1, \dots, x_n\rangle$ of rank n given by $\eta(x_i) = x_{i+1}$, $i = 1, \dots, n-1$, $\eta(x_n) = x_1$, and $w \in \mathbb{F}_n$ is a cyclically reduced word.

The polynomial associated with the cyclically presented group $G_n(w)$ is given by

$$f_w(t) = \sum_{i=1}^{n} a_i t^{i-1},$$

where a_i is the exponent sum of x_i in w, $1 \leq i \leq n$. We will say that the word w is *admissable* if $|f_w(1)| = 1$.

Obviously, η induces an automorphism $\Phi : G_n(w) \to G_n(w)$ given by $\Phi(x_i) = x_{i+1}$, $i = 1, \dots, n-1$ with $\Phi(x_n) = x_1$. Let us define the *natural HNN-extension* $\mathcal{G}_n(w)$ of the cyclically presented group $G_n(w)$:

$$\mathcal{G}_n(w) = \{G_n(w), t \mid t^{-1}gt = \Phi(g), \quad g \in G_n(w)\}.$$

The presentation $\langle x_1, x_2, \dots \mid r_1, r_2, \dots\rangle$ of a group G is said to be a *Wirtinger presentation*, if each relator r is of the form $r = x_i^{-1} w_{ij}^{-1} x_j w_{ij}$, where x_i and x_j are some generators, and w_{ij} is a word in G. According to [14], a group G can be realized as $\pi_1(S^{\ell+2} \setminus M^\ell)$ ($\ell \geq 2$), where M^ℓ is a closed, orientable, connected ℓ-manifold tamely embedded in the $(\ell+2)$-sphere $S^{\ell+2}$, if and only if G satisfies the following: (1) G is finitely presented; (2) $G/G' \cong \mathbb{Z}$; (3) G is of weight one, i.e. there exists $t \in G$ such that $G/ << t >> = \{1\}$ (here $<< t >>$ denotes the normal close of t in G); (4) G has a Wirtinger presentation.

If we replace the condition (4) by the property $H_2(G) = 0$, we obtain Kervaire's conditions [13, Section 11D] for a group to be a ℓ-knot group ($\ell \geq 3$). It was shown in [14] that any ℓ-knot group has a Wirtinger presentation.

A group G is said to be a LOG (labelled oriented graph) group if it can be encoded by a labelled oriented graph in the following way [6]. Consider a finite connected graph Γ with the set of vertices $V = V(\Gamma)$ and the set of edges $E = E(\Gamma)$, and three maps $i, \tau, \lambda : E \to V$ which are respectively, the *initial vertex* map, the *terminal vertex* map, and the *labelling* map. The maps i and τ together define an oriented graph structure on (V, E) in which loops and multiply edges are allowed.

A labelled oriented graph Γ determines the group presentation

$$G(\Gamma) \;=\; \langle V(\Gamma) \,|\, [\tau(e)]^{-1}\, [\lambda(e)]^{-1}\, i(e)\, \lambda(e), \quad e \in E(\Gamma)\rangle.$$

A group G is said to be *LOG group* if it has a presentation $G(\Gamma)$ for some labelled oriented graph Γ. As we see, LOG presentation is a Wirtinger presentation. So, the fundamental group of the complement to a codimension two closed, orientable, tamely embedded ℓ-manifold is a LOG group. Not every labelled oriented graph gives rise to such a group. However, it was shown in [8] that for each integer $\ell \geq 2$, every labelled oriented tree gives rise to the presentation of a ribbon ℓ-disc complement. Moreover, it was shown in [9] that if Γ is a labelled oriented tree of diameter at most 3 then the corresponding group $G(\Gamma)$ is an HNN extension with finitely presented base.

2 The groups with admissable defining word

Denote by $A_n(w) = G_n(w)^{ab}$ the abelianization of the group $G_n(w)$.

Theorem 1. *Let $\mathcal{G}_n(w)$ be the natural HNN extension of a cyclically presented group $G_n(w)$. Assume that the abelianization $A_n(w)$ is finite. Then $\mathcal{G}_n(w)$ is a ℓ-knot group ($\ell \geq 3$) if and only if the word w is admissable.*

Proof: For the proof we shall check the Kervaire's conditions (cf. Introduction). First we prove that the automorphism Φ is *meridional* if and only if $|f(1)| = 1$; in other words, that the normal closure $C_n(w)$ in $G_n(w)$ of $\{g^{-1}\Phi(g) \mid g \in G_n(w)\}$ is $G_n(w)$ only for $k = 1$, where $k = |f(1)|$ (cf. [7, p. 123]). Indeed, there is an epimorphism

$$h : G_n(w) \to \mathbb{Z}_k \; = < \gamma \mid \gamma^k = 1 >$$

given by $h(x_i) = \gamma$ for all $i = 1, \dots, n$. It is immediate from the definition of Φ that $C_n(w) \subset \ker h$. Hence $H_1(\mathcal{G}_n(w)) = \mathbb{Z}$ if and only if $k = 1$. In fact, if $k = 1$ then $C_n(w) = \ker h$ and the abelianization of $\mathcal{G}_n(w)$ is equal to $\mathbb{Z}$. In opposite side, if $k \neq 1$ then $G \neq \ker h$ and hence the abelianization of the HNN extension has non-trivial torsions. That gives contradiction. So, from now we assume that $k = 1$ in our considerations.

Second we show that $\mathcal{G}_n(w)$ is a normal closure of the element $b = t^{-1}x_1$. In fact,

$$t^{-1}x_2 \;=\; t^{-1}(tx_1t^{-1}) \;=\; t(t^{-1}x_1)t^{-1} \;=\; tbt^{-1},$$

and for $i = 1, \ldots, n$ we get

$$t^{-1}x_i \;=\; t^{-1}(t^{i-1}x_1t^{-(i-1)}) \;=\; t^{i-1}(t^{-1}x_1)t^{-(i-1)} \;=\; t^{i-1}bt^{-(i-1)}.$$

Let $\mathcal{B}$ be a normal closure of b in $\mathcal{G}_n(w)$. Consider the canonical projection

$$\rho : \mathcal{G}_n(w) \;\to\; \mathcal{G}_n(w)/\mathcal{B},$$

and denote $\rho(x) = \bar{x}$ for $x \in \mathcal{G}_n(w)$. Then $\bar{x}_1 = \ldots = \bar{x}_n = \bar{t}$. The assumption $|f(1)| = 1$ holds $\bar{t} = e$, the neutral element. Therefore $\bar{x}_1 = \ldots = \bar{x}_n = \bar{t} = e$ and $\mathcal{G}_n(w) = \mathcal{B}$. Thus $\mathcal{G}_n(w)$ has weight 1.

Third from the short exact sequence of groups

$$1 \;\to\; G_n(w) \;\to\; \mathcal{G}_n(w) \;\to\; \mathbb{Z} \;\to\; 1$$

we have the Hochschild–Serre spectral sequence [1, p. 171]:

$$E^2_{p,q} \;=\; H_p(\mathbb{Z}, H_q(G_n(w))).$$

So, it is enough to prove that

$$E^2_{2,0} \;=\; H_2(\mathbb{Z}, H_0(G_n(w))) \;=\; 0,$$

$$E^2_{1,1} \;=\; H_1(\mathbb{Z}, H_1(G_n(w))) \;=\; H_1(G_n(w))_{\mathbb{Z}} \;=\; 0,$$

and

$$E^2_{0,2} \;=\; H_0(\mathbb{Z}, H_2(G_n(w))) \;=\; 0.$$

The first equality is obvious. For the proof of the next we shall use the Hopf's formula to get a 5-term exact sequence [1, p. 47]

$$H_2(\mathcal{G}_n(w)) \;\to\; H_2(\mathbb{Z}) \;\to\; H_1(G_n(w))_{\mathbb{Z}} \;\to\; H_1(\mathcal{G}_n(w)) \;\to\; H_1(\mathbb{Z}) \;\to\; 0,$$

where the $\mathbb{Z}$-action on $H_1(G_n(w))$ is induced by the conjugation action of $\mathcal{G}_n(w)$ on G. We have $H_2(\mathbb{Z}) = 0$, $H_1(\mathbb{Z}) = \mathbb{Z}$. It was shown on the first step that $H_1(\mathcal{G}_n(w)) = \mathbb{Z}$. Thus

$$0 \to H_1(G_n(w))_{\mathbb{Z}} \to \mathbb{Z} \to \mathbb{Z} \to 0,$$

and

$$H_1(G_n(w))_{\mathbb{Z}} = 0.$$

We remark that $H_2(G_n(w)) = 0$. In fact, from Hopf's formula [1, p. 46] the number of generators of the group $H_2(G_n(w))$ is equal to $r - g + s$, where g is a number of generators, r is a number of relations of the group G, and $s = \text{rank}(H_1(G_n(w))$. In our case $g = r = n$, and from the assumption about $A_n(w)$ we get $s = 0$. So, $H_2(G_n(w)) = 0$ and $E^2_{0,2} = 0$. Summing up we prove that $H_2(\mathcal{G}_n(w)) = 0$.

$\square$

Due to [10] there is a simple way to check if the abelianization $A_n(w)$ of a cyclically presented group $G_n(w)$ is finite: $A_n(w)$ is infinite if and only if $f_w(t)$ has a root in common with $t^n - 1$. Using it we get a new version of the Theorem 1.

Theorem 1′. *Assume that $f_w(t)$ has no roots in common with $t^n - 1$. Then $\mathcal{G}_n(w)$ is a ℓ-knot group ($\ell \geq 3$) if and only if $|f_w(1)| = 1$.*

Question: *Let $G_n(w)$ be a cyclically presented group with an infinite abelianization. ¿From the proof of Theorem 1 the condition $|f_w(1)| = 1$ is necessary for $\mathcal{G}_n(w)$ to be a ℓ-knot group. When this condition is sufficient?*

Example 1. Let us consider the group

$$\Gamma_n^k = G_n(x_1 x_2 \dots x_{n-1} x_n^{-k})$$

defined in [17]. If $k = n$ or $k = n - 2$ then assumptions of the theorem are satisfied. So, the natural HNN extensions of groups Γ_n^n and Γ_n^{n-2} are ℓ-knot groups $\ell \geq 3$. Remark that Γ_n^k are the fundamental groups of Seifert fibred spaces whose Seifert invariants are described in [17].

Example 2. All groups presented in [5, Table 1] have admissable defining word. Moreover, corresponding polynomials $f_w(t)$ are Alexander polynomials of 1-knots.

Example 3. For integers $n \geq 2$, $r \geq 1$, and k consider sets of integers $q_1, \ldots, q_r$, and $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_r$. Define a family of groups with the cyclic presentations by the following:

$$\begin{aligned} & G^r_{n,k}(q_1, \ldots, q_r; \varepsilon_0, \ldots \varepsilon_r) \\ & \quad = G_n \left(x_1^{\varepsilon_0} \, x_{1+q_1}^{\varepsilon_1} \cdots x_{1+q_{r-1}}^{\varepsilon_{r-1}} \, x_{1+q_r}^{\varepsilon_r} \, (x_{1+k}^{\varepsilon_0} \, x_{1+k+q_1}^{\varepsilon_1} \cdots x_{1+k+q_{r-1}}^{\varepsilon_{r-1}})^{-1} \right), \end{aligned}$$

where all suffices are by modulo n.

This family contains some well-known particular cases.

(1) The groups

$$G^1_{n,1}(n-1; -1, -1) \; = \; G_n(x_1^{-1} \, x_n^{-1} \, x_2) \; = \; G_n(x_1 \, x_2 \, x_3^{-1})$$

are the Fibonacci groups $F(2, n)$ introduced by Conway. These groups with even number of generators are fundamental groups of n-fold cyclic coverings of the 3-sphere branched over the figure-eight knot [16]. It is well known that these groups have a finite abelianization (cf. [10]).

(2) The groups

$$G^1_{n,1}(m; 1, 1) \; = \; G_n(x_1 \, x_{1+m} \, x_2^{-1})$$

were introduced in [6] (and notated by $H(n, m)$), where their HNN-extensions and asphericity were studied. In particular, the group $G^1_{n,1}(m; 1, 1)$ has infinite abelianization if and only if 6 divides n and $m = 2$ modulo 6 [11].

(3) The groups

$$G^1_{n,k}(m; 1, 1) \; = \; G_n(x_1 \, x_{1+m} \, x_{1+k}^{-1})$$

were introduced in [3]. It is not difficult to see that the polynomial associated with this group is equal to $f_w(t) = t^m - t^k + 1$. By direct considerations one can see that $f_w(t)$ has a common root with $t^n - 1$ if and only if 6 divides n and $m = k + 1$ modulo 6.

(4) The groups

$$G^r_{n,1}(2, 4, \ldots, 2r; 1, 1, \ldots, 1) \; = \; G_n \left(x_1 x_3 \cdots x_{1+2r} \, (x_2 x_4 \cdots x_{2r}^{-1} \right),$$

are the generalized Sieradski groups $S(r + 1, n)$ investigated in [2]. These groups are fundamental groups of n-fold cyclic coverings of the 3-sphere branched over the torus knots $T(2r+1,2)$. (cf. [2]). For $r=1$ we get the Sieradski groups notated by $S(n)$ in [2]. The polynomial associated with this group is equal to

$$f_w(t) \;=\; \sum_{i=0}^{2r}(-1)^i t^i.$$

It is easy to see that $f_w(t) \mid (t^{4r}-1)$. Hence for $4r$ does not divide by n the abelianization of the generalized Sieradski group is finite.

(5) The groups

$$\begin{aligned} &G^r_{n,k-1}(q,2q,\ldots,q(r-1);1,1,\ldots,1)\\ &\quad= G_n\left(x_1x_{1+q}\cdots x_{1+q(r-1)}\left(x_kx_{k+q}\cdots x_{k+q(r-2)}\right)^{-1}\right), \end{aligned}$$

were considered in [12] (and have notations $P(r,n,k,r-1,q)$), where asphericity and atoricity of groups with similar presentations were studied.

(6) The groups

$$\begin{aligned} &G^4_{n,1}(1,2,3,3;-(k-2),k-1,-(k-2),k-2,1)\\ &\quad= G_n\left(x_1^{-(k-2)}x_2^{k-1}x_3^{-(k-2)}x_4^{k-2}x_4\,(x_2^{-(k-2)}x_3^{k-1}x_4^{-(k-2)}x_5^{k-2})^{-1}\right), \end{aligned}$$

were considered in [4] (and notated by $G^3_n(k)$). These groups are fundamental groups of n-fold cyclic coverings of the 3-sphere branched over the 2-bridge knots $\dfrac{8k-13}{2k-3}$.

(7) The groups

$$\begin{aligned} &G^4_{n,1}(1,2,3,4;-(k-1),k-2,-(k-1),k-2,-1)\\ &\quad= G_n\left(x_1^{-(k-1)}x_2^{k-2}x_3^{-(k-1)}x_4^{k-2}x_5^{-1}\,(x_2^{-(k-1)}x_3^{k-2}x_4^{-(k-1)}x_5^{k-2})^{-1}\right) \end{aligned}$$

were considered in [4] (and notated by $G^4_n(k)$). These groups are fundamental groups of n-fold cyclic coverings of the 3-sphere branched over the 2-bridge knots $(2k+1)_4$.

Further to simplify notations we will write $G^r_{n,k}(\bar q;\bar\varepsilon)$, where $\bar q=(q_1,\ldots,q_r)$ and $\bar\varepsilon=(\varepsilon_0,\ldots,\varepsilon_r)$. It is an easy observation that the word w is admissable if $|\varepsilon_r|=1$. As one can see, it holds for the above cases $(1)-(7)$.

Corollary 1. *Let $G=G^r_{n,k}(\bar q;\bar\varepsilon)$ with $|\varepsilon_r|=1$. If the abelianization G/G' is finite (or, equivalently, the polynomial*

$$f_w(t) \;=\; \varepsilon_r t^{q_r} + (1-t^k)(\varepsilon_{r-1}t^{q_{r-1}} + \cdots + \varepsilon_1 t^{q_1} + \varepsilon_0)$$

has no roots in common with t^n-1), then $\mathcal{G}^r_{n,k}(\bar q;\bar\varepsilon)$ is a ℓ-knot group ($\ell\geq 3$).

3 LOG groups

In this section we will demonstrate that groups $\mathcal{G}^r_{n,1}(\bar{q};\bar{\varepsilon})$ with $|\varepsilon_i| = 1, i = 1,\ldots,r$ have Wirtinger presentations and so, they are LOG groups. Therefore these groups are codimension two manifold groups.

Theorem 2. *A group $\mathcal{G}^r_{n,k}(\bar{q};\bar{\varepsilon})$ has the following presentation:*

$$\begin{aligned}\langle b_1,\ldots,b_n,t \quad | \quad & b_{i+1} = t^{-1}b_i t, \quad i=1,\ldots,n,\\ & b^{\varepsilon_r}_{1-k+q_r-q_{r-1}} = \left(\prod_{i=0}^{r-1} b^{\varepsilon_i}_{1+k(r-1-i)+q_r-q_{r-1}}\right)^{-1} t^{-1} \left(\prod_{i=0}^{r-1} b^{\varepsilon_i}_{1+k(r-1-i)+q_r-q_{r-1}}\right)\rangle\end{aligned} \quad (1)$$

where all suffices are by modulo n.

Proof: By the definition of the group, we have

$$\begin{aligned}\mathcal{G}^r_{n,k}(\bar{q};\bar{\varepsilon}) \;=\; \langle x_1,\ldots,x_n,t\,| \quad & x_{i+1} \;=\; t^{-1}x_i t, \quad i=1,\ldots,n\\ & x_i^{\varepsilon_0}\, x^{\varepsilon_1}_{i+q_1}\cdots x^{\varepsilon_{r-1}}_{i+q_{r-1}}\, x^{\varepsilon_r}_{i+q_r} \;=\; x^{\varepsilon_0}_{i+k}\, x^{\varepsilon_1}_{i+k+q_1}\cdots x^{\varepsilon_{r-1}}_{i+k+q_{r-1}} \quad i=1,\ldots,n\rangle\end{aligned} \quad (2)$$

The relations of the second type in (2) are pairwise conjugated. So, all of them are equivalent to the relation

$$x_1^{\varepsilon_0}\, x^{\varepsilon_1}_{1+q_1}\cdots x^{\varepsilon_{r-1}}_{1+q_{r-1}}\, x^{\varepsilon_r}_{1+q_r} \;=\; x^{\varepsilon_0}_{1+k}\, x^{\varepsilon_1}_{1+k+q_1}\cdots x^{\varepsilon_{r-1}}_{1+k+q_{r-1}}. \quad (3)$$

Using $x^{\varepsilon_j}_{1+k+q_j} \;=\; t^{-k}x^{\varepsilon_j}_{1+q_j}t^k$, from (3) we get

$$x^{\varepsilon_r}_{1+q_r} \;=\; \left(x_1^{\varepsilon_0}\, x^{\varepsilon_1}_{1+q_1}\cdots x^{\varepsilon_{r-1}}_{1+q_{r-1}}\right)^{-1} t^{-k} \left(x_1^{\varepsilon_0}\, x^{\varepsilon_1}_{1+q_1}\cdots x^{\varepsilon_{r-1}}_{1+k+q_{r-1}}\right) t^k,$$

so, $x^{\varepsilon_r}_{1+q_r}\, t^{-k} \;=\; v^{-1}\, t^{-k}\, v$, where $v = x_1^{\varepsilon_0}\, x^{\varepsilon_1}_{1+q_1}\cdots x^{\varepsilon_{r-1}}_{1+q_{r-1}}$. Consider elements $b_i = t^{-k}x_i$, $i=1,\ldots,n$, whence $b_{i+1} = t^{-1}b_i t$. Using $x^{\varepsilon_r}_{1+q_r} = t^k b^{\varepsilon_r}_{1+q_r}$ we get

$$t^k\, b^{\varepsilon_r}_{1+q_r}\, t^{-k} \;=\; v^{-1}\, t^{-k}\, v$$

that is $b^{\varepsilon_r}_{1-k+q_r} \;=\; v^{-1}\, t^{-k}\, v$. As

$$v \;=\; x_1^{\varepsilon_0}\, x^{\varepsilon_1}_{1+q_1}\cdots x^{\varepsilon_{r-2}}_{1+q_{r-2}}\, t^{-q_{r-1}}\, x_1^{\varepsilon_{r-1}}\, t^{q_{r-1}},$$

denoting $\widetilde{w} = v\, t^{-q_{r-1}}$, we get

$$b^{\varepsilon_r}_{1-k+q_r} \;=\; t^{-q_{r-1}}\, \widetilde{w}^{-1}\, t^{-k}\, \widetilde{w}\, t^{q_{r-1}},$$

whence

$$b^{\varepsilon_r}_{1-k+q_r-q_{r-1}} \;=\; \widetilde{w}^{-1}\, t^{-k}\, \widetilde{w}. \tag{4}$$

Next we will find the expression of $\widetilde{w}$ in terms of b_i's.

Suppose that $q_0 = 0$. By the direct calculations:

$$\begin{aligned}
\widetilde{w} &= x_1^{\varepsilon_0}\, x_{1+q_1}^{\varepsilon_1} \cdots x_{1+q_{r-2}}^{\varepsilon_{r-2}}\, t^{-q_{r-1}}\, x_1^{\varepsilon_{r-1}} \\
&= t^k\, b_1^{\varepsilon_0}\, t^k\, b_{1+q_1}^{\varepsilon_1} \cdots t^k\, b_{1+q_{r-2}}^{\varepsilon_{r-2}}\, t^{-(q_{r-1}-k)}\, b_1^{\varepsilon_{r-1}} \\
&= t^k\, b_1^{\varepsilon_0}\, t^k\, b_{1+q_1}^{\varepsilon_1} \cdots t^k\, b_{1+q_{r-3}}^{\varepsilon_{r-3}}\, t^k\, t^{\pm(q_{r-1}-k)} b_{1+q_{r-2}}^{\varepsilon_{r-2}}\, t^{-(q_{r-1}-k)}\, b_1^{\varepsilon_{r-1}} \\
&= t^k\, b_1^{\varepsilon_0}\, t^k\, b_{1+q_1}^{\varepsilon_1} \cdots t^k\, b_{1+q_{r-3}}^{\varepsilon_{r-3}}\, t^{-(q_{r-1}-2k)}\, b_{1+k+q_{r-2}-q_{r-1}}^{\varepsilon_{r-2}}\, b_1^{\varepsilon_{r-1}} \\
&= t^k\, b_1^{\varepsilon_0}\, t^k\, b_{1+q_1}^{\varepsilon_1} \cdots t^k\, b_{1+q_{r-3}}^{\varepsilon_{r-3}}\, t^{-(q_{r-1}-2k)} \prod_{i=r-2}^{r-1} b_{1+k(r-1-i)+q_i-q_{r-1}}^{\varepsilon_i} \\
&\cdots \\
&= t^k\, b_1^{\varepsilon_0}\, t^k\, b_{1+q_1}^{\varepsilon_1}\, t^{-(q_{r-1}-(r-2)k)} \prod_{i=2}^{r-1} b_{1+k(r-1-i)+q_i-q_{r-1}}^{\varepsilon_i} \\
&= t^k\, b_1^{\varepsilon_0}\, t^{-(q_{r-1}-(r-1)k)} \prod_{i=1}^{r-1} b_{1+k(r-1-i)+q_i-q_{r-1}}^{\varepsilon_i} \\
&= t^{rk-q_{r-1}} \prod_{i=0}^{r-1} b_{1+k(r-1-i)+q_i-q_{r-1}}^{\varepsilon_i}.
\end{aligned} \tag{5}$$

Denote $w = \prod_{i=0}^{r-1} b_{1+k(r-1-i)+q_i-q_{r-1}}^{\varepsilon_i}$. Then from (4) and (5) we have

$$b^{\varepsilon_r}_{1-k+q_r-q_{r-1}} \;=\; \widetilde{w}^{-1}\, t^{-k}\, \widetilde{w} \;=\; w^{-1}\, t^{-k}\, w,$$

and the theorem is proven.

□

Corollary 2. *If $k = 1$ and $|\varepsilon_i| = 1$ for $i = 1, \ldots, n$ then the group $\mathcal{G}^r_{n,1}(\bar{q}; \bar{\varepsilon})$ has the following Wirtinger presentation:*

$$\begin{aligned}
\langle b_1^{\varepsilon_r}, \ldots, b_n^{\varepsilon_r}; t_0, t_1, \ldots, t_{r-1} \mid \quad & b_{i+1}^{\varepsilon_r}, = t_0 b_i^{\varepsilon_r}, t_0^{-1}, \quad i = 1, \ldots, n, \\
t_{j+1} \;=\; & b^{-\varepsilon_j}_{r-j+q_j-q_{r-1}} t_j b^{\varepsilon_j}_{r-j+q_j-q_{r-1}}, \quad j = 0, \ldots, r-2, \\
& b^{\varepsilon_r}_{q_r-q_{r-1}} \;=\; b_1^{-\varepsilon_{r-1}} t_{r-1} b_1^{\varepsilon_{r-1}} \rangle.
\end{aligned} \tag{6}$$

where all indices are by modulo n.

Proof: Suppose $k = 1$ in (1) and denote $c_j = b^{\varepsilon_j}_{r-j+q_j-q_{r-1}}$ (in particular $c_{r-1} = b_1^{\varepsilon_{r-1}}$). Considering $t_0 = t^{-1}$, $t_1 = c_0^{-1}t_0c_0$, …, $t_{r-1} = c_{r-2}^{-1}t_{r-2}c_{r-2}$. we will get the presentation (6).

□

It follows from the presentation (6) that the group $\mathcal{G}^r_{n,1}(\bar{q};\bar{\varepsilon})$ with $|\varepsilon_i| = 1$ is a LOG group. Corresponding labelled oriented graph is pictured in Figure 1.

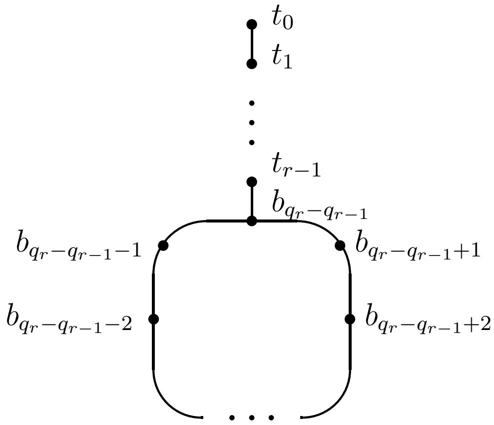

Fig. 1. The graph defining the group.

According to Introduction we get

Corollary 3. *Any group $\mathcal{G}^r_{n,1}(\bar{q};\bar{\varepsilon})$ with $|\varepsilon_i| = 1, i = 1, \dots, r$ is the fundamental group of the complement $S^{\ell+2} \setminus M^\ell$ where M^ℓ is a closed orientable connected tamely embedded ℓ-manifold ($\ell \geq 2$).*

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