ON THE LOCATION OF THE ZEROS OF FABER POLYNOMIALS

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Abstract. We present non-convex sets for which the zeros of the corresponding Faber polynomials are outside the convex hull of the sets. Unlike the sets that have been used for this purpose before, our sets have nonempty interiors and analytic boundaries.

AMS Subject Classifications (2000). 30C15, 30C20.

1. Introduction

Suppose that $\Omega \subset \mathbf{C}$ is a compact set containing more than one point. Further suppose that its complement $\Omega^C := \hat{\mathbf{C}} \setminus \Omega$ is simply connected in the extended plane $\hat{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$. Let $\mathbf{E} := \{z \, : \, |z| \leq 1\}$ denote the closed unit disk with boundary $\partial \mathbf{E}$. Then there exists a unique conformal map $\Psi : \mathbf{E}^C \to \Omega^C$, which satisfies $\Psi(\infty) = \infty$ and $\Psi'(\infty) =: t > 0$. Ψ and its inverse can be expanded as

$$\Psi(w) = t\left(w + \alpha_0 + \frac{\alpha_1}{w} + \ldots\right), \quad \text{and} \quad \Psi^{-1}(z) = \frac{z}{t} + \beta_0 + \frac{\beta_1}{z} + \ldots . \qquad (1)$$

The part in the expansion of $[\Psi^{-1}(z)]^n$, $n \geq 0$, which contains only the nonnegative powers of z is called the nth Faber polynomial $F_n(z)$ for Ω. Obviously, $F_0(z) \equiv 1$ and $F_n(z)$ is of exact degree n with leading term $(z/t)^n$. The Faber polynomials have the determinant representation $t^n F_n(z) = \det(zI - H_n)$, where

$$H_n = \begin{bmatrix} a_0 & 1 & 0 & \cdots & 0 \\ 2a_1 & a_0 & 1 & \ddots & \vdots \\ 3a_2 & a_1 & a_0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ na_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{bmatrix}, \quad a_k := t^{k+1}\alpha_k, \; k \geq 0 \qquad (2)$$

(see e.g. [4]). Thus, the zeros of $F_n(z)$, $n \geq 1$, are the eigenvalues of the matrix H_n. For more information on Faber polynomials see e.g. [1, 8].

The zeros of Faber polynomials have been the subject of active research in the last decades [2, 3, 4, 5, 7, 10]. Here we are concerned with a conjecture on their location: Kövari and Pommerenke [7] show that the zeros of the Faber polynomials for a convex set which has nonempty interior are located in the interior of that set. It was later conjectured that the zeros of the Faber polynomials for any set are always in the convex hull of that set (cf. [10]). This conjecture is shown to be false by examples of Goodman [3].

Goodman's example sets are circular arcs and thus have empty interiors and nonanalytic boundaries. In this paper we use results of [6] to present non-convex sets with nonempty interiors and analytic boundaries for which the above mentioned conjecture is false. So far we are not aware of such examples in the literature.

2. The 'bratwurst' shape sets revisited

The examples we are going to construct are derived from the class of non-convex 'bratwurst' shape sets $\mathbf{B}(\lambda_m, \phi)$ introduced in [6].

DEFINITION 1 *Suppose that* $\lambda_m \in \partial\mathbf{E}$ *and* $\phi \in (0, 2\pi)$ *are given. Define*

$$\epsilon_{max} := \tan\frac{\phi}{4}\left(1 + \tan\frac{\phi}{8}\right), \quad P := \tan\frac{\phi}{4} + \left(\cos\frac{\phi}{4}\right)^{-1}, \tag{3}$$

and, for each $\epsilon \in [0, \epsilon_{max})$*,*

$$\Psi_\epsilon(w) := t_\epsilon \frac{(z - \lambda_m N_\epsilon)(z - \lambda_m M_\epsilon)}{z - \lambda_m S_\epsilon}, \tag{4}$$

where

$$N_\epsilon := \frac{1}{2}\left(\frac{P}{1+\epsilon} + \frac{1+\epsilon}{P}\right), \quad M_\epsilon := \frac{(1+\epsilon)^2 - 1}{2\tan\frac{\phi}{4}(1+\epsilon)}, \tag{5}$$

$$t_\epsilon := \frac{1}{N_\epsilon - M_\epsilon}, \quad \textit{and} \;\; S_\epsilon := \frac{1 - N_\epsilon M_\epsilon}{N_\epsilon - M_\epsilon}. \tag{6}$$

Then the class of sets $\mathbf{B}(\lambda_m, \phi)$ *is defined by*

$$\mathbf{B}(\lambda_m, \phi) := \left\{\Omega_\epsilon \;:\; \Omega_\epsilon = \left(\Psi_\epsilon(\mathbf{E}^C)\right)^C, \; \epsilon \in [0, \epsilon_{max})\right\}. \tag{7}$$

Examples of sets $\Omega_\epsilon \in \mathbf{B}(-1, \phi)$ for different ϵ and ϕ are shown in Figures 1 and 3. We recall some results from [6]:

The elements of $\mathbf{B}(\lambda_m, \phi)$ are compact and simply connected in $\mathbf{C}$. Therefore the Faber polynomials for each $\Omega_\epsilon \in \mathbf{B}(\lambda_m, \phi)$ are well-defined. For $\epsilon \in (0, \epsilon_{max})$, the boundary $\partial\Omega_\epsilon = \Psi_\epsilon(\partial\mathbf{E})$ is an analytic Jordan curve. Each $\Omega_\epsilon \in \mathbf{B}(\lambda_m, \phi)$ contains the circular arc $\left\{\lambda_m e^{i\beta} \;:\; \phi/2 \le \beta \le 2\pi - \phi/2\right\}$, and $\{0, \lambda_m\} \not\subset \Omega_\epsilon$. For $\epsilon \in (0, \epsilon_{max})$, the set Ω_ϵ has a nonempty interior (in particular the arc lies in the interior of Ω_ϵ), while Ω_0 is identical to the arc. Using results of [6, Section 4], the relation

$$0 \le M_\epsilon < S_\epsilon < 1 < N_\epsilon \;\text{ for all }\; \epsilon \in [0, \epsilon_{max}) \tag{8}$$

can be derived by elementary means.

To construct our examples, we need the following proposition, which reveals further properties of the elements of $\mathbf{B}(\lambda_m, \phi)$.

PROPOSITION 2 *Suppose that $\lambda_m \in \partial\mathbf{E}$ and $\phi \in (0, 2\pi)$ are given. Define ϵ_{max}, $\Psi_\epsilon(w)$ and M_ϵ as in Definition 1. Then for each $\epsilon \in [0, \epsilon_{max})$,*

$$|\Psi_\epsilon(\lambda_m)| = \frac{1-M_\epsilon}{1+M_\epsilon} \leq |\Psi_\epsilon(w)| \leq \frac{1+M_\epsilon}{1-M_\epsilon} = |\Psi_\epsilon(-\lambda_m)|, \text{ for all } w \in \partial\mathbf{E}. \quad (9)$$

Hence, $\Omega_\epsilon \in \mathbf{B}(\lambda_m, \phi)$ is contained in an annulus with inner radius $(1-M_\epsilon)/(1+M_\epsilon)$ and outer radius $(1+M_\epsilon)/(1-M_\epsilon)$.

Proof. Without loss of generality we can assume that $\lambda_m = 1$. We denote $f_X := f(X_\epsilon, \varphi) := 1 + X_\epsilon^2 - 2X_\epsilon \cos\varphi$, and compute $\left|\Psi_\epsilon\left(e^{i\varphi}\right)\right|^2 = t_\epsilon^2 f_N f_M / f_S$. Since $f_S = t_\epsilon^2 \left((1+M_\epsilon)^2 f_N + 2M_\epsilon(1+N_\epsilon)^2(\cos\varphi - 1)\right)$, we get

$$\left|\Psi_\epsilon\left(e^{i\varphi}\right)\right|^2 = \frac{f_N f_M}{(1+M_\epsilon)^2 f_N + 2M_\epsilon(1+N_\epsilon)^2(\cos\varphi - 1)} =: g(\varphi).$$

Obviously, $g(\varphi)$ is periodic with respect to 2π and differentiable for all $\varphi \in \mathbf{R}$. Note that (9) is satisfied for $\epsilon = 0$, since this implies $M_\epsilon = 0$ and $g(\varphi) \equiv 1$ for all φ. It remains to show the assertion for $\epsilon \in (0, \epsilon_{max})$.

The two equalities in (9) follow directly from evaluating $g(0)$ and $g(\pi)$. It therefore suffices to show that $\varphi = 0, \pi, 2\pi$ are the only zeros of $g'(\varphi)$ in $[0, 2\pi]$. $g'(\varphi) = 0$ is equivalent to $2M_\epsilon h(\varphi)\sin\varphi = 0$, where

$$h(\varphi) := (1+M_\epsilon)^2 f_N^2 + (1+N_\epsilon)^2 f_N f_M + 2(1+N_\epsilon)^2[N_\epsilon f_M + M_\epsilon f_N](\cos\varphi - 1).$$

Hence $g'(\varphi) = 0$ if and only if either $\varphi = k\pi$, $k \in \mathbf{Z}$, or $h(\varphi) = 0$. To complete the proof we show that $h(\varphi) > 0$ for all $\varphi \in [0, 2\pi]$.

We first compute

$$h(2k\pi) = (1+M_\epsilon)^2(1-N_\epsilon)^4 + (1+N_\epsilon^2)(1-N_\epsilon)^2(1-M_\epsilon)^2$$

and

$$h((2k+1)\pi) = 2\left((1+N_\epsilon+4N_\epsilon^2+N_\epsilon^3+N_\epsilon^4)(1+M_\epsilon^2) - 2N_\epsilon M_\epsilon(1-N_\epsilon)^2\right),$$

for $k \in \mathbf{Z}$. $h(2k\pi) > 0$ and $h((2k+1)\pi) > 0$ are both obvious from (8). Next,

$$h'(\varphi) = 4N_\epsilon \sin\varphi \left((1+N_\epsilon^2)(1+M_\epsilon^2) - 2\left(N_\epsilon(1+M_\epsilon)^2 - M_\epsilon(1+N_\epsilon^2)\right)\cos\varphi\right)$$

shows that $h'(\varphi) = 0$ if either $\varphi = k\pi$, $k \in \mathbf{Z}$, or

$$\cos\varphi = \frac{(1+N_\epsilon^2)(1+M_\epsilon^2)}{2\left(N_\epsilon(1+M_\epsilon)^2 - M_\epsilon(1+N_\epsilon^2)\right)}.$$

Since $S_\epsilon = (1-N_\epsilon M_\epsilon)/(N_\epsilon - M_\epsilon) > 0$ (cf. (8)),

$$N_\epsilon(1+M_\epsilon)^2 - M_\epsilon(1+N_\epsilon^2) = (1-N_\epsilon M_\epsilon)(N_\epsilon - M_\epsilon) + 2N_\epsilon M_\epsilon > 0.$$

From

$$\begin{aligned}(1+N_\epsilon^2)(1+M_\epsilon^2) - 2\left(N_\epsilon(1+M_\epsilon)^2 - M_\epsilon(1+N_\epsilon^2)\right) &= \\ \left((1-N_\epsilon M_\epsilon) - (N_\epsilon - M_\epsilon)\right)^2 &> 0,\end{aligned}$$

it then follows that $h'(\varphi) \neq 0$ for $\varphi \neq k\pi$, $k \in \mathbf{Z}$. Thus $h(\varphi) > 0$ for all $\varphi \in \mathbf{R}$, which completes the proof.

3. The examples

In this section we show that two of the three zeros of $F_3(z)$ for certain sets $\Omega_\epsilon \in \mathbf{B}(-1, \phi)$ are outside $co(\Omega_\epsilon)$, the convex hull of Ω_ϵ. Considering only the case $\lambda_m = -1$ is no restriction: It is easy to see that if z_j, $1 \leq j \leq 3$, are the zeros of $F_3(z)$ for $\Omega_\epsilon \in \mathbf{B}(-1, \phi)$, then $-\lambda_m z_j$, $1 \leq j \leq 3$, are the zeros of $F_3(z)$ for $\Omega_\epsilon \in \mathbf{B}(\lambda_m, \phi)$. Hence, if for fixed ϵ and ϕ, zeros of $F_3(z)$ for $\Omega_\epsilon \in \mathbf{B}(-1, \phi)$ are outside $co(\Omega_\epsilon)$, the same holds for all $F_3(z)$ for $\Omega_\epsilon \in \mathbf{B}(\lambda_m, \phi)$.

Expanding the function $\Psi_\epsilon(w)$ as in (1) and using the determinant relation shows that the Faber polynomial $F_3(z)$ of each $\Omega_\epsilon \in \mathbf{B}(-1, \phi)$ satisfies

$$t_\epsilon^3 F_3(z) = (z - a_0)^3 - 3(z - a_0)a_1 - 3a_2,$$

where $a_0 = t_\epsilon(N_\epsilon + M_\epsilon - S_\epsilon)$, $a_1 = t_\epsilon^2(S_\epsilon - N_\epsilon)(S_\epsilon - M_\epsilon)$ and $a_2 = -t_\epsilon S_\epsilon a_1$. Since $a_1 < 0$ (cf. (8)), it is easy to see that $F_3(z)$ has exactly one real zero z_1 and two complex conjugate zeros z_2 and z_3.

Note that Goodman's example sets in [3] are elements of the class $\mathbf{B}(\lambda_m, \phi)$. In our notation, Goodman shows that when $\phi > 4 \arccos 3^{-1/2} \approx 3.8213$, then z_2 and z_3, the two complex conjugate zeros of $F_3(z)$ for the circular arc $\Omega_0 \in \mathbf{B}(-1, \phi)$, have moduli larger than one, and therefore $\{z_2, z_3\} \not\subset co(\Omega_\epsilon)$. We next point out that the eigenvalues of the matrix H_3 in (2), i.e. the zeros of $F_3(z)$, depend continuously on the parameters ϵ and ϕ. Thus, for each $\phi_1 \in (4 \arccos 3^{-1/2}, 2\pi)$, there exist some $\epsilon_1 > 0$, so that z_2 and z_3 of $F_3(z)$ for each set $\Omega_\epsilon \in \mathbf{B}(-1, \phi_1)$, $0 \leq \epsilon < \epsilon_1$, are outside $co(\Omega_\epsilon)$. By construction, each set Ω_ϵ with positive ϵ has a nonempty interior and an analytic boundary.

To check if a certain choice of parameters leads to zeros outside $co(\Omega_\epsilon)$, we computed the eigenvalues of H_3 using MATLAB 5.3 [9], and compared their moduli with the upper bound given in Proposition 2. We remark that $|z_2| > (1 + M_\epsilon)/(1 - M_\epsilon)$ is sufficient but not necessary for $\{z_2, z_3\} \not\subset co(\Omega_\epsilon)$. Here are two examples for which zeros are outside $co(\Omega_\epsilon)$:

ϵ	ϕ	z_1	z_2, z_3	$\lvert z_2\rvert = \lvert z_3\rvert$	$\frac{1+M_\epsilon}{1-M_\epsilon}$
0.001	$3\pi/2$	0.99310	$0.78402 \pm 0.62416i$	1.00213	1.00083
0.001	$5\pi/4$	0.96751	$0.55364 \pm 0.83508i$	1.00194	1.00134

A picture of the first example is shown in Figure 1. The close-up in Figure 2 reveals the distance between z_2 and $\partial\Omega_{0.001}$. In general we noted that zeros are outside $co(\Omega_\epsilon)$ only for very small ϵ. A typical example for a larger ϵ is shown in Figure 3.

We were unsuccessful in proving analytic expressions for the ranges of ϵ and ϕ for which $\{z_2, z_3\} \not\subset co(\Omega_\epsilon)$. However, we derived the following lower bound on $|z_2|$:

$$|z_2| > \sqrt{-3a_1 + a_0^2 + t_\epsilon S_\epsilon \gamma a_0 + (t_\epsilon S_\epsilon \gamma)^2},$$

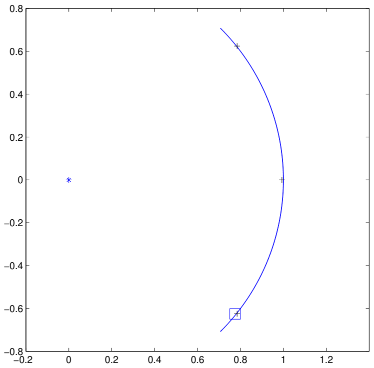

Figure 1: $\Omega_{0.001} \in \mathbf{B}(-1, 3\pi/2)$, zeros of $F_3(z)$ (pluses), and the origin (star).

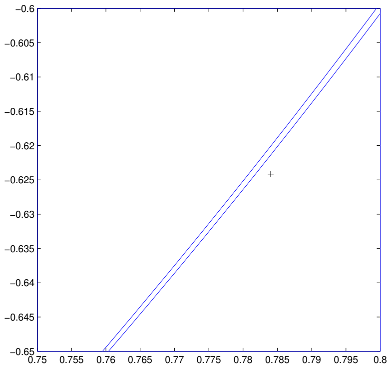

Figure 2: Close-up of the small box in Figure 1.

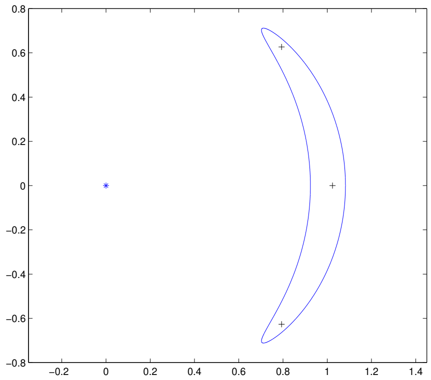

Figure 3: $\Omega_{0.1} \in \mathbf{B}(-1, 3\pi/2)$, zeros of $F_3(z)$ (pluses), and the origin (star).

where $\gamma := 1 - S_\epsilon^2/(3N_\epsilon M_\epsilon - 3S_\epsilon(N_\epsilon + M_\epsilon))$. Our bound is usually very sharp. For the above two examples the bound is $|z_2| > 1.00211$ and $|z_2| > 1.00186$, respectively. In both cases the bound detects that $|z_2| > (1 + M_\epsilon)/(1 - M_\epsilon)$, and thus shows $\{z_2, z_3\} \not\subset co(\Omega_\epsilon)$.

Acknowledgments

This work was supported by the SFB 343 at the University of Bielefeld, Germany. Thanks to Prof. Ludwig Elsner, Dr. Tino Koch and Prof. Richard S. Varga for helpful comments.

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