Coxeter Decompositions of Hyperbolic Pyramids and Triangular Prisms.

A.Felikson

Introduction

Let P be a polyhedron in a hyperbolic space H^3.

Definition 1. *A polyhedron P is a* **Coxeter polyhedron** *if all dihedral angles of P are the integer parts of π.*

Definition 2. *A polyhedron P admits a* **Coxeter decomposition** *if P can be tiled by N Coxeter polyhedra F_i,...,F_N ($1 < N < \infty$), such that any two tiles F_i and F_j with a common face are symmetric with respect to this face.*

In this paper we classify all Coxeter decompositions of bounded convex pyramids and triangular prisms in hyperbolic space $\mathbb{H}^3$.

Basic definitions

All polyhedra F_i in a Coxeter decomposition are obviously congruent. We call these polyhedra **fundamental** and denote by F. We say that P **admits a Coxeter decomposition** and that P is **quasi-Coxeter** polyhedron.

A plane containing a face of a fundamental polyhedron will be called a **mirror** if it contains no face of P.

Remark. A Coxeter decomposition of any polyhedron can be prolonged to the decomposition of the whole hyperbolic space (to define a Coxeter decomposition of the space take definition 2 without the condition $N < \infty$).

Any polyhedron in this paper is bounded by the mirrors of a Coxeter decomposition of some polyhedron or the space.

Definition 3. *Fix a Coxeter decomposition of a polyhedron P.*

- *A* **vertex** *of P is called* **fundamental** *if it does not belong to any mirror.*
- *A* **dihedral angle** *of P made up by faces α and β is called* **fundamental** *if no mirror contains $\alpha \cap \beta$. In this case the* **edge** *$\alpha \cap \beta$ of P is called* **fundamental** *too.*
- *A* **face** *α of P is called* **fundamental** *if no mirror intersects the inner part of α.*

Lemma 1. *(Obvious).*

(1) If P is a quasi-Coxeter polyhedron, then any polyhedron P' inside P is quasi-Coxeter too.

(2) If all dihedral angles of a quasi-Coxeter polyhedron P are fundamental, then P is a Coxeter polyhedron.

(3) Let α be a mirror of a Coxeter decomposition of $\mathbb{H}^3$. Then α is tiled by faces of fundamental polyhedron (in general, this tiling is not a Coxeter decomposition).

The following lemma was proved in [3]:

Lemma 2. *If $ABCD$ is bounded hyperbolic quasi-Coxeter tetrahedron, then $ABCD$ is decomposed as shown in one of the figures 1(a)–1(h).*

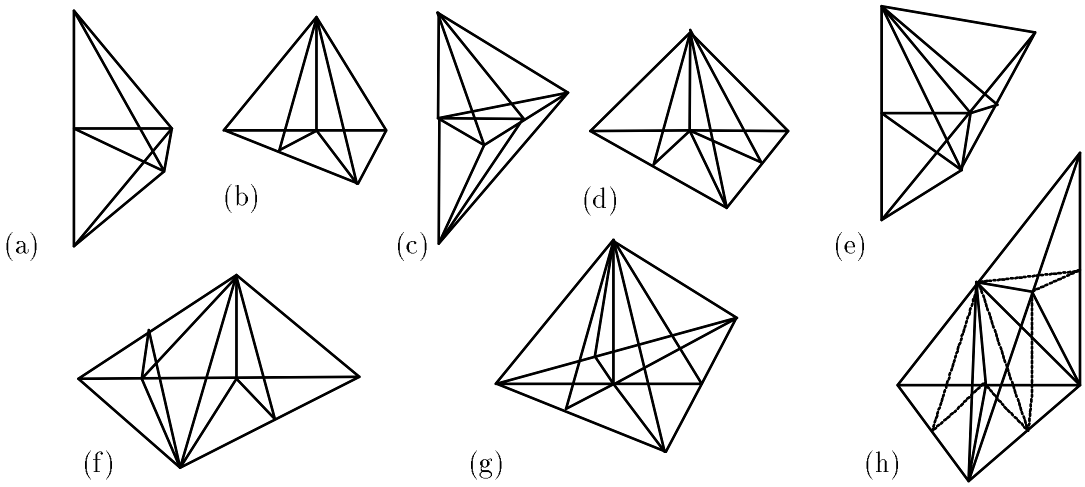


Figure 1: Coxeter decompositions of bounded tetrahedra.

1 Fundamental polyhedron

From now on a bounded quasi-Coxeter polyhedron P is a pyramid or a triangular prism in $I\!H^3$.

Notation. Denote by $\alpha \cap \beta$ an intersection of sets α and β in the *inner* part of $\mathbf{H}^3$.
If α is a face of a polyhedron, denote by $\overline{\alpha}$ a plane containing α.

Lemma 3. *Let F be a Coxeter polyhedron.*
Then either F is a tetrahedron or F has two faces α and β such that $\overline{\alpha} \cap \overline{\beta} = \emptyset$.

Proof. It is well known that for any polyhedron without obtuse angles

$$\alpha \cap \beta = \emptyset \quad \Rightarrow \quad \overline{\alpha} \cap \overline{\beta} = \emptyset.$$

Coxeter polyhedra have no obtuse angles. Thus, it is enough to prove that F has faces α and β such that $\alpha \cap \beta = \emptyset$.

Suppose that $\alpha \cap \beta \neq \emptyset$ for any faces α and β of F. Let A be an arbitrary vertex of F. Let α_1,...,α_k be a complete set of faces of F such that $A \in \alpha_i$. Let β be any face of P such that $A \notin \beta$. By assumption, $\alpha_i \cap \beta \neq \emptyset \quad \forall i = 1, ..., k$. Therefore, F has no faces except α_i and β, i.e. F is a pyramid.

Suppose that the vertex A is not ideal vertex. It is well known that any non-ideal vertex of a Coxeter polyhedron is trivalent. Thus, the pyramid F is a tetrahedron and everything is proved.

Suppose now that A is ideal vertex and the pyramid F is not a tetrahedron. Take two faces (say, α_1 and α_3) having no common edges. Since A is not an inner point of $\mathbf{H}^3$, $\alpha_1 \cap \alpha_3 = \emptyset$.

□

Lemma 4. *If P is a pyramid then F is a tetrahedron.*

Proof. Suppose that F is not a tetrahedron. By lemma 3, F has two faces α and β such that $\overline{\alpha} \cap \overline{\beta} = \emptyset$. Take any fundamental polyhedron F_0 in P; let α_0 and β_0 be its disjoint faces. Consider a sequence of fundamental polyhedra $F_i \in P$, $i \in Z\!\!\!Z$, such that $\alpha_i = \alpha_{i+1}$, if i is odd, and $\beta_i = \beta_{i+1}$, if i is even (see figure 2).

The sequence is finite, since P contains only finitely many fundamental polyhedra. Let F_k and F_s be the endpoints of the sequence. Then α_k or β_k belongs to some face γ of P. Similarly, α_s or β_s belongs to some face δ of P. Obviously, $\overline{\alpha}_i \cap \overline{\beta}_j = \emptyset \quad \forall i, j$. But γ intersects δ, since P is a pyramid. The contradiction shows that F is a tetrahedron.

□

Lemma 5. *Let α be a mirror of a Coxeter decomposition of the triangular prism P. Then α goes as it is shown in one of the figures 3.1–3.16.*

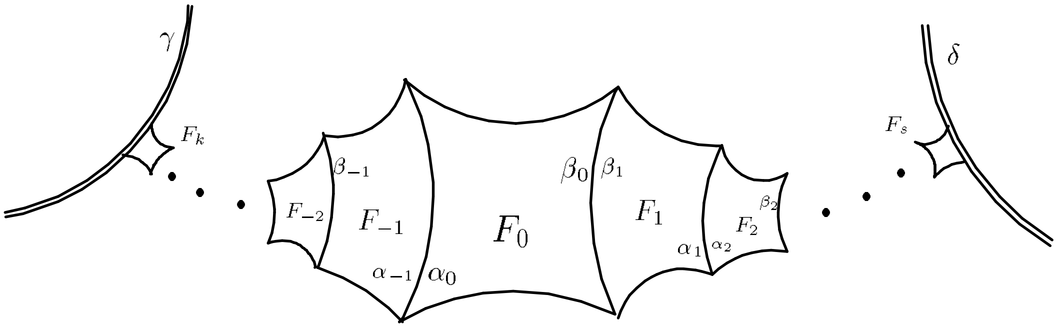


Figure 2: A sequence of the fundamental polyhedra.

Proof. It is easy to see that there are no other possibilities. □

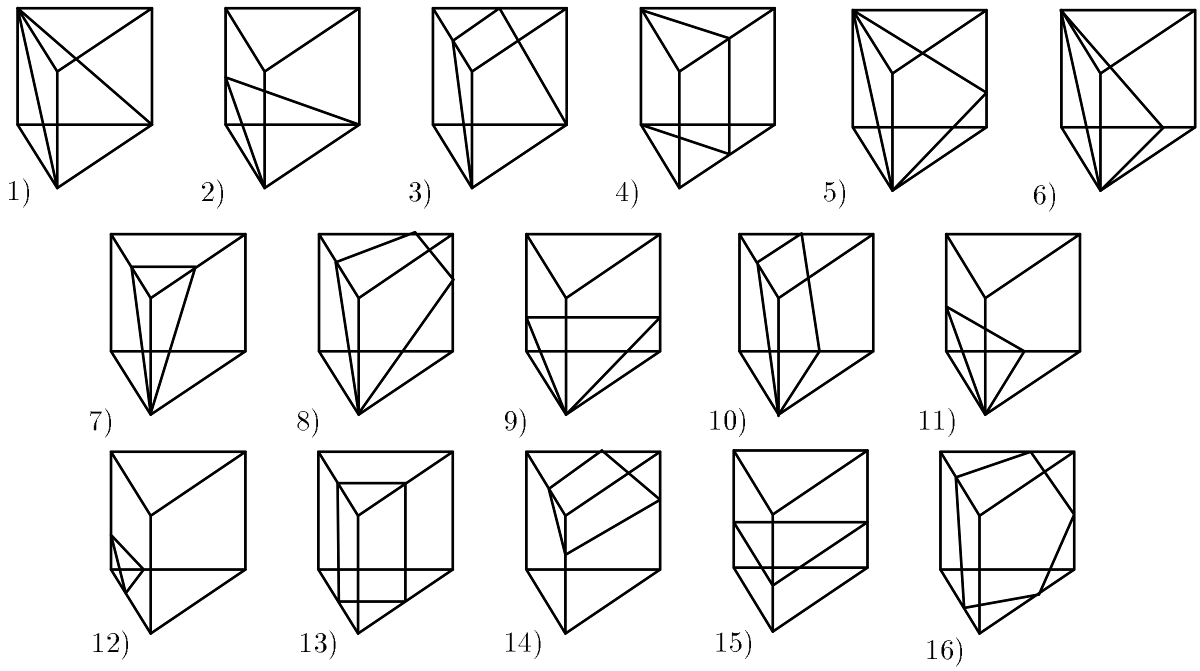


Figure 3: Mirrors in the prism.

Definition 4. *We say that a mirror is* **pentagonal** *if it goes as shown in figure 3.16.*

Definition 5. *We say that a triangular quasi-Coxeter prism P is* **minimal** *if any prism P' inside P is fundamental.*

Lemma 6. *Let P be a minimal prism and F be a fundamental polyhedron. Suppose that F is neither tetrahedron nor triangular prism. Then any mirror is pentagonal and any dihedral angle of P is fundamental.*

Proof. Suppose that the decomposition contains a tetrahedron or a pyramid. Then by lemma 4 F is a tetrahedron. Hence P contains neither pyramid nor tetrahedron. Since P is minimal, P contains no smaller triangular prisms. Therefore any mirror is pentagonal (this follows from lemma 5). Clearly, all dihedral angles of P are fundamental. □

Lemma 7. *If P is a triangular prism then F is either tetrahedron or triangular prism.*

Proof. It is sufficient to prove the lemma for a minimal prism P. Suppose that F is neither tetrahedron nor triangular prism. Then by previous lemma any mirror is pentagonal and any dihedral angle of P is fundamental.

Let $ABCDE$ be any pentagonal mirror (see figure 4). This mirror separates the points M_1, N_1, N_3 from the points M_2, M_3, N_2.

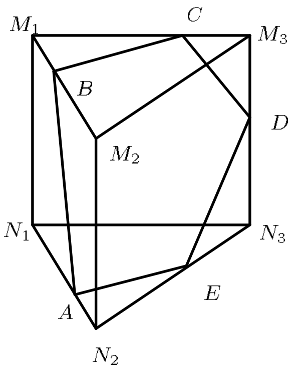


Figure 4: A pentagonal mirror decomposes a prism into two combinatorially equivalent polyhedra.

We say that a polyhedron W is **special** if it contains the points M_2, M_3, N_2 and it can be cut out of P by a single pentagonal mirror. Let W be a minimal special polyhedron (i.e. no polyhedron inside W is special). The minimal polyhedron exists, since the decomposition contains finitely many mirrors. Suppose that W is $M_2M_3N_2ABCDE$ (see figure 4).

Suppose that W has a non-fundamental dihedral angle. By lemma 6 this angle is formed by $ABCDE$ and some face of P. Let Π be a mirror decomposing this angle. Clearly, Π does not separate the points M_1, N_1 and N_3 one from another. Since Π is pentagonal, it separates all the points above from the points M_2, M_3, N_2. Therefore Π cut out of P a special polyhedron contained in W. This contradicts to the minimal property of W.

Therefore any dihedral angle of W is fundamental. By lemma 1 W is a Coxeter polyhedron. It is well known property of the Coxeter polyhedra that the prolongations of two disjoint faces cannot intersect each other. This property is broken in the polyhedron W (consider the faces AEN_2 and CDM_3 whose prolongations have a common line N_1N_3). The contradiction shows that F is either tetrahedron or triangular prism.

□

2 The decompositions of the pyramids.

This section describes how to classify all the Coxeter decompositions of the bounded hyperbolic pyramids.

At first suppose that a pyramid P has only five vertices $OA_1A_2A_3A_4$ (where $A_1A_2A_3A_4$ is a base). By lemma 4 F is a tetrahedron. Suppose that any edge OA_i is fundamental. Consider a small sphere centered in O. Its intersection with the boundary of P is a spherical quadrilateral q. Any angle of q is fundamental. This contradicts to the fact that the sum of the angles for a spherical quadrilateral should be greater then 2π.

Hence, we can assume that there is a mirror through OA_1. This mirror decomposes the pyramid into two smaller pyramids. One of the smaller pyramids is a tetrahedron and another is either tetrahedron or small quadrangular pyramid. Thus, any minimal quadrangular pyramid consists of two tetrahedra (not necessary fundamental). The decompositions of these tetrahedra are known from lemma 2. Therefore we can find all the decompositions of the minimal quadrangular pyramids. Using the decompositions of the minimal pyramids we can find the decompositions of the greater pyramids too. So, it is possible to find all the decompositions of quadrangular pyramids.

Analogically, it is possible to classify the decompositions of the pyramids with a bigger number of vertices. Any pyramid $OA_1...A_n$ has a decomposed edge OA_i. So, the pyramid is decomposed into two smaller pyramids whose possible decomposition we should enumerate at the previous steps.

Being realized by a computer, this procedure leads to a big list of the quadrangular pyramids, several pentagonal pyramids and exactly one hexagonal pyramid. See the Table 2 for the result.

3 Decompositions of the triangular prisms whose fundamental polyhedron is a triangular prism

In this section both P and F are triangular prisms.
For any prism we say that a triangular face is a **base** and a quadrilateral face is a **side**.

Lemma 8. *No base of F belongs to a side of P.*

Proof. Suppose that a side α of P contains a base of fundamental prism F_1. Consider a sequence of the fundamental prisms F_1, F_2,...,F_n,..., such that $F_i \in P$ and F_i and F_{i+1} has a common base. This chain is finite, since P contains finitely many fundamental prisms. The prolongations of the two bases of F_1 have no common points, since F_1 is a Coxeter prism. So, the prolongations of the bases of two different prisms F_i have no common points. Therefore the prisms F_1, F_2,...,F_n, cannot make a cycle and the sequence has an endpoint F_n. One of the bases of F_n belongs to some face β of P (otherwise the sequence has a prolongation). The face α cannot intersect the face β, since these faces are prolongations of the bases of F_i. This contradicts to the fact that a side α intersects each face of P.

□

Lemma 9. *No side of F belongs to a base of P.*

Proof. Denote by α and β the bases of P. Suppose that a side α_0 of a fundamental prism F_0 belongs to α. As in the previous lemma consider a sequence of fundamental prisms F_{-k},...,F_0,...F_l, such that $F_i \in P$ and F_i and F_{i+1} have a common base. Let α_{-k} and β_l be the bases of the endpoints F_{-k} and F_l. By lemma 8 the bases α_{-k} and β_l belong to two different bases of P. If α_{-k} belongs to the base α then the prolongations of α_{-k} and α_0 have a common point (see figure 5). This is impossible.

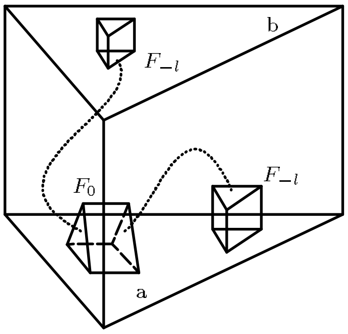


Figure 5: A sequence of fundamental prisms.

□

Lemma 10. *Let α be a base of P and α_1 and α_2 be the bases of F. Suppose that α is non-fundamental. Then a tiling of α by the faces of F is a Coxeter decomposition. A fundamental triangle of this decomposition is a triangle α_1 or α_2.*

Proof. By previous lemma all the tiles are triangles. The adjacent tiles of the tiling are the faces of F having a common edge. Thus, if one of the tiles is a base α_1 then the other tiles are the copies of this base. Clearly, this tiling is a Coxeter decomposition.

□

By lemma 8 any side α of P is tiled by sides of F. This tiling is not necessary Coxeter decomposition, but lemmas 11 and 12 show that this tiling is very simple.

Lemma 11. *Let α be a side of P. Then the tiling of α (see figure 6a) satisfies the following properties:*

(1) any vertex of the face α belongs to a unique tile;

(2) any point at the edge of the quadrilateral α belongs to one or two tiles;

(3) any point inside α belongs to one, two or four tiles.

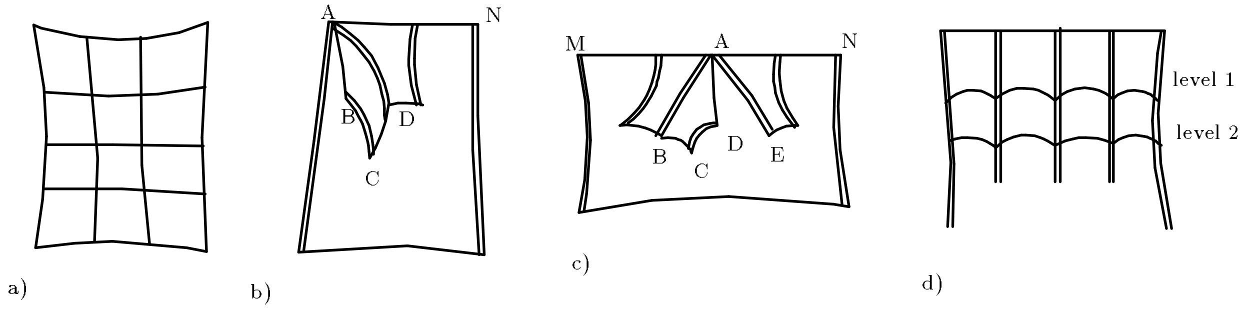


Figure 6: Any side is tiled as shown in figure (a); (b) and (c) are impossible.

Proof. We say that an edge AB of a tile is **horizontal** if AB belongs to a base of a fundamental prism; otherwise we say that AB is **vertical**. Let $ABCD$ be any tile whose vertical edges are AB and CD. Consider a sequence of the tiles such that this sequence contains $ABCD$ and any two neighboring tiles have a common horizontal edge. We say that this sequence is **vertical**. Analogically construct a **horizontal** sequence where the neighboring tiles have a common vertical edge. Evidently, the vertical sequence ends on two different horizontal edges of α and a horizontal sequence ends on two different vertical edges. Let us show that this condition is broken if the lemma is false.

(1) Let A be a vertex of the face α. Suppose that A belongs to more then one tile (see figure 6b). Consider a vertical sequence through the tile $ABCD$. No pair of tiles from this sequence can be separated from each other by the line AD. Therefore this sequence cannot reach a horizontal edge AN.

(2) Let A be an inner point of a horizontal edge MN. Suppose that A belongs to more then two tiles. Then one of these tiles has no edges on MN (see figure 6c). At least two lines through A are vertical edges of some tiles. Consider a vertical sequence through the tile $ABCD$. It is evident that the angle $\angle BAE$ contains any tile from this sequence. Thus, the sequence cannot reach a horizontal edge MN.

   The same reason is good for the points on the vertical edges.

(3) Let A be an inner point of the tile α. Clearly, A cannot be incident with exactly three tiles (see the third part of lemma 1). Suppose that A is a vertex of five or more tiles.

   Let α_1 be any tile and α_1,...,α_k be an upper part of vertical sequence (i.e. α_k is upper endpoint of the sequence). We say that α_1 is a tile of level k. We say that a point A is a point of level k if A is an inner point of a tile of level k. If A belongs to several tiles we define the level of A as a maximum.

   The part (2) of this proof establishes the lemma for the points of level 1. The same reason as in part (2) shows that if lemma is true for the tiles of level k then it is true for the tiles of level $k+1$ (see figure 6d). Therefore the lemma is true for any tile.

□

Lemma 12. *Any side of P is decomposed as shown in figure 7.*

Proof. Let α be a side of P and b_1, b_2, b_3 be the sides of F. Since F is a Coxeter polyhedron, no angle of b_i is obtuse. Therefore any horizontal line is perpendicular to any vertical line (where the word "line" means an intersection of α with some mirror). By the same reason the lines are perpendicular to the edges of α. At most

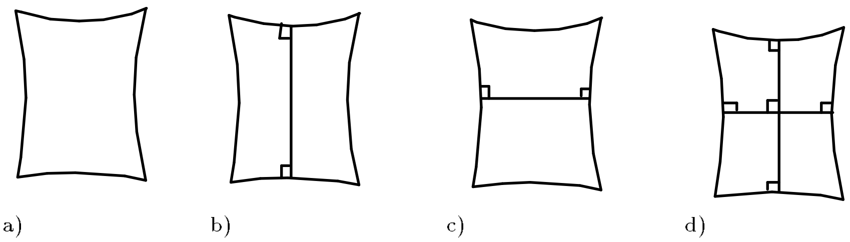


Figure 7: Four possibilities for a tiling of a side.

one line may be perpendicular to a pair of opposite edges of a quadrilateral in $\mathbb{H}^2$. Thus, there is at most one vertical line and at most one horizontal line.

□

Lemma 13. *The bases of P are tiled as shown in figure 8.*

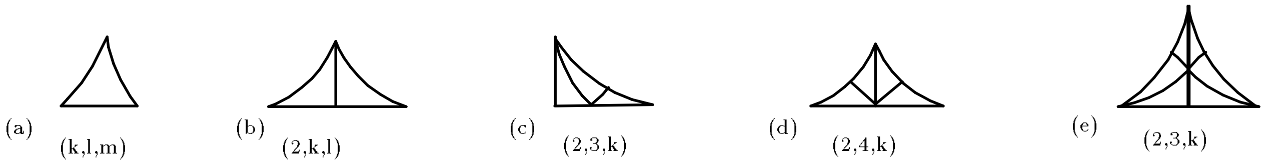


Figure 8: Five possibilities for the tiling of a base
(see the angles of the fundamental triangles under the decompositions).

Proof. Let α be a base of P. By lemma 10 α is either fundamental or quasi-Coxeter. The Coxeter decompositions of the triangles are classified in [2] (see the list in figure 9). By lemma 12 no edge of the triangle α may be decomposed into more then two parts. The decompositions satisfying this condition are listed in figure 8.

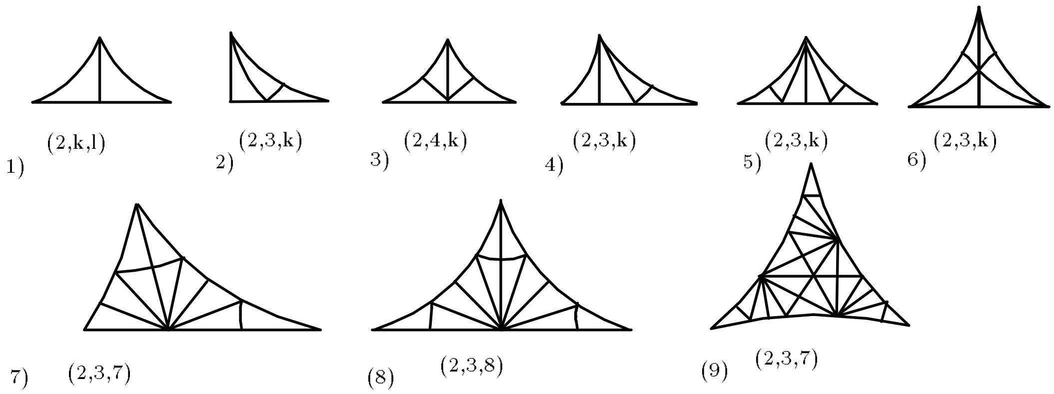


Figure 9: Coxeter decompositions of the triangles.

□

Theorem 1. *Let P be a quasi-Coxeter triangular prism such that F is a triangular prism. Then the decomposition is one of the decompositions listed in Table 1.*

Proof. It follows from lemma 11 that two bases of P are decomposed in the same way. By lemma 13 there are five possibilities for the decompositions of the bases. Consider two cases.

(1) Suppose that the bases are fundamental. Then the sides are either fundamental or decomposed like in figure 7(c). Obviously all the sides of P are decomposed in the same way. This leads to a decomposition shown in the left of Table 1.

(2) Suppose that the bases are decomposed. Each decomposition of the base corresponds to two possible decompositions of the sides — with horizontal lines or without them. Thus, each decomposition of the base corresponds to two decompositions of P. Some of the dihedral angles of F are uniquely determined by the combinatorial structure of the decomposition. We need only to check if we can define the rest dihedral angles of F to satisfy the Andreev theorem (see [1]). If the bases are decomposed as in figure 8(b) or 8(d) then it is possible to satisfy the Andreev theorem. Otherwise, it is impossible.

□

4 Decompositions of the triangular prisms whose fundamental polyhedron is a tetrahedron

In this section P is a triangular prism and F is a tetrahedron.

Any face of P is obviously tiled by the triangles, but in most part of cases this tiling is not a Coxeter decomposition. At first we derive some information about the tilings of the bases.

Definition 6. *An edge AB of a fundamental tetrahedron $ABCD$ is called k-***edge** *if a dihedral angle formed by ABC and ABD equals $\frac{\pi}{k}$.*

Lemma 14. *Let p be a base of P. If no flat angle of p is decomposed then the tiling of p consists of a unique triangle. If p has a flat angle decomposed into three pars then p is bounded by 2-edge, 3-edge and 5-edge.*

Proof. Let f be a face of F and $\bar{f}$ be a plane containing f. Obviously, $\bar{f}$ is tiled by the triangles congruent to the faces of F. To find this tiling consider a face f and the adjacent to f triangles f_1, f_2 and f_3 at the plane $\bar{f}$. These triangles f_i are congruent either to f or to the other faces of F (we obtain the same face if the dihedral angle is equal to $\frac{\pi}{k}$, where k is even, otherwise we obtain another face). Adding face by face we can prolong the tiling. The only problem is how many triangles are incident with a fixed vertex.

To solve this problem suppose that A is a vertex of F incident with k-edge, l-edge and m-edge. Suppose that $\bar{f}$ contains A. Then a decomposition of a small three-dimensional spherical neighborhood of A is similar to a Coxeter decomposition of a sphere with fundamental triangle $(\frac{\pi}{k}, \frac{\pi}{l}, \frac{\pi}{m})$. An intersection of the neighborhood with $\bar{f}$ corresponds to a spherical line in the decomposition of the sphere. So, to find how many rays starting from A belong to $\bar{f}$, it is enough to count a number of the vertices incident with a corresponding spherical line.

Thus, for any face of any Coxeter tetrahedron we can find a tiling of the corresponding plane. In these tilings any triangle with fundamental angles is a face of F. This proves the first part of the lemma. The triangle with an angle decomposed into three parts was found only in one of these tilings. That was a tiling corresponding to a face of the Coxeter tetrahedron having a number 8 in the Table 1. Such a triangle is bounded by 2-edge, 3-edge and 5-edge and tiled as shown in figure 10.

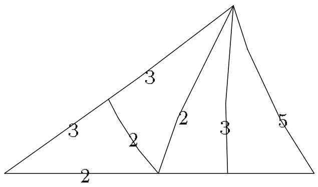


Figure 10: The only triangle whose angle is decomposed into three parts.

□

Lemma 15. *Let A be a non-fundamental vertex of P. Then there exists a non-fundamental edge incident with A.*

Proof. Suppose that any edge incident with A is fundamental (i.e. three dihedral angles incident with A are fundamental). Consider a small sphere s centered in A. The decomposition of P restricted to s is a Coxeter

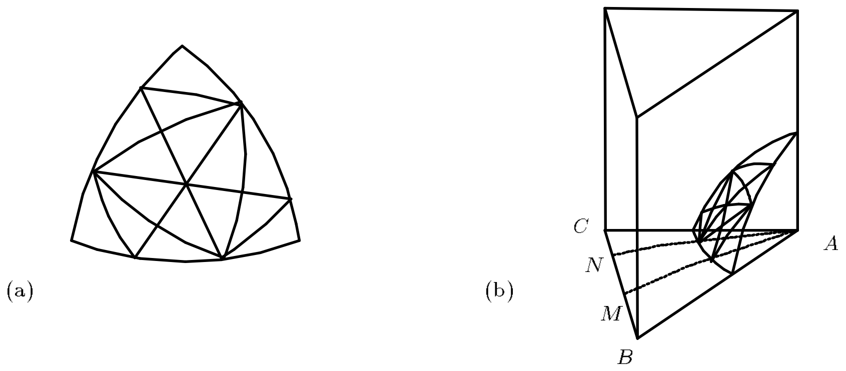


Figure 11: If A is non-fundamental, but AB, AC and AD are fundamental, then the neighborhood of A is decomposed as shown in figure (b).

decomposition of a spherical triangle $p = s \cap P$. Evidently, any angle of p is fundamental. It is easy to check (see [3]) that in this condition p is decomposed as shown in figure 11(a). Any angle of p is a right angle.

Thus the neighborhood of A is decomposed as shown in figure 11(b). Any edge of P incident with A is a 2-edge. Consider a base ABC of the prism P. The angle A of this triangle is decomposed into three parts by the lines AN and AM. It follows from lemma 14 that the sides of ABC should be 2-edge, 3-edge and 5-edge. This contradicts to a fact that AB and AC are 2-edges.

□

Lemma 16. *Let P be a triangular prism admitting a Coxeter decomposition into tetrahedra. Then P has a non-fundamental dihedral angle.*

Proof. Suppose that any dihedral angle of P is fundamental. Then any vertex of P is fundamental and by lemma 14 any base of P is congruent to a single face of F. Consider a fundamental tetrahedron F_0 containing a base α. Since any dihedral angle of P is fundamental, the sides of P are the faces of F_0. This is impossible.

□

The procedure classifying the decompositions

We have already proved that any prism has a non-fundamental dihedral angle. It follows from figure 3(1)–3(4) that any prism consists of a tetrahedron and either smaller prism or quadrangular pyramid. If we know a decompositions of the smaller parts we can find a decomposition of the whole prism.

Recall that a prism P is **minimal** if P is non-fundamental and any prism inside P is fundamental.

Definition 7. *We say that a minimal prism is a* **prism of level 0***. A non-fundamental prism P is a* **prism of level** $k+1$ *if P contains a prism of level k but any prism inside P contains no prism of level k.*

At first we will classify the decompositions of the minimal prisms. Evidently, minimal prism contains no mirrors shown in figures 3(2)–3(4). Thus, it contains a mirror shown in figure 3(1). By lemma 15 any minimal prism has at least two non-fundamental dihedral angles (see figure 12 where A_1A_2 and B_2B_3 are non-fundamental edges). Therefore the prism is decomposed into three tetrahedra. The decompositions of the tetrahedra are classified, so we can classify the decompositions of the minimal prisms.

Suppose that we know the classification of decompositions for the prisms of levels smaller than $k+1$. Show the we can find a classification for the prisms of level $k+1$. Suppose that P has a dihedral angle decomposed as in figure 3(2)–3(4). By the definition any prismatic part is a prism of level smaller than $k+1$. So, we know a list of possible decompositions of the prismatic parts. The decompositions of the tetrahedral parts are known too. Therefore we can classify all possible decompositions of P. Suppose now that P has no mirrors shown in figures 3(2)–3(4). Then P has at least two mirrors shown in figure 3(1) and these mirrors decompose P into three tetrahedra (we use lemma 15 again). Combining the decompositions of these tetrahedra we can find all possible decompositions of P.

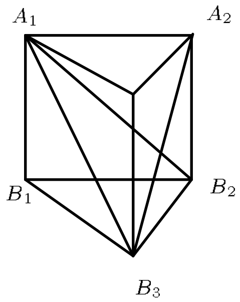


Figure 12: Any minimal prism consists of three (possibly non-fundamental) tetrahedra.

Thus, we can classify the decompositions of the prisms whose level is smaller then any fixed number. In fact there is no prism of level eight; hence, there cannot be prism of greater level. Therefore this algorithm classifies all the decompositions of the triangular prisms. See Table 2 for the classification.

Tables

Table 1 contains a list of the decompositions for the triangular prisms whose fundamental polyhedron is a triangular prism. Table 2 contains all the decompositions of the convex bounded pyramids and triangular prisms whose fundamental polyhedron is a tetrahedron. Here we describe the structure of Table 2.

- The horizontal lines separate the polyhedra with different fundamental tetrahedra. For any fundamental tetrahedron the table contains three columns. The left column contains a list of the decompositions for the bounded tetrahedra (the tetrahedron number 0 is a fundamental one). The right column lists the decompositions of the bounded triangular prisms. The column in the middle contains the bounded pyramids. At first we list the quadrangular pyramids which can be combined from two (possibly non-fundamental) tetrahedra. Then, after a dotted line, we list the rest quadrangular pyramids. After a new dotted line we list the pentagonal pyramids and at last we list the hexagonal pyramids. (There is no Coxeter decomposition of the heptagonal pyramid).

The polyhedra and their decompositions are represented in Table 2 as described below.

- A tetrahedron with the dihedral angles $\frac{k_i\pi}{q_i}$ $i=1,...,6$ is represented by the following diagram: take a Coxeter diagram of the tetrahedron whose dihedral angles equals to $\frac{\pi}{q_i}$ $i=1,...,6$ and subdivide the edges corresponding to the dihedral angles of the size $\frac{k_i\pi}{q_i}$ into k_i parts.
  The faces of the tetrahedron are numbered by the following way: the nodes of the scheme have numbers 0,1,2,3 from left to right. (The numeration will be used below).
- A quadrangular pyramid $OA_1A_2A_3A_4$ with a base $A_1A_2A_3A_4$ is represented by eight dihedral angles $(\widehat{A_1A_2},\widehat{A_2A_3},\widehat{A_3A_4},\widehat{A_4A_1};\widehat{OA_1},\widehat{OA_2},\widehat{OA_3},\widehat{OA_4})$, where a numerator q means that a corresponding dihedral angle is decomposed into q parts (we always omit a multiple π). For example, a fraction $\frac{2}{4}$ denotes a right angle decomposed into two parts. A triangular face OA_iA_{i+1} has a number i.
- In the same way we represent the pentagonal and hexagonal pyramids (we use ten and twelve angles correspondingly).
- A triangular prism $A_1A_2A_3B_1B_2B_3$ (where $A_1A_2A_3$ and $B_1B_2B_3$ are bases) is represented by nine dihedral angles $(\widehat{A_3B_3},\widehat{A_1B_1},\widehat{A_2B_2};\widehat{A_1A_2};\widehat{A_2A_3},\widehat{A_3A_1};\widehat{B_1B_2},\widehat{B_2B_3},\widehat{B_3B_1})$. A nominator is a number of the parts in the dihedral angle; the multiple π is omitted. The faces are numbered in the following way: $A_1A_2A_3$ has number 0, $B_1B_2B_3$ has number 1, and $A_iA_{i+1}B_{i+1}B_i$ has number $i+1$.

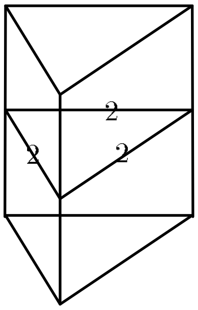
Remark. In this notation each pyramid or triangular prism can be written by several ways (for instance, the order of the angles changes while we rotate the polyhedron). To unify the notation we choose a record which stands first in the alphabetical order.

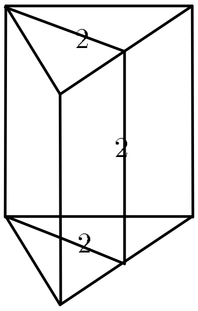
- For any polyhedron except fundamental tetrahedra we need some way to reconstruct the decomposition. For this aim we put some numbers under the symbol of the polyhedron. These seven numbers (denoted by t,k,l and m,n,p,q) shows the following:

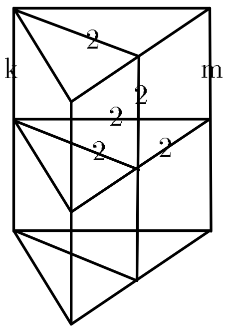
  - t is a number of the polyhedron (for each fundamental tetrahedron we start a new numeration);
  - k is a quantity of fundamental tetrahedra in the decomposition;
  - l is a quantity of glueings used to obtain this decomposition (if the decomposition was obtained by a gluing together of two polyhedra with l_1 and l_2, then $l=1+\max\{l_1,l_2\}$);
  - m and n are numbers of the polyhedra which should be glued together to obtain the decomposition;
  - p and q are numbers of the glued faces of the polyhedra m and n respectively.

  The numbers m and n are accompanied with the labels: we write "tet", "pyr" or "pri" to show that a corresponding polyhedron is a tetrahedron, a quadrangular pyramid or a triangular prism. (Any quasi-Coxeter tetrahedron consists of two smaller tetrahedra, so we omit the label "tet" in the left column.)

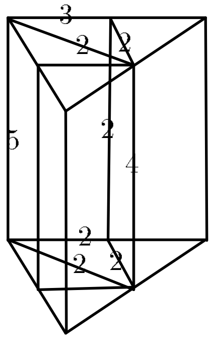
Table 1: Coxeter decompositions of the prisms into prisms.







k,m=4 or 5



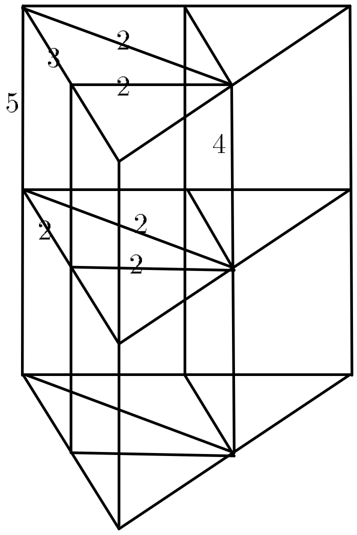
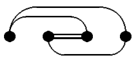
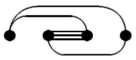
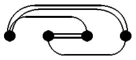
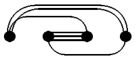
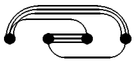
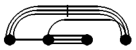
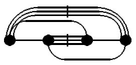


Table 2. Coxeter decompositions of the pyramids and prisms where a fundamental polyhedron is a tetrahedron

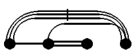
tetrahedra	pyramids	triangular prisms
1 0 	$\left(\frac{1}{3}\frac{1}{3}\frac{1}{4}\frac{1}{4};\frac{2}{3}\frac{1}{2}\frac{2}{3}\frac{1}{2}\right)$ 1, 2, 2, tet0, tet0, 0, 0 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3};\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{2}{4}\right)$ 2, 2, 2, tet0, tet0, 1, 1	$\left(\frac{1}{3}\frac{1}{3}\frac{1}{4};\frac{2}{3}\frac{1}{4}\frac{1}{2};\frac{1}{4}\frac{2}{3}\frac{1}{2}\right)$ 1, 3, 3, tet0, pyr1, 0, 1 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{4};\frac{1}{2}\frac{1}{3}\frac{2}{3};\frac{1}{2}\frac{2}{4}\frac{1}{3}\right)$ 2, 3, 3, tet0, pyr1, 1, 3 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{4};\frac{1}{2}\frac{2}{4}\frac{1}{3};\frac{1}{4}\frac{2}{3}\frac{1}{2}\right)$ 3, 4, 3, tet0, pri1, 1, 0 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{4};\frac{1}{2}\frac{2}{4}\frac{1}{3};\frac{2}{4}\frac{1}{2}\frac{1}{3}\right)$ 4, 5, 3, tet0, pri3, 1, 1
2 0 	$\left(\frac{1}{3}\frac{1}{3}\frac{1}{5}\frac{1}{5};\frac{2}{3}\frac{1}{2}\frac{2}{3}\frac{1}{2}\right)$ 1, 2, 2, tet0, tet0, 0, 0 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3};\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{2}{5}\right)$ 2, 2, 2, tet0, tet0, 1, 1	$\left(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{2}{3}\frac{1}{5}\frac{1}{2};\frac{1}{5}\frac{2}{3}\frac{1}{2}\right)$ 1, 3, 3, tet0, pyr1, 0, 1 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{3};\frac{1}{2}\frac{2}{5}\frac{1}{3}\right)$ 2, 3, 3, tet0, pyr1, 1, 3 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{2}{5}\frac{1}{3};\frac{1}{5}\frac{2}{3}\frac{1}{2}\right)$ 3, 4, 3, tet0, pri1, 1, 0 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{2}{5}\frac{1}{3};\frac{2}{5}\frac{1}{2}\frac{1}{3}\right)$ 4, 5, 3, tet0, pri3, 1, 1
3 0 	$\left(\frac{1}{3}\frac{1}{3}\frac{1}{4}\frac{1}{4};\frac{2}{4}\frac{1}{2}\frac{2}{3}\frac{1}{2}\right)$ 1, 2, 2, tet0, tet0, 0, 0	$\left(\frac{1}{3}\frac{1}{4}\frac{1}{4};\frac{1}{2}\frac{1}{3}\frac{2}{4};\frac{1}{2}\frac{2}{4}\frac{1}{3}\right)$ 1, 3, 3, tet0, pyr1, 0, 3
4 0 	$\left(\frac{1}{3}\frac{1}{3}\frac{1}{5}\frac{1}{5};\frac{2}{4}\frac{1}{2}\frac{2}{3}\frac{1}{2}\right)$ 1, 2, 2, tet0, tet0, 0, 0 $\left(\frac{1}{3}\frac{1}{3}\frac{1}{4}\frac{1}{4};\frac{2}{5}\frac{1}{2}\frac{2}{3}\frac{1}{2}\right)$ 2, 2, 2, tet0, tet0, 1, 1	$\left(\frac{1}{3}\frac{1}{4}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{4};\frac{1}{2}\frac{2}{5}\frac{1}{3}\right)$ 1, 3, 3, tet0, pyr1, 1, 3
5 0 	$\left(\frac{1}{3}\frac{1}{3}\frac{1}{5}\frac{1}{5};\frac{2}{5}\frac{1}{2}\frac{2}{3}\frac{1}{2}\right)$ 1, 2, 2, tet0, tet0, 0, 0	$\left(\frac{1}{3}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{1}{2}\frac{2}{5}\frac{1}{3}\right)$ 1, 3, 3, tet0, pyr1, 0, 3
6 0  1 2,1 0, 0, 0, 0  2 4,2 1, 1, 2, 2	$\left(\frac{1}{2}\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{2}{5}\frac{1}{2}\frac{2}{3}\frac{1}{2}\right)$ 1, 2, 2, tet0, tet0, 1, 1 $\left(\frac{1}{2}\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{3}{5}\frac{1}{2}\frac{2}{3}\frac{1}{3}\right)$ 2, 3, 3, tet1, tet0, 0, 1 $\left(\frac{1}{2}\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{4}{5}\frac{1}{3}\frac{2}{3}\frac{1}{3}\right)$ 3, 4, 3, tet1, tet1, 0, 0 - - - - - - - - - - - - $\left(\frac{1}{2}\frac{1}{5}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{2}{3}\frac{2}{3}\frac{1}{3}\right)$ 4, 5, 4, tet1, pyr2, 0, 1 - - - - - - - - - - - - $\left(\frac{1}{2}\frac{1}{2}\frac{1}{5}\frac{1}{5}\frac{1}{5};\frac{4}{5}\frac{1}{2}\frac{2}{3}\frac{2}{3}\frac{1}{2}\right)$ 1, 4, 2, tet0, pyr2, 1, 1 $\left(\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5};\frac{2}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3}\right)$ 2, 10, 5, pyr4, pyr4, 1, 1	$\left(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{2}\frac{2}{5};\frac{1}{2}\frac{2}{5}\frac{1}{2}\right)$ 1, 3, 3, tet0, pyr1, 1, 3 $\left(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{2}\frac{2}{5};\frac{1}{3}\frac{3}{5}\frac{1}{2}\right)$ 2, 4, 4, tet0, pyr2, 1, 3 $\left(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{3}\frac{1}{2}\frac{3}{5};\frac{1}{3}\frac{3}{5}\frac{1}{2}\right)$ 3, 5, 4, tet1, pyr2, 0, 3 $\left(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{2}\frac{2}{5};\frac{2}{3}\frac{1}{3}\frac{1}{5}\right)$ 4, 6, 4, tet1, pri2, 0, 1 $\left(\frac{1}{5}\frac{1}{5}\frac{2}{5};\frac{1}{2}\frac{1}{2}\frac{2}{5};\frac{1}{2}\frac{1}{3}\frac{1}{2}\right)$ 5, 7, 4, pri1, pri2, 3, 3 $\left(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{3}\frac{1}{2}\frac{3}{5};\frac{2}{3}\frac{1}{3}\frac{1}{5}\right)$ 6, 7, 4, tet1, pri3, 0, 0 $\left(\frac{1}{5}\frac{1}{5}\frac{2}{5};\frac{1}{2}\frac{1}{3}\frac{1}{2};\frac{1}{3}\frac{1}{3}\frac{3}{5}\right)$ 7, 9, 5, tet1, pri5, 2, 0 $\left(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{2}{3}\frac{1}{3}\frac{1}{5};\frac{2}{3}\frac{1}{5}\frac{1}{3}\right)$ 8, 9, 6, tet1, pri6, 0, 0

tetrahedra	pyramids	triangular prisms

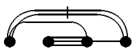
tetrahedra

7

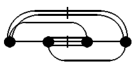
0



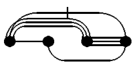
1 2,1 0, 0, 0, 0



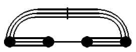
2 2,1 0, 0, 3, 3



3 4,2 1, 1, 2, 2



4 4,2 2, 2, 0, 0



5 8,3 4, 4, 0, 0

pyramids

$(\frac12\frac12\frac14\frac14;\frac25\frac12\frac23\frac12)$
1, 2, 2, tet0, tet0, 1, 1

$(\frac12\frac12\frac15\frac15;\frac24\frac12\frac23\frac12)$
2, 2, 2, tet0, tet0, 2, 2

$(\frac12\frac12\frac14\frac14;\frac35\frac12\frac23\frac13)$
3, 3, 3, tet1, tet0, 0, 1

$(\frac12\frac12\frac14\frac14;\frac45\frac13\frac23\frac13)$
4, 4, 3, tet1, tet1, 0, 0

$(\frac12\frac12\frac15\frac15;\frac34\frac12\frac23\frac13)$
5, 3, 3, tet2, tet0, 0, 2

$(\frac13\frac13\frac13\frac13;\frac23\frac25\frac23\frac25)$
6, 8, 4, tet3, tet3, 0, 0

$(\frac12\frac24\frac13\frac15;\frac12\frac23\frac12\frac25)$
7, 6, 4, tet4, tet2, 0, 2

$(\frac13\frac24\frac13\frac15;\frac13\frac23\frac12\frac35)$
8, 8, 4, tet4, tet3, 0, 1

$(\frac15\frac15\frac15\frac15;\frac23\frac23\frac23\frac23)$
9, 8, 4, tet4, tet4, 1, 1

$(\frac13\frac13\frac13\frac15;\frac23\frac25\frac12\frac35)$
10, 12, 5, tet5, tet4, 0, 2

$(\frac13\frac15\frac13\frac15;\frac12\frac35\frac12\frac35)$
11, 16, 5, tet5, tet5, 0, 0

$(\frac15\frac15\frac25\frac25;\frac23\frac12\frac25\frac12)$
12, 16, 5, tet5, tet5, 1, 1

$(\frac13\frac13\frac15\frac15;\frac25\frac12\frac45\frac12)$
13, 16, 5, tet5, tet5, 3, 3

- -

$(\frac12\frac14\frac14\frac14;\frac12\frac23\frac23\frac13)$
14, 5, 5, tet1, pyr3, 0, 1

$(\frac12\frac15\frac15\frac15;\frac12\frac23\frac23\frac12)$
15, 4, 4, tet0, pyr5, 2, 1

- -

$(\frac12\frac12\frac14\frac14\frac14;\frac45\frac12\frac23\frac23\frac12)$
1, 4, 2, tet0, pyr3, 1, 1

$(\frac13\frac24\frac13\frac15\frac15;\frac23\frac23\frac12\frac45\frac12)$
2, 12, 4, tet4, pyr8, 0, 1

$(\frac14\frac14\frac14\frac14\frac14;\frac23\frac23\frac23\frac23\frac23)$
3, 10, 6, pyr14, pyr14, 1, 1

triangular prisms

$(\frac12\frac14\frac15;\frac12\frac12\frac24;\frac12\frac25\frac12)$
1, 3, 3, tet0, pyr1, 2, 3

$(\frac12\frac14\frac15;\frac12\frac12\frac24;\frac13\frac35\frac12)$
2, 4, 4, tet0, pyr3, 2, 3

$(\frac12\frac14\frac15;\frac12\frac25\frac12;\frac13\frac12\frac34)$
3, 4, 4, tet0, pyr5, 1, 3

$(\frac12\frac13\frac15;\frac12\frac12\frac34;\frac12\frac25\frac15)$
4, 7, 5, tet0, pyr7, 2, 2

$(\frac12\frac14\frac15;\frac13\frac12\frac34;\frac13\frac35\frac12)$
5, 5, 4, tet1, pyr5, 0, 3

$(\frac24\frac15\frac15;\frac15\frac23\frac23;\frac25\frac12\frac12)$
6, 10, 5, tet2, pyr9, 2, 1

$(\frac24\frac15\frac15;\frac15\frac23\frac23;\frac35\frac13\frac13)$
7, 12, 5, tet3, pyr9, 1, 1

$(\frac13\frac13\frac15;\frac12\frac13\frac25;\frac13\frac23\frac24)$
8, 16, 6, tet3, pyr10, 1, 1

$(\frac13\frac13\frac15;\frac12\frac13\frac25;\frac23\frac25\frac13)$
9, 20, 6, tet4, pyr13, 2, 3

$(\frac13\frac13\frac15;\frac12\frac13\frac25;\frac12\frac35\frac15)$
10, 24, 6, tet5, pyr11, 0, 2

$(\frac15\frac15\frac25;\frac25\frac25\frac12;\frac25\frac25\frac12)$
11, 24, 6, tet5, pyr12, 1, 1

$(\frac24\frac15\frac15;\frac12\frac12\frac12;\frac25\frac12\frac12)$
12, 6, 3, pri1, pri1, 2, 2

$(\frac12\frac14\frac15;\frac12\frac12\frac24;\frac23\frac13\frac14)$
13, 6, 3, tet1, pri2, 0, 1

$(\frac14\frac14\frac25;\frac12\frac12\frac24;\frac12\frac13\frac12)$
14, 7, 3, pri1, pri2, 3, 3

$(\frac24\frac15\frac15;\frac12\frac12\frac12;\frac35\frac13\frac13)$
15, 8, 3, pri2, pri2, 2, 2

$(\frac12\frac14\frac15;\frac12\frac25\frac12;\frac23\frac15\frac12)$
16, 5, 4, tet0, pri3, 2, 1

$(\frac12\frac14\frac15;\frac13\frac35\frac12;\frac23\frac15\frac12)$
17, 6, 4, tet0, pri5, 2, 0

$(\frac12\frac14\frac15;\frac13\frac12\frac34;\frac23\frac13\frac14)$
18, 7, 4, tet1, pri5, 0, 1

$(\frac14\frac14\frac25;\frac12\frac13\frac12;\frac13\frac13\frac34)$
19, 9, 4, pri3, pri5, 3, 3

$(\frac24\frac15\frac15;\frac25\frac12\frac12;\frac25\frac12\frac12)$
20, 12, 5, tet2, pri6, 2, 0

$(\frac24\frac15\frac15;\frac25\frac12\frac12;\frac35\frac13\frac13)$
21, 14, 5, tet3, pri6, 1, 0

$(\frac24\frac15\frac15;\frac35\frac13\frac13;\frac35\frac13\frac13)$
22, 16, 5, tet3, pri7, 1, 0

$(\frac13\frac13\frac15;\frac12\frac13\frac25;\frac13\frac12\frac25)$
23, 32, 6, tet5, pri10, 0, 1

$(\frac15\frac15\frac25;\frac12\frac12\frac12;\frac25\frac25\frac12)$
24, 12, 6, pri12, pri12, 2, 2

$(\frac12\frac14\frac15;\frac12\frac25\frac12;\frac12\frac25\frac12)$
25, 6, 2, tet0, pri16, 1, 1

$(\frac12\frac14\frac15;\frac12\frac25\frac12;\frac13\frac35\frac12)$
26, 7, 3, tet1, pri16, 0, 1

$(\frac12\frac14\frac15;\frac23\frac13\frac14;\frac23\frac15\frac12)$
27, 8, 3, tet1, pri17, 0, 0

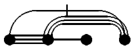
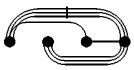
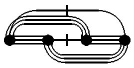
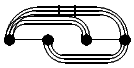
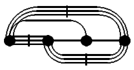
$(\frac12\frac14\frac15;\frac13\frac35\frac12;\frac13\frac35\frac12)$
28, 8, 3, tet1, pri17, 0, 1

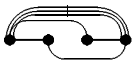
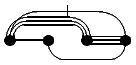
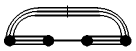
$(\frac12\frac14\frac15;\frac12\frac25\frac12;\frac23\frac13\frac14)$
29, 9, 3, tet1, pri27, 0, 1

$(\frac14\frac14\frac25;\frac12\frac13\frac12;\frac13\frac12\frac12)$
30, 14, 1, pri27, pri27, 3, 3

$(\frac12\frac14\frac15;\frac13\frac35\frac12;\frac23\frac13\frac14)$
31, 10, 3, tet1, pri28, 0, 1

$(\frac14\frac14\frac25;\frac12\frac13\frac12;\frac12\frac13\frac12)$
32, 14, 1, pri26, pri29, 3, 3

tetrahedra	pyramids	triangular prisms	
8 0  1 2,1 0, 0, 0, 0  2 3,2 1, 0, 0, 1  3 4,2 1, 1, 2, 2  4 5,3 2, 1, 0, 0  5 6,3 2, 2, 1, 1	$(\frac{1}{2}\frac{1}{2}\frac{1}{3}\frac{1}{3};\frac{2}{3}\frac{1}{2}\frac{2}{5}\frac{1}{2})$ 1, 2, 2, tet0, tet0, 1, 1 $(\frac{1}{2}\frac{1}{2}\frac{1}{3}\frac{1}{5};\frac{2}{3}\frac{1}{2}\frac{3}{5}\frac{1}{3})$ 2, 4, 4, tet2, tet0, 0, 1 $(\frac{1}{2}\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{2}{3}\frac{1}{3}\frac{4}{5}\frac{1}{3})$ 3, 6, 4, tet2, tet2, 0, 0 $(\frac{1}{2}\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{2}{3}\frac{2}{5}\frac{2}{3}\frac{2}{5})$ 4, 6, 4, tet2, tet2, 2, 2 $(\frac{1}{2}\frac{1}{3}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{3}{5}\frac{2}{3}\frac{2}{5})$ 5, 7, 4, tet3, tet2, 0, 3 $(\frac{1}{2}\frac{1}{3}\frac{1}{5}\frac{1}{3};\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{2}{5})$ 6, 8, 5, tet4, tet2, 0, 3 $(\frac{1}{2}\frac{1}{5}\frac{1}{5}\frac{1}{5};\frac{1}{3}\frac{4}{5}\frac{2}{3}\frac{2}{5})$ 7, 9, 5, tet4, tet3, 0, 0 $(\frac{1}{3}\frac{1}{3}\frac{1}{5}\frac{1}{5};\frac{2}{5}\frac{3}{5}\frac{2}{3}\frac{3}{5})$ 8, 10, 5, tet4, tet4, 1, 1 $(\frac{1}{3}\frac{1}{5}\frac{1}{3}\frac{2}{5};\frac{1}{2}\frac{2}{3}\frac{1}{3}\frac{3}{5})$ 9, 11, 5, tet5, tet4, 0, 0 - $(\frac{1}{3}\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5};\frac{3}{5}\frac{2}{3}\frac{4}{5}\frac{2}{3}\frac{3}{5})$ 1, 14, 5, tet4, pyr7, 1, 1 $(\frac{1}{3}\frac{1}{3}\frac{1}{5}\frac{1}{3}\frac{1}{5};\frac{4}{5}\frac{1}{2}\frac{2}{3}\frac{2}{3}\frac{1}{2})$ 2, 16, 5, tet4, pyr9, 0, 4 - $(\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5};\frac{2}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3})$ 1, 18, 6, pyr7, pyr7, 1, 1	$(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{2}{5}\frac{2}{3}\frac{1}{2})$ 1, 11, 6, tet1, pyr7, 1, 3 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{3}\frac{1}{2}\frac{2}{3};\frac{1}{3}\frac{2}{3}\frac{1}{2})$ 2, 7, 5, tet2, pyr2, 0, 3 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{3}\frac{1}{2}\frac{2}{3};\frac{2}{3}\frac{2}{5}\frac{1}{5})$ 3, 10, 5, tet2, pyr5, 0, 2 $(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{2}{5}\frac{1}{2};\frac{2}{5}\frac{1}{2}\frac{1}{2})$ 4, 10, 5, tet2, pyr5, 3, 3 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{3}{5}\frac{1}{2}\frac{1}{3})$ 5, 12, 6, tet2, pyr7, 3, 3 $(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{2}{5}\frac{1}{2};\frac{3}{5}\frac{1}{3}\frac{2}{5})$ 6, 13, 6, tet2, pyr8, 3, 3	$(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{2}{3}\frac{1}{5}\frac{2}{5};\frac{2}{3}\frac{2}{5}\frac{1}{5})$ 7, 13, 6, tet3, pyr7, 0, 2 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{4}{5}\frac{1}{3}\frac{1}{5})$ 8, 14, 6, tet4, pyr7, 0, 3 $(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{3}\frac{3}{5}\frac{2}{5};\frac{3}{5}\frac{1}{3}\frac{2}{5})$ 9, 16, 6, tet4, pyr9, 1, 2 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{1}{3}\frac{1}{2}\frac{2}{3})$ 10, 15, 5, tet4, pri3, 0, 1 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{2}{3}\frac{1}{5}\frac{2}{5})$ 11, 18, 6, tet4, pri7, 0, 0 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{1}{2}\frac{2}{5}\frac{1}{3})$ 12, 23, 5, tet4, pri11, 0, 1

tetrahedra	pyramids	triangular prisms
9 0   1 2,1 0, 0, 1, 1  2 2,1 0, 0, 2, 2  3 4,2 2, 2, 0, 0	$(\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3};\frac{2}{3}\frac{2}{5}\frac{2}{3}\frac{2}{5})$ 1, 4, 3, tet1, tet1, 1, 1 $(\frac{1}{2}\frac{1}{2}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{2}{3}\frac{1}{2}\frac{2}{5})$ 2, 3, 3, tet2, tet0, 0, 0 $(\frac{1}{2}\frac{1}{3}\frac{1}{5}\frac{1}{3};\frac{1}{3}\frac{3}{5}\frac{1}{2}\frac{2}{3})$ 3, 4, 3, tet2, tet1, 0, 0 $(\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5};\frac{2}{3}\frac{2}{3}\frac{2}{3}\frac{2}{3})$ 4, 4, 3, tet2, tet2, 1, 1 $(\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{2}{3}\frac{2}{5}\frac{1}{2}\frac{3}{5})$ 5, 6, 4, tet3, tet2, 0, 2 $(\frac{1}{3}\frac{1}{5}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{3}{5}\frac{1}{2}\frac{3}{5})$ 6, 8, 4, tet3, tet3, 0, 0 $(\frac{1}{5}\frac{1}{5}\frac{2}{5}\frac{2}{5};\frac{2}{3}\frac{1}{2}\frac{2}{5}\frac{1}{2})$ 7, 8, 4, tet3, tet3, 1, 1 $(\frac{1}{3}\frac{1}{3}\frac{1}{5}\frac{1}{5};\frac{2}{5}\frac{1}{2}\frac{4}{5}\frac{1}{2})$ 8, 8, 4, tet3, tet3, 3, 3 - $(\frac{1}{2}\frac{1}{3}\frac{1}{5}\frac{1}{5}\frac{1}{3};\frac{2}{3}\frac{1}{2}\frac{4}{5}\frac{1}{2}\frac{2}{3})$ 1, 6, 3, tet2, pyr3, 0, 2	$(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{5}\frac{2}{3}\frac{2}{3};\frac{2}{5}\frac{1}{2}\frac{1}{2})$ 1, 5, 4, tet0, pyr4, 0, 1 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{1}{5}\frac{2}{3}\frac{2}{3};\frac{3}{5}\frac{1}{3}\frac{1}{3})$ 2, 6, 4, tet1, pyr4, 0, 1 $(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{1}{3}\frac{2}{3}\frac{1}{2})$ 3, 8, 5, tet1, pyr5, 0, 1 $(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{2}{3}\frac{2}{5}\frac{1}{3})$ 4, 10, 5, tet2, pyr8, 2, 3 $(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{1}{2}\frac{3}{5}\frac{1}{5})$ 5, 12, 5, tet3, pyr6, 0, 2 $(\frac{1}{5}\frac{1}{5}\frac{2}{5};\frac{2}{5}\frac{2}{5}\frac{1}{2};\frac{2}{5}\frac{2}{5}\frac{1}{2})$ 6, 12, 5, tet3, pyr7, 1, 1 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{2}{5}\frac{1}{2}\frac{1}{2};\frac{2}{5}\frac{1}{2}\frac{1}{2})$ 7, 6, 4, tet0, pri1, 0, 0 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{2}{5}\frac{1}{2}\frac{1}{2};\frac{3}{5}\frac{1}{3}\frac{1}{3})$ 8, 7, 4, tet1, pri1, 0, 0 $(\frac{1}{2}\frac{1}{5}\frac{1}{5};\frac{3}{5}\frac{1}{3}\frac{1}{3};\frac{3}{5}\frac{1}{3}\frac{1}{3})$ 9, 8, 4, tet1, pri2, 0, 0 $(\frac{1}{3}\frac{1}{3}\frac{1}{5};\frac{1}{2}\frac{1}{3}\frac{2}{5};\frac{1}{3}\frac{1}{2}\frac{2}{5})$ 10, 16, 5, tet3, pri5, 0, 1

References

[1] E.M.Andreev, On convex polyhedra in Lobachevskii space, Math. USSR, Sbornik, 10 (1970), 413-440.

[2] A.Felikson, Coxeter decompositions of hyperbolic polygons. European J. of Combinatorics. 1998.

[3] A.Felikson, Coxeter decompositions of hyperbolic tetrahedra. Preprint,1998.