REPRESENTATIONS OF SEMISIMPLE LIE ALGEBRAS IN POSITIVE CHARACTERISTIC AND QUANTUM GROUPS AT ROOTS OF UNITY

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1. Introduction

1.1. If A is a finite dimensional algebra then its blocks are in one-to-one correspondence with its primitive central idempotents. The aim of this paper is to study this interaction for a class of noetherian algebras arising naturally in representation theory. This class includes the universal enveloping algebra of a reductive Lie algebra in positive characteristic and its quantised counterpart, the quantised enveloping algebra of a Borel subalgebra and the quantised function algebra of a semisimple algebraic group at roots of unity.

1.2. More generally this paper is concerned with the role the centre of these algebras plays in their representation theory. The techniques used fall into two categories: local and global. The local approach is concerned principally with the behaviour of certain finite dimensional factors of these noetherian algebras whilst the global approach focuses on general properties of these algebras. The aim in both cases is to understand the structure of these finite dimensional factor algebras. In the first case we use a little deformation theory to piece things together whilst in the second case we can use some geometric tools before passing to the factors.

1.3. In Section 2 we introduce the class of algebras we wish to study and present some general properties these have in common. In the following three sections we apply this theory to the study of enveloping algebras and quantised enveloping algebras of Lie algebras and to quantised function algebras. We end with an appendix on the structure of the centre of a quantised Borel algebra. Most of this paper surveys results from the articles [5] and [6]. The approach to Theorem 3.6 using deformation theory is new whilst several results in Section 5 tie up loose ends from [6].

2. Generalities

2.1. Throughout K denotes an algebraically closed field. We consider a triple of K-algebras

$$R \subseteq Z \subseteq H$$

where H is a prime Hopf algebra with centre Z and R is an affine sub-Hopf algebra of H over which H (and hence Z) are finitely generated modules. We have four examples in mind.

(A) Let K have positive characteristic p and let $\mathfrak{g}$ be a finite dimensional restricted Lie algebra over K. Then H is $U(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$, and R the p-centre of H.

(B) Let $K = \mathbb{C}$, let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$ and let $\epsilon \in \mathbb{C}$ be a primitive ℓ^{th} root of unity, for ℓ an odd integer greater than 1. Then H is the quantised enveloping algebra $U_\epsilon(\mathfrak{g})$ and R the ℓ-centre of H.

(C) Let K, $\mathfrak{g}$ and ℓ be as above. Then H is $U_\epsilon^{\leq 0}$, the subalgebra of $U_\epsilon(\mathfrak{g})$ corresponding to a Borel subalgebra of $\mathfrak{g}$, and R the ℓ-centre of H.

(D) Let $K = \mathbb{C}$, let G be a simply-connected, semisimple algebraic group over $\mathbb{C}$ and let ℓ be as above. Then H is $\mathcal{O}_\epsilon[G]$, the quantised function algebra of G, and R the ℓ-centre of H.

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2.2. It is straightforward to show that there is an upper bound on the dimension of the simple H-modules, namely the PI degree of H, [4, Proposition 3.1]. In particular, each simple H-module is annihilated by a maximal ideal of R. As a result the family of finite dimensional algebras

$$\left\{ \frac{H}{\mathfrak{m}H} : \mathfrak{m} \text{ a maximal ideal in } R \right\}$$

captures an important slice of the representation theory of H: each simple H-module is a simple module for exactly one algebra in this family.

2.3. To firm up this notion of a "family of finite dimensional algebras" recall the following definition of the *variety of n-dimensional algebras* over K, [19]. Let

$$\mathrm{Bil}(n) = \{ \text{ bilinear maps } m : K^n \times K^n \longrightarrow K^n \} \cong \mathbb{A}_K^{n^3}$$

and

$$\mathrm{Alg}(n) = \{ \text{ associative, bilinear } m \text{ which have an identity} \} \subseteq \mathrm{Bil}(n).$$

It can be shown that $\mathrm{Alg}(n)$ is an affine variety, locally closed in $\mathrm{Bil}(n)$.

Let $Q(R)$ be the quotient field of R and let $Q(H) = H \otimes_R Q(R)$. Since H is a finitely generated R-module there is an integer n such that $Q(H)$ is an n-dimensional $Q(R)$-module.

Lemma. *Let n be as above. There is a morphism of varieties*

$$\alpha : \mathrm{Maxspec}(R) \longrightarrow \mathrm{Alg}(n),$$

sending $\mathfrak{m}$ to $H/\mathfrak{m}H$.

Proof. By [35] our hypotheses in 2.1 ensure that H is a free R-module of rank n. Let $\{x_1, \dots , x_n\}$ be a basis for H over R and define $c_{ij}^k \in R$ for $1 \leq i, j, k \leq n$ by the following equations,

$$x_i x_j = \sum_k c_{ij}^k x_k.$$

For any maximal ideal, $\mathfrak{m}$, of R the structure constants of $H/\mathfrak{m}H$ with respect to the basis $\{x_i + \mathfrak{m}H\}$ are given by the scalars $(c_{ij}^k + \mathfrak{m}) \in R/\mathfrak{m}R \cong K$. It follows that α is a morphism of varieties, as required. □

In 2.2 we could have equally considered the family of algebras $\{H/MH : M \text{ a maximal ideal of } Z\}$. This family, however, does not behave very well in general since the extension $Z \subseteq H$ need not be flat. For instance in examples (A), (B), (D) and often in (C) the presence of singular points in $\mathrm{Maxspec}(Z)$ prevents flatness since H has finite global dimension, [4].

2.4. **Müller's Theorem.** The first result on the block structure of the algebras $H/\mathfrak{m}H$ is a striking analogue of the finite dimensional case.

Theorem. *The blocks of $H/\mathfrak{m}H$ are in one-to-one correspondence with the maximal ideals of Z lying over $\mathfrak{m}$.*

From now on we write $\mathcal{B}_M$ to denote the block of $H/(M \cap R)H$ corresponding to M.

Remark. This result first appears in a different and more general context in [28, Theorem 7]; the interpretation here is discussed in [5, 2.10].

2.5. There is also a result on the local level about the number of blocks of $H/\mathfrak{m}H$.

Proposition. [19, Proposition 2.7] *Let $s \in \mathbb{N}$. The set*

$$X_s = \{\mathfrak{m} \in \mathrm{Maxspec}(R) : H/\mathfrak{m}H \textit{ has no more than } s \textit{ blocks}\}$$

is closed in $\mathrm{Maxspec}(R)$.

2.6. There is one type of block that is controlled by Z, a block corresponding to a point on the *Azumaya locus* of H:

$$\mathcal{A}_H = \{M \in \operatorname{Maxspec}(Z) : \mathcal{B}_M \text{ has a simple module of maximal dimension}\}.$$

This definition is not standard, but under the hypotheses of 2.1 is equivalent to the usual notion, see [5, 2.5].

Proposition. [5, Proposition 2.5] *Let M be a maximal ideal of Z belonging to $\mathcal{A}_H$ and let $\mathfrak{m} = M \cap R$. There is an algebra isomorphism*

$$\mathcal{B}_M \cong \operatorname{Mat}_n\left(\frac{Z_M}{\mathfrak{m}Z_M}\right)$$

where $Z_M/\mathfrak{m}Z_M$ is the primary component of $Z/\mathfrak{m}Z$ associated to M.

Recall a block is *primary* if it has a unique simple module. The proposition shows that blocks corresponding to Azumaya points are primary.

2.7. Being Azumaya is a generic condition, that is $\mathcal{A}_H$ is a dense open set in $\operatorname{Maxspec}(Z)$, [34, Section 1.9]. Under the hypotheses of 2.1 $\mathcal{A}_H$ is contained in the smooth locus of $\operatorname{Maxspec}(Z)$, [4, Lemma 3.3]. In sufficiently well-behaved situations the converse holds.

Theorem. [4, Theorem 3.8] *Suppose H has finite global dimension. If* $\operatorname{codim}(\operatorname{Maxspec}(Z) \setminus \mathcal{A}_H) \geq 2$ *then $\mathcal{A}_H$ equals the smooth locus of* $\operatorname{Maxspec}(Z)$.

It is not true in general that $\operatorname{Maxspec}(Z)$ is Azumaya in codimension one, [6, Proposition 2.6].

2.8. We finish this section with a comparison of $Z/\mathfrak{m}Z$ and $Z(H/\mathfrak{m}H)$ for $\mathfrak{m} \in \operatorname{Maxspec}(R)$. Quite generally there is an homomorphism

$$\iota : \frac{Z}{\mathfrak{m}Z} \longrightarrow Z\left(\frac{H}{\mathfrak{m}H}\right).$$

Lemma. *The map ι is generically an isomorphism. Moreover, if K has characteristic zero then it is always injective.*

Proof. The morphism $\pi : \operatorname{Maxspec}(Z) \longrightarrow \operatorname{Maxspec}(R)$ is finite since Z is a finitely generated R-module. Since finite morphisms are closed the set $\mathcal{F}_H = \operatorname{Maxspec}(R) \setminus \pi(\operatorname{Maxspec}(Z) \setminus \mathcal{A}_H)$ is a dense open set containing precisely those maximal ideals, $\mathfrak{m}$, of R whose fibre $\pi^{-1}(\mathfrak{m})$ is contained in $\mathcal{A}_H$. It follows from Proposition 2.6 that for any $\mathfrak{m} \in \mathcal{F}_H$ we have an isomorphism

$$\frac{H}{\mathfrak{m}H} \cong \operatorname{Mat}_n\left(\frac{Z}{\mathfrak{m}Z}\right), \tag{1}$$

proving the first claim.

Now suppose that K has characteristic zero. Then there is a Z-module map, the reduced trace, $Tr : H \longrightarrow Z$ splitting the inclusion, [33, 9.8 and Theorem 10.1]. Thus Z is a direct summand of H and so $\mathfrak{m}H \cap Z = \mathfrak{m}Z$ for all $\mathfrak{m} \in \operatorname{Maxspec}(R)$, as required. □

2.9. Under the hypotheses of 2.1 the algebras $H/\mathfrak{m}H$ are all Frobenius, [20, Theorem 3.4].

Lemma. [13, cf. I.3.9] *Suppose that $H/\mathfrak{m}H$ is a symmetric algebra. Then*

$$\dim(\operatorname{soc}(Z(H/\mathfrak{m}H))) \geq \text{the number of simple } H/\mathfrak{m}H\text{-modules.}$$

In particular, if $\iota(Z/\mathfrak{m}Z)$ is self-injective then ι is surjective only if all blocks of $H/\mathfrak{m}H$ are primary.

Proof. Write A for $H/\mathfrak{m}H$. Let $\{S_1, \dots, S_t\}$ be a complete set of representatives for the isomorphism classes of simple A-modules and let P_i be the projective cover of S_i for $1 \leq i \leq t$. Since Morita equivalence preserves symmetry and the centre of an algebra, [2, Volume I, Proposition 2.2.7], without loss of generality we may assume that A is basic.

The homomorphism

$$\theta_i : P_i \longrightarrow S_i \longrightarrow P_i$$

defines an element in $A \cong \mathrm{End}_A(\oplus P_i)^{\mathrm{op}}$. Write an arbitrary element of A as $f = \sum_i \lambda_{f,i} id_{P_i} + f'$ where f' is a radical morphism. Then, by construction, $\theta_i f = \lambda_{f,i}\theta_i = f\theta_i$. In particular θ_i is central and lies in the socle of $Z(H/\mathfrak{m}H)$.

Let $Z' = \iota(Z/\mathfrak{m}Z)$ and decompose Z' into primary components. By hypothesis each of these is self-injective and so Frobenius, [1, Example IV.3]. In particular each component has a simple socle. Suppose that $Z' = Z(A)$. If S_i and S_j belong to the same block of A then θ_i and θ_j belong to the same primary component of Z', contradicting the simplicity of the socle. □

Remark. Let K have positive characteristic p and let $\mathfrak{l}$ be the Heisenberg Lie algebra over K, that is the Lie algebra with basis $\{x, y, z\}$ such that $[x, y] = z$ and z is central. Let $H = U(\mathfrak{l})$ and $R = Z = K[z, x^p, y^p]$. Let $\mathfrak{m} = (z, x^p, y^p)Z$, a maximal ideal of Z. Then $H/\mathfrak{m}H$ is a truncated polynomial ring, showing that the converse of the second claim is false in general. This also shows that in general not all primary blocks are Azumaya.

2.10. The following proposition provides a partial adjunct to Lemma 2.9.

Proposition. [19, Proposition 2.7] *Let $s \in \mathbb{N}$. The set*

$$Y_s = \{\mathfrak{m} \in \mathrm{Maxspec}(R) : \dim Z(H/\mathfrak{m}H) \geq s\}$$

is closed in $\mathrm{Maxspec}(R)$.

3. Enveloping algebras

3.1. We follow the notation used in [25]. Let G be a connected, reductive algebraic group over K, an algebraically closed field of positive characteristic p, and let $\mathfrak{g} = \mathrm{Lie}(G)$. We assume that G satisfies the following hypotheses:

1. The derived group $\mathcal{D}G$ of G is simply-connected;
2. The prime p is good for $\mathfrak{g}$;
3. There exists a G-invariant non-degenerate bilinear form on $\mathfrak{g}$.

More details can be found in [25, Section 6]. Let $T \subseteq B = U.T$ be a maximal torus contained in a Borel subgroup of G and let $\mathfrak{h} = \mathrm{Lie}(T)$, $\mathfrak{n} = \mathrm{Lie}(U)$ and $\mathfrak{b} = \mathrm{Lie}(B)$. Let X be the character group of T and let $\Lambda = X/pX$. Let Φ be the set of roots associated with $\mathfrak{g}$ and let Φ^+ be set of positive roots corresponding to the choice of B. Let W be the Weyl group of G. We will be interested only in the "dot action" of W on X (and hence on Λ). By definition this is just an affine translation of the natural action of W on X. Given a K-vector space V, let $V^{(1)}$ denote the twist of V along the automorphism of K which sends λ to λ^p.

3.2. Being the Lie algebra of G, $\mathfrak{g}$ has a restriction map $x \longmapsto x^{[p]}$. We have a triple

$$Z_0 \subseteq Z \subseteq U(\mathfrak{g})$$

where $U(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$ and $Z_0 = K[x^p - x^{[p]} : x \in \mathfrak{g}]$ is the p-centre of $\mathfrak{g}$, a central sub-Hopf algebra of $U(\mathfrak{g})$. Standard arguments with the PBW theorem imply that $Z_0 \cong \mathcal{O}(\mathfrak{g}^{*(1)})$, the ring of regular functions on $\mathfrak{g}^{*(1)}$ and that $U(\mathfrak{g})$ is a free Z_0-module of rank $p^{\dim \mathfrak{g}}$, [25, Proposition 2.3]. Thus $U(\mathfrak{g})$ satisfies the hypotheses of 2.1. By Lemma 2.3 we have a morphism of varieties

$$\alpha : \mathfrak{g}^{*(1)} \longrightarrow \mathrm{Alg}(p^{\dim \mathfrak{g}})$$

sending χ to the algebra $U_\chi = U(\mathfrak{g})/(x^p - x^{[p]} - \chi(x))$, a *reduced enveloping algebra*. Note that if $\chi = 0$ then U_0 is the restricted enveloping algebra of $\mathfrak{g}$. It is straightforward to check that $U_{g.\chi} \cong U_\chi$ for $g \in G$ acting on $\mathfrak{g}^{*(1)}$ by the coadjoint action, [25, 2.9].

3.3. Hypothesis 3.1.3 yields a G-equivariant isomorphism $\theta : \mathfrak{g}^{*(1)} \longrightarrow \mathfrak{g}^{(1)}$. In particular, given $\chi \in \mathfrak{g}^{*(1)}$ let $y = \theta(\chi)$ and write $y = y_s + y_n$, the Jordan decomposition of y in $\mathfrak{g}$. Then $\chi = \chi_s + \chi_n$ where $\chi_s = \theta^{-1}(y_s)$ and $\chi_n = \theta^{-1}(y_n)$. We call $\chi = \chi_s + \chi_n$ the *Jordan decomposition* of χ.

Let $\mathfrak{z}_{\mathfrak{g}}(\chi) = \{x \in \mathfrak{g} : \chi([x, \mathfrak{g}]) = 0\}$ and $Z_G(\chi) = \{g \in G : g.\chi = \chi\}$. Under the hypotheses in 3.1 we have that $Z_G(\chi_s)$ is a connected, reductive algebraic group such that $\mathfrak{z}_{\mathfrak{g}}(\chi_s) = \mathrm{Lie}(Z_G(\chi_s))$ and $Z_G(\chi_s)$ satisfies 3.1.1,3.1.2, and 3.1.3, [25, 6.5 and 7.4]. Note that χ can be considered as an element of $\mathfrak{z}_{\mathfrak{g}}(\chi_s)^{*(1)}$.

3.4. **Reduction Theorem.** It is reasonable to be concerned mostly with almost simple G and simple $\mathfrak{g}$. The following reduction theorem in conjunction with 3.3, however, justifies the general hypotheses of 3.1.

Theorem. [37, Theorem 2], [18, Theorem 3.2] *Let $\chi = \chi_s + \chi_n \in \mathfrak{g}^{*(1)}$ be the Jordan decomposition. Let $d = \frac{1}{2}(\dim G.\chi_s)$. Then there is an algebra isomorphism*

$$U_\chi(\mathfrak{g}) \cong \mathrm{Mat}_{p^d}\left(U_{\chi_n}(\mathfrak{z}_{\mathfrak{g}}(\chi_s))\right).$$

3.5. Thanks to Theorem 3.4, without loss of generality we can work under the hypothesis $\chi = \chi_s + \chi_n$ where $\mathfrak{z}_{\mathfrak{g}}(\chi_s) = \mathfrak{g}$. Since there is a finite number of nilpotent orbits in $\mathfrak{g}$, [21, Chapter 3], the classification of simple $\mathfrak{g}$-modules essentially becomes a finite problem.

3.6. **Blocks of $U(\mathfrak{g})$.** Recall that we consider the dot action of the Weyl group W on $\Lambda = X/pX$.

Theorem. *Let χ be as in 3.5. Then U_χ has $|\Lambda/W|$ blocks.*

Proof. Let $\mathcal{N} = \{\eta \in \mathfrak{g}^{*(1)} : \theta(\eta) \text{ nilpotent}\} \subseteq g^{*(1)}$ be the nilpotent cone in $\mathfrak{g}^{*(1)}$. By 3.5 affine translation $\mathcal{N} \longrightarrow \chi_s + \mathcal{N}$ is a G-equivariant isomorphism of varieties. In particular $\chi_s + \mathcal{N}$ is irreducible and has a unique dense orbit consisting of regular elements, that is of elements whose centraliser has minimal dimension, [21, Chapter 4]. Moreover, every G-orbit in $\chi_s + \mathcal{N}$ contains χ_s in its closure, [31, Theorem 2.5].

Let

$$\mathcal{O} = \{\eta \in \chi_s + \mathcal{N} : U_\eta \text{ has } |\Lambda/W| \text{ blocks}\}.$$

Clearly $\mathcal{O}$ is G-stable and by Proposition 2.5 $\mathcal{O}$ is locally closed in $\chi_s + \mathcal{N}$. By [25, Section 10] $\mathcal{O}$ contains both χ_s and the regular orbit. The result follows. □

This theorem first appeared in [5], confirming a conjecture of Humphreys in [22]. The proof given in [5], however, was based on Müller's Theorem, 2.4, and less representation theoretic than the above. Moreover the case $p = 2$ was omitted.

Henceforth we write

$$U_\chi = \bigoplus_{\lambda \in \Lambda/W} \mathcal{B}_{\chi,\lambda}.$$

We often abuse notation by writing $\mathcal{B}_{\chi,\lambda}$ for $\lambda \in \Lambda$ or even $\lambda \in X$.

3.7. **Baby Verma modules.** Let $\chi = \chi_s + \chi_n \in \mathfrak{g}^{*(1)}$ be as in 3.5. We can assume without loss of generality that $\chi(\mathfrak{n}) = 0$, [25, Lemma 6.6]. Then any element $\lambda \in \Lambda$ gives rise to K_λ, a one dimensional representation of $U_\chi(\mathfrak{b})$, a reduced enveloping algebra of the Lie algebra of $\mathfrak{b}$. Indeed, by [25, 11.1] and [5, 3.19] there is a W-equivariant isomorphism

$$\Lambda \cong \{\lambda \in \mathfrak{h}^* : \lambda(h)^p - \lambda(h^{[p]}) = \chi(h) \text{ for all } h \in \mathfrak{h}\}.$$

The induced module $V_\chi(\lambda) = U_\chi \otimes_{U_\chi(\mathfrak{b})} K_\lambda$, a *baby Verma module*, plays an important role in the representation theory of U_χ. For instance it follows from [25, 10.11] and Theorem 3.6 that $V_\chi(\lambda)$ belongs to a block of U_χ and further that we can choose the labelling of blocks such that $V_\chi(\lambda)$ belongs to $\mathcal{B}_{\chi,\lambda}$. In particular $V_\chi(\lambda)$ and $V_\chi(w{\bullet}\lambda)$ belong to the same block for all $w \in W$.

3.8. **Primary Blocks.** We can describe when a block $\mathcal{B}_{\chi,\lambda}$ is primary.

Proposition. *Assume that $p \neq 5$ if R is of type E_7. Then the block $\mathcal{B}_{\chi,\lambda}$ is primary if and only if it corresponds to an Azumaya point of* $\mathrm{Maxspec}(Z)$.

Proof. Sufficiency follows from 2.6. For the converse, let $\lambda \in X$ and suppose that L is the unique simple module in the block $\mathcal{B}_{\chi,\lambda}$ and let $P(L)$ be its projective cover. By [23, B.12(2)] we have

$$\dim P(L) = p^N |W{\bullet}(\lambda + pX)| [V_\chi(\lambda) : L],$$

where $N = |\Phi^+|$, the number of positive roots. On the other hand, by [23, C.2], there is a projective $\mathcal{B}_{\chi,\lambda}$-module P such that

$$\dim P = p^N |W{\bullet}\lambda|.$$

We deduce that for $\mu \in W_\bullet\lambda + pX$

$$[V_\chi(\mu) : L] \text{ divides } \frac{|W_\bullet\mu|}{|W_\bullet(\mu + pX)|}. \tag{2}$$

Moreover $[V_\chi(\mu) : L] = p^i$ since $V_\chi(\mu)$ belongs to $\mathcal{B}_{\chi,\lambda}$. So we must find an element $\mu \in W_\bullet\lambda + pX$ which forces i to be zero.

Arguing as in [23, H.1 Remarks] we can assume without loss of generality that G is almost simple.

Let

$$C_0 = \{\lambda \in X : 0 \le \langle \lambda + \rho, \beta^\vee \rangle \le p \text{ for all } \beta \in \Phi^+\}$$

and

$$C_0' = \{\lambda \in X : 0 \le \langle \lambda + \rho, \beta^\vee \rangle < p \text{ for all } \beta \in \Phi^+\}.$$

Since C_0 is a fundamental domain for the dot action of the affine Weyl group $W \ltimes pX$ on X, we can assume without loss of generality that $\mu \in C_0$. By [23, C.1 Lemma] if $\mu \in C_0'$ then $|W_\bullet\mu| = |W_\bullet(\mu + pX)|$ and so, by (2), $i = 0$ as required. By [23, H.1 Proposition], if Φ is not exceptional then we can choose $\mu \in C_0'$ finishing the proof. Hence we need only consider the case $\mu \in C_0 \setminus C_0'$ and Φ exceptional. In particular if p is prime to the order of W then (2) forces $i = 0$. The only remaining cases are $p = 5$ and Φ of type E_6, E_7 or, E_8 and $p = 7$ and Φ of type E_7 or E_8. A case-by-case analysis shows that the only possible exception to $i = 0$ occurs when Φ has type E_7, $p = 5$ and $\mu = \varpi_2 + \varpi_5$ (where we've followed the numbering of [3]). □

3.9. The previous proposition has a consequence for the structure of the centre of U_χ. This phenomenon was observed in the case $\chi = 0$ by Premet and noted in [5, 3.17].

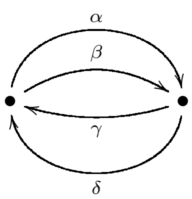
Corollary. *Suppose that $p \neq 2$ and that $p \neq 5$ if Φ has a component of type E_7. Then the centre of U_χ is isomorphic to $Z/\mathfrak{m}_\chi Z$ if and only if χ is regular.*

Proof. By [36] and [18, 1.2] the algebras U_χ are symmetric. By [5, Theorem 3.5(4)] Z is a complete intersection ring (this is where we require $p \neq 2$), so in particular Gorenstein. By [5, Theorem 3.5(6)] Z is a free Z_0-module so, since Z_0 is smooth, standard commutative algebra implies that each factor $Z/\mathfrak{m}_\chi Z$ is Gorenstein, or, in other words, self-injective. Premet has proved the following, [32],

$$\text{the natural map } \iota : Z/\mathfrak{m}_\chi Z \longrightarrow Z(U_\chi) \text{ is injective.}$$

The result follows from Lemma 2.9, Proposition 3.8 and [5, Proposition 3.15]. □

3.10. **Example.** Let $G = SL_2(K)$ where K has odd characteristic. If χ satisfies the condition in 3.5 then either χ is regular nilpotent or $\chi = 0$. The algebra U_χ has $(p+1)/2$ blocks, labelled by the set of integers $\{-1, 0, \ldots, (p-3)/2\}$. In all cases the block corresponding to -1 is associated with a simple projective module, the *Steinberg module*, and is isomorphic to $\mathrm{Mat}_p(K)$. If χ is regular then $\mathcal{B}_{\chi,i} \cong \mathrm{Mat}_p(K[X]/(X^2))$ for $0 \le i \le (p-3)/2$. This follows, for instance, from [5, Proposition 3.16]. If $\chi = 0$ then it is shown in [16] that for $0 \le i \le (p-3)/2$ the block $\mathcal{B}_{0,i}$ is Morita equivalent to the path algebra of the quiver



with relations $\alpha\delta = \beta\gamma = 0$; $\delta\alpha = \gamma\beta = 0$; $\alpha\gamma = \beta\delta$; $\gamma\alpha = \delta\beta$, an eight-dimensional algebra. In this case the centre of the block is spanned by the linearly independent elements $1, \alpha\gamma$ and $\gamma\alpha$. In particular we have

$$\dim Z(U_0) = 3\left(\frac{p-1}{2}\right) + 1 = \frac{3p-1}{2},$$

whilst $\dim Z/\mathfrak{m}_0 Z = p$.

3.11. Let's conclude with a number of remarks.
1) The structure of $Z/\mathfrak{m}_\chi Z$ is known, [27]. To the best of my knowledge, however, the structure of $Z(U_\chi)$ is, in general, unknown.
2) Proposition 3.8 completes the classification of blocks of finite representation type begun in [29]. Indeed, thanks to [15, Theorem 3.2], any block of finite representation type is uniserial so in particular primary. Then Proposition 3.8 tells us that the block is Azumaya. Consequently the results in [29] and [5] can be applied.
3) The structure of Z has a number of further consequences for the representation theory of U thanks to Theorem 2.7 which is valid in this situation, [4, Theorem 4.10] and [5, Theorem 2.9.1]. In particular see [5, 3.11 and 3.13].

4. Quantised enveloping algebras

4.1. We follow the notation of [11]. Let G be a simply-connected, semisimple algebraic group of rank r over $\mathbb{C}$ and let $\mathfrak{g} = \mathrm{Lie}(G)$. Let B^+ be a Borel subgroup of G and let B^- be the Borel subgroup opposite to B^+ in G. Let $T = B^+ \cap B^-$, a maximal torus of G, so that $B^\pm = T.U^\pm$. Let $W = N_G(T)/T$ be the Weyl group of G.

Let $\epsilon \in \mathbb{C}$ be a primitive ℓ^{th} root of unity where ℓ is an odd integer greater than 1 and prime to 3 if G has a component of type G_2.

4.2. Let $U_\epsilon(\mathfrak{g})$ be the simply-connected quantised enveloping algebra of $\mathfrak{g}$ at a root of unity ϵ, as defined in [8]. In particular $U_\epsilon(\mathfrak{g})$ is a Hopf algebra. We have a triple

$$Z_0 \subseteq Z \subseteq U_\epsilon(\mathfrak{g}),$$

where Z_0 is ℓ-centre of $U_\epsilon(\mathfrak{g})$, a central sub-Hopf algebra generated by the ℓ^{th} powers of certain generators of $U_\epsilon(\mathfrak{g})$, [8]. Then $U_\epsilon(\mathfrak{g})$ is a free Z_0-module of rank $\ell^{\dim \mathfrak{g}}$. Thanks to Lemma 2.3 we have a map

$$\mathrm{Maxspec}(Z_0) \longrightarrow \mathrm{Alg}(\ell^{\dim \mathfrak{g}})$$

where $\chi \in \mathrm{Maxspec}(Z_0)$ is sent to $U_{\epsilon,\chi}$.

Remark. The algebra $U_\epsilon(\mathfrak{g})$ is not the quantised enveloping algebra considered by Lusztig in [26]. Lusztig's algebra is a quantum analogue of the hyperalgebra of $\mathfrak{g}$ whilst $U_\epsilon(\mathfrak{g})$ is a quantum analogue of the enveloping algebra of $\mathfrak{g}$. However the reduced quantum group of Lusztig, a finite dimensional sub-Hopf algebra of Lusztig's quantised enveloping algebra, is isomorphic to a skew group extension of the algebra $U_{\epsilon,1}$, where $1 \in \mathrm{Maxspec}(Z_0)$ is the augmentation ideal of Z_0.

4.3. The structure of Z_0 is well-understood, [8]. In particular there is an unramified covering of degree 2^r

$$\pi : \mathrm{Maxspec}(Z_0) \cong (U^- \times U^+) \rtimes T \longrightarrow B^-B^+ \subseteq G.$$

Moreover, given $\tilde{\chi} \in (U^- \times U^+) \rtimes T$ there exists $\chi = \chi_u\chi_s \in U^- \rtimes T = B^-$ such that $\pi(\chi_u)\pi(\chi_s)$ is the Jordan decomposition of $\pi(\chi)$ in G and $U_{\epsilon,\chi} \cong U_{\epsilon,\tilde{\chi}}$.

4.4. The algebra $U_\epsilon(\mathfrak{g})$ can be defined as the specialisation of an integral form of $U_q(\mathfrak{g})$, the quantised enveloping algebra of $\mathfrak{g}$ at a transcendental parameter q. As a result we can find a central subalgebra $Z_1 \subseteq U_\epsilon(\mathfrak{g})$, the specialisation of the centre of $U_q(\mathfrak{g})$.

Theorem. [11, Section 21] *The algebra Z is a complete intersection ring and there is an isomorphism $Z \cong Z_0 \otimes_{Z_0 \cap Z_1} Z_1$. Moreover, Z is a free Z_0-module of rank ℓ^r.*

4.5. **Blocks of $U_\epsilon(\mathfrak{g})$.** Theorem 4.4 allows us to determine the structure of the algebra $Z/\mathfrak{m}_\chi Z$ for $\chi = \chi_u\chi_s \in B^-$, [5, Theorem 4.5]. In particular we find that the primary components of $Z/\mathfrak{m}_\chi Z$ are in natural bijection with the elements of $R_\chi = \{t \in T : t^\ell \in W\chi_s^2\}/W$. Theorem 2.4 gives the following result.

Theorem. [5, Theorem 4.8] *Let $\chi = \chi_u\chi_s \in B^-$ be as above. Then the blocks of $U_{\epsilon,\chi}$ naturally correspond to elements of R_χ.*

If we have $\chi_s = 1$ then $R_\chi = \{t \in T : t^\ell = 1\}/W$ and we have the quantum analogue of Theorem 3.6.

4.6. We conclude with a number of remarks.
1) There is an analogue of Theorem 3.4 for the algebras $U_{\epsilon,\chi}$ which shows that the classification of the simple $U_\epsilon(\mathfrak{g})$-modules is a finite problem, [7]. The reduction theorem, however, is difficult to use in practise: there is no guarantee that we remain in the class of simply-connected quantised enveloping algebras and it is not enough in general to consider only unipotent central characters.
2) The algebra $U_{\epsilon,1}$ is a Hopf algebra whose antipode squared is inner, [24, 4.9(1)]. Since $U_{\epsilon,1}$ has a unique one dimensional module it follows from [15, Lemma 3.1] and [17, Lemma 1.5] that $U_{\epsilon,1}$ is symmetric. Since the block containing this module is never primary we deduce from Lemma 2.9 and Theorem 4.4 that $Z/\mathfrak{m}_1 Z$ is a proper subalgebra of $Z(U_{\epsilon,1})$. Note that $Z/\mathfrak{m}_\chi Z$ is always a subalgebra of $Z(U_{\epsilon,\chi})$ thanks to Lemma 2.8.
3) More details on the influence of Z on the representation theory of $U_\epsilon(\mathfrak{g})$ can be found in [5, Section 4].

5. Quantised function algebras and quantum Borels

5.1. Let G, B^+, B^-, T and W be as in 4.1. Let X be the character group of T and let Φ be the root system of G. Let Φ^+ be the positive roots of Φ with respect to B^+ and let $\{\alpha_i : 1 \leq i \leq r\}$ be the simple roots in Φ^+ and $\{\varpi_i : 1 \leq i \leq r\}$ be the fundamental weights in X. Let $(\, , \,)$ be the natural pairing between the root lattice, Q, and the weight lattice, P.

There is a stratification of G

$$G = \coprod_{w_1, w_2 \in W} X_{w_1, w_2}$$

where $X_{w_1,w_2} = B^+ w_1 B^+ \cap B^- w_2 B^-$. This restricts to a stratification of B^+

$$B^+ = \coprod_{w \in W} X_{e,w}.$$

For $w \in W$ let $\ell(w)$, respectively $s(w)$, equal the minimal length of an expression for w as a product of simple, respectively arbitrary, reflections.

Let w_0 be the longest word in W and let $N = \ell(w_0)$. Since w_0 sends Φ^+ to $-\Phi^+$ there is an involution σ on the set of integers $[1, r]$ defined by $w_0 \alpha_i = -\alpha_{\sigma(i)}$. It can be shown that the number of fixed points of σ equals $2s(w_0) - r$.

5.2. Let $\epsilon \in \mathbb{C}$ be a primitive ℓ^{th} root of unity where ℓ is an odd integer greater than one and good, that is ℓ is prime to the bad primes of Φ.

Let $U_\epsilon^{\leq 0}$ be the non-positive subalgebra of $U_\epsilon(\mathfrak{g})$ as defined in [9], the quantised enveloping algebra of a Borel subalgebra of $\mathfrak{g}$. We have a triple

$$Z_0^{\leq 0} \subseteq Z(U_\epsilon^{\leq 0}) \subseteq U_\epsilon^{\leq 0}$$

where $Z_0^{\leq 0}$ is the ℓ-centre of $U_\epsilon^{\leq 0}$, that is the intersection of the ℓ-centre of $U_\epsilon(\mathfrak{g})$ with $U_\epsilon^{\leq 0}$. By [10] $U_\epsilon^{\leq 0}$ is a free $Z_0^{\leq 0}$-module of rank $\ell^{\dim B^+}$. By [10] $Z_0^{\leq 0}$ is isomorphic as a Hopf algebra to $\mathcal{O}[B^+]$, the ring of regular functions on B^+. Given $b \in B^+$ we let $\mathfrak{m}_b$ denote the corresponding maximal ideal of $Z_0^{\leq 0}$. By Theorem 2.3 we have a morphism of varieties

$$\alpha : \text{Maxspec}(Z_0^{\leq 0}) \longrightarrow \text{Alg}(\ell^{\dim B^+}).$$

We denote $\alpha(\mathfrak{m}_b)$ by $U_\epsilon^{\leq 0}(b)$.

Let $\mathcal{O}_\epsilon[G]$ be the quantised function algebra of G at a root of unity ϵ, as defined in [10]. We have a triple

$$Z_0 \subseteq Z(\mathcal{O}_\epsilon[G]) \subseteq \mathcal{O}_\epsilon[G],$$

where Z_0 is the ℓ-centre of $\mathcal{O}_\epsilon[G]$, [10]. By [10] and [35] $\mathcal{O}_\epsilon[G]$ is a free Z_0-module of rank $\ell^{\dim G}$. By [10] Z_0 is isomorphic as a Hopf algebra to $\mathcal{O}[G]$. Given $g \in G$ we let $\mathfrak{m}_g$ denote the corresponding maximal ideal of Z_0. By Theorem 2.3 we have a morphism of varieties

$$\beta : \text{Maxspec}(Z_0) \longrightarrow \text{Alg}(\ell^{\dim G}).$$

We denote $\beta(\mathfrak{m}_g)$ by $\mathcal{O}_\epsilon[G](g)$.

5.3. The following theorem shows us that there is only a finite number of isomorphism classes of algebras $U_\epsilon^{\leq 0}(b)$ and $\mathcal{O}_\epsilon[G](g)$. It also demonstrates that the PI degree of $U_\epsilon^{\leq 0}$, respectively of $\mathcal{O}_\epsilon[G]$, is $\ell^{\frac{1}{2}(N+s(w_0))}$, respectively ℓ^N.

Theorem. [9, Theorem 4.4], [10, Section 9], [12, Theorem 4.4 and Proposition 4.10] *Let $w, w_1, w_2 \in W$ and let $b, b' \in X_{e,w}$ and $g, g' \in X_{w_1,w_2}$.*
(a)(i) There is an algebra isomorphism $U_\epsilon^{\leq 0}(b) \cong U_\epsilon^{\leq 0}(b')$.
(ii) There are precisely $\ell^{r-s(w)}$ simple $U_\epsilon^{\leq 0}(b)$-modules, each having dimension $\ell^{\frac{1}{2}(\ell(w)+s(w))}$.
(b)(i) There is an algebra isomorphism $\mathcal{O}_\epsilon[G](g) \cong \mathcal{O}_\epsilon[G](g')$.
(ii) There are precisely $\ell^{r-s(w_2^{-1}w_1)}$ simple $\mathcal{O}_\epsilon[G](g)$-modules, each having dimension $\ell^{\frac{1}{2}(\ell(w_1)+\ell(w_2)+s(w_2^{-1}w_1))}$.

5.4. **Centres.** We begin by introducing some distinguished elements in $\mathcal{O}[B^+]$ and $\mathcal{O}[G]$. Let $V(\varpi_i)$ be the simple G-module with highest weight ϖ_i and let $V(\varpi_i)^*$ be its dual. For each i choose a highest weight vector v_i, respectively a lowest weight vector v_i', of $V(\varpi_i)$. Let f_i, respectively f_i', be the unique weight vector in $V(\varpi_i)^*$ dual to v_i, respectively v_i'. For each i we have elements a_i, b_i and c_i of $\mathcal{O}[G]$ defined by

$$a_i(g) = f'_{\sigma(i)}(gv'_{\sigma(i)}), \quad b_i(g) = f_i'(gv_i), \quad c_i(g) = f_{\sigma(i)}(gv'_{\sigma(i)}).$$

These elements restrict to elements of $\mathcal{O}[B^+]$ which we will denote by a_i, b_i and c_i too.

Given a subset $I \subseteq [1, r]$ let $S(I)$ be the subalgebra of $\mathbb{C}[X_1, Y_1, \ldots, X_r, Y_r]$ generated by $X_i^k Y_i^{\ell-k}$ for $i \in I$ and $0 \leq k \leq \ell$. Then $S(I)$ is a free module of rank $\ell^{|I|}$ over $S_0(I) = \mathbb{C}[X_i^\ell, Y_i^\ell : i \in I]$.

Theorem. *a) Let $\tilde{I} = \{1 \leq i \leq r : \sigma(i) \neq i\}$ and let I be a set of orbit representatives for the σ-action on $\tilde{I}$. Then the centre of $U_\epsilon^{\leq 0}$ is isomorphic to the algebra*

$$\mathcal{O}[B^+] \otimes_{S_0(I)} S(I),$$

where $S_0(I) \longrightarrow \mathcal{O}[B^+]$ sends X_i^ℓ to $a_i c_i$ and Y_i^ℓ to $a_{\sigma(i)} c_{\sigma(i)}$. In particular $Z(U_\epsilon^{\leq 0})$ is a free $Z_0^{\leq 0}$-module of rank $\ell^{r-s(w_0)}$.
b) The centre of $\mathcal{O}_\epsilon[G]$ is isomorphic to the algebra

$$\mathcal{O}[G] \otimes_{S_0([1,r])} S([1, r]),$$

where $S_0([1, r]) \longrightarrow \mathcal{O}[G]$ is obtained by sending X_i^ℓ to b_i and Y_i^ℓ to c_i. In particular $Z(\mathcal{O}_\epsilon[G])$ is a free Z_0-module of rank ℓ^r.

Proof. Part (a) can be found in the appendix: it follows the method of proof of (b) in [14]. □

Remark. If G only has components of type B_r, C_r, D_r (r even),E_7, E_8, F_4 or G_2 then $s(w_0) = r$ so $\tilde{I} = \emptyset$ and consequently $Z(U_\epsilon^{\leq 0}) \cong \mathcal{O}[B^+]$. It can be shown that this is the only case when one of the algebras above is Gorenstein.

5.5. In constrast to Section 3 the centre of $U_\epsilon^{\leq 0}(b)$ or $\mathcal{O}_\epsilon[G](g)$ is easy to describe.

Lemma. *a) Let $b \in B^+$. Then $Z(U_\epsilon^{\leq 0}(b)) \cong Z/\mathfrak{m}_b Z$.*
b) Let $g \in G$. Then $Z(\mathcal{O}_\epsilon[G](g)) \cong Z/\mathfrak{m}_\mathfrak{g} Z$.

Proof. (a) By Lemma 2.8 the natural map $\iota : Z/\mathfrak{m}_b Z \longrightarrow Z(U_\epsilon^{\leq 0}(b))$ is injective, so by Theorem 5.4(a), it is enough to show that $\dim Z(U_\epsilon^{\leq 0}(b)) = \ell^{r-s(w_0)}$ for all $b \in B^+$. Since $\overline{X_{e,w}} = \coprod_{w' \preccurlyeq w} X_{e,w'}$, where $\preccurlyeq$ denotes the Bruhat-Chevalley order on W, the identity of B^+ is contained in the closure of any cell $X_{e,w}$. By Proposition 2.10 $\dim Z(U_\epsilon^{\leq 0}(b)) \leq \dim Z(U_\epsilon^{\leq 0}(1))$ for all $b \in B^+$. We will prove that $\dim Z(U_\epsilon^{\leq 0}(1)) = \ell^{r-s(w_0)}$.

Let $\overline{U}_1 = U_\epsilon^{\leq 0}(1)$. Fix a reduced expression $w_0 = s_{i_1} \ldots s_{i_N}$ and hence an ordering $\beta_1 < \ldots < \beta_N$ in R^+. The algebra $\overline{U}_1$ is generated by the elements F_{β_j} $(1 \leq j \leq N)$ and K_i $(1 \leq i \leq r)$ subject to the relations

$$K_i K_j = K_j K_i, \quad K_i^\ell = 1 \tag{3}$$

$$K_i F_{\beta_j} = \epsilon^{-(\beta_j, \varpi_i)} F_{\beta_j} K_i, \quad F_{\beta_j}^\ell = 0 \tag{4}$$

$$F_{\beta_j} F_{\beta_k} = \epsilon^{-(\beta_j, \beta_k)} F_{\beta_k} F_{\beta_j} + p_{jk}^1 \quad j < k \tag{5}$$

where p_{jk}^1 is a polynomial in the variables F_{β_h} for $j < h < k$, [11, Theorem 9.3].

Suppose we have defined inductively an $\ell^{\dim B^+}$-dimensional algebra $\overline{U}_m$ with generators F_{β_j} and K_i satisfying (3), (4) and

$$F_{\beta_j} F_{\beta_k} = \epsilon^{-(\beta_j,\beta_k)} F_{\beta_k} F_{\beta_j} + p_{jk}^m \tag{6}$$

where p_{jk}^m is a polynomial in the variables F_{β_h} for $j < h < \min\{k, N+1-m\}$. We have a morphism

$$\mathbb{C} \longrightarrow \mathrm{Alg}(\ell^{\dim B^+})$$

where $t \in \mathbb{C}$ is sent to the algebra $\overline{U}_m(t)$ with generators F_{β_j} and K_i satisfying relations (3), (4) and (6′) obtained from (6) by replacing $F_{\beta_{N+1-m}}$ with $tF_{\beta_{N+1-m}}$. Then $\overline{U}_m(t) \cong \overline{U}_m$ for $t \in \mathbb{C}^*$ and, by definition, $\overline{U}_{m+1} = \overline{U}_m(0)$.

The algebra $\overline{U}_N$ is generated by F_{β_j} and K_i subject to relations (3), (4) and

$$F_{\beta_j} F_{\beta_k} = \epsilon^{-(\beta_j,\beta_k)} F_{\beta_k} F_{\beta_j}.$$

By a repeated application of Proposition 2.10 we find that $\dim Z(U_\epsilon^{\leq 0}(1)) \leq \dim Z(\overline{U}_N)$. Using the techniques of [11, Chapter 2 and Section 10] we can see that the dimension of $Z(\overline{U}_N)$ is $\ell^{r-s(w_0)}$ as required.

(b) This is proved in a similar manner using [20, Section 2.9]. □

5.6. **Blocks.** Given $w, w_1, w_2 \in W$ let

$$\mathcal{S}(w) = \{1 \leq i \leq r : \sigma(i) \neq i \text{ and } w_0 w, w w_0 \in \mathrm{Stab}_W(\varpi_i)\}$$

and

$$\mathcal{T}(w_1, w_2) = \{1 \leq i \leq r : w_0 w_1, w_0 w_2 \in \mathrm{Stab}_W(\varpi_i)\}.$$

Note that $\mathcal{S}(w)$ is σ-stable.

Theorem. *(a) Let $b \in X_{e,w}$. Then $U_\epsilon^{\leq 0}(b)$ has $\ell^{\frac{1}{2}|\mathcal{S}(w)|}$ blocks.*
(b) Let $g \in X_{w_1,w_2}$. Then $\mathcal{O}_\epsilon[G](g)$ has $\ell^{|\mathcal{T}(w_1,w_2)|}$ blocks.

Proof. (a) Recall that I is a set of representatives for the σ-action on $\tilde{I} = \{1 \leq i \leq r : \sigma(i) \neq i\}$. The number of blocks of $U_\epsilon^{\leq 0}(b)$ equals the number of maximal ideals of $Z/\mathfrak{m}_b Z$, either by Theorem 2.4 or by Lemma 5.5. The description of $Z(U_\epsilon^{\leq 0})$ shows that $Z/\mathfrak{m}_b Z$ is the tensor product of a discrete family of ℓ-dimensional algebras indexed by elements of I, where

(i) if $a_i c_i(b) \neq 0 \neq a_{\sigma(i)} c_{\sigma(i)}(b)$ then $Z_i \cong \mathbb{C}^\ell$;
(ii) if $a_i c_i(b) \neq 0$ and $a_{\sigma(i)} c_{\sigma(i)}(b) = 0$ (or vice-versa) then $Z_i \cong \mathbb{C}[X]/(X^\ell)$;
(iii) if $a_i c_i(b) = 0 = a_{\sigma(i)} c_{\sigma(i)}(b)$ then $Z_i \cong \mathbb{C}[X_1, \dots X_{\ell-1}]/(X_j X_k : 1 \leq j,k \leq \ell - 1)$.

It follows that $U_\epsilon^{\geq 0}(b)$ has ℓ^d blocks where $d = |\{i \in I : a_i c_i(b) \neq 0 \neq a_{\sigma(i)} c_{\sigma(i)}(b)\}|$.

By definition a_i is a co-ordinate function on the maximal torus T so is non-vanishing on B^+. By [6, Lemma 7.2] c_i does not vanish on $X_{e,w}$ if and only if $w_0 w \in \mathrm{Stab}_W(\varpi_i)$, or equivalently $w w_0 \in \mathrm{Stab}_W(\varpi_{\sigma(i)})$. The first part follows.

(b) This is proved similarly. Details can be found in [6, Section 7]. □

5.7. **Azumaya locus.** The precise structure of $Z(U_\epsilon^{\leq 0}(b))$ and $Z(\mathcal{O}_\epsilon[G](g))$ can be determined using Lemma 5.5 and the methods in the proof of Theorem 5.6. The result, however, is a little awkward to present in general, but in the Azumaya case things are simpler.

Theorem. *(a) Let $w \in W$ be such that $\ell(w) + s(w) = N + s(w_0)$. Let $n = r - s(w)$ and $d = \frac{1}{2}(N + s(w_0))$. Then for $b \in X_{e,w}$ we have*

$$U_\epsilon^{\leq 0}(b) \cong \bigoplus^{\ell^n} \mathrm{Mat}_{\ell^d}\left(\frac{\mathbb{C}[X_1, \dots, X_{s(w)-s(w_0)}]}{(X_1^\ell, \dots, X_{s(w)-s(w_0)}^\ell)}\right).$$

(b) Let $w_1, w_2 \in W$ be such that $\ell(w_1) + \ell(w_2) + s(w_2^{-1} w_1) = 2N$. Let $n = r - s(w_2^{-1} w_1)$. Then for $g \in X_{w_1,w_2}$ we have

$$\mathcal{O}_\epsilon[G](g) \cong \bigoplus^{\ell^n} \mathrm{Mat}_{\ell^N}\left(\frac{\mathbb{C}[X_1, \dots, X_{s(w_2^{-1} w_1)}]}{(X_1^\ell, \dots, X_{s(w_2^{-1} w_1)}^\ell)}\right).$$

Proof. (a) By Theorem 5.3 every simple $U_\epsilon^{\leq 0}(b)$-module has maximal dimension so it follows from (1) that $U_\epsilon^{\leq 0}(b) \cong \mathrm{Mat}(Z/\mathfrak{m}_b Z)$. The argument of [6, Proposition 3.2] shows that a point $(b,x) \in \mathrm{Maxspec}(Z(U_\epsilon^{\leq 0}))$ such that $a_i c_i(b) = a_{\sigma(i)} c_{\sigma(i)}(b) = 0$ for some $i \in I$ is singular. Since the Azumaya locus is contained in the smooth locus of $\mathrm{Maxspec}(Z(U_\epsilon^{\leq 0}))$ it follows that only cases (i) and (ii) in the proof of Theorem 5.6 can occur. The result follows from 5.3 by counting the number of simple $Z/\mathfrak{m}_b Z$-modules.
(b) This is proved similarly. Details can be found in [6, Section 3]. □

Remark. This theorem is the crucial step in the determination of the representation type of the algebras $U_\epsilon^{\leq 0}(b)$ and $\mathcal{O}_\epsilon[G](g)$, see [6, Section 4].

Appendix A.

A.1. We would like to describe the centre of $U_\epsilon^{\leq 0}$. We follow the ideas of [14] and use the notation of [12] without further ado. We recall that there is an isomorphism of Hopf algebras

$$U_\epsilon^{\leq 0} \cong \mathcal{O}_\epsilon[B^+]$$

where $\mathcal{O}_\epsilon[B^+]$ is the factor algebra of $\mathcal{O}_\epsilon[G]$ essentially obtained by restricting functions from $U_q(\mathfrak{g})$ to $U_q^{\geq 0}$, [10]. Let

$$\pi : \mathcal{O}_\epsilon[G] \longrightarrow \mathcal{O}_\epsilon[B^+]$$

be the canonical map. We often abuse notation by continuing to write x for $\pi(x)$.

A.2. For $1 \leq i \leq r$ the following elements of $\mathcal{O}_\epsilon[B^+]$ are defined as matrix coefficients (compare 5.4)

$$x_i = c^{V(-w_0\varpi_i)}_{\phi^{w_0\varpi_i}, v_{-\varpi_i}}, \quad y_i = c^{V(-w_0\varpi_i)}_{\phi^{\varpi_i}, v_{-\varpi_i}}.$$

We have a general commutation rule, [12, 1.2]

$$c_{\phi,v} c_{\psi,w} = \epsilon^{-(\mu_1,\mu_2)+(\nu_1,\nu_2)} c_{\psi,w} c_{\phi,v} + \sum_j c_{\psi_j,w_j} c_{\phi_j,v_j}, \tag{7}$$

where $\psi_j \otimes \phi_j = p_j(M_j(E) \otimes M_j(F))\psi \otimes \psi$, $w_j \otimes v_j = p_j'(M_j'(E) \otimes M_j'(F)) w \otimes v$ and p_j, p_j' are scalars and M_j, M_j' are monomials, at least one of which is non-constant.

A.3. We define

$$z_{i,k} = (x_i y_i)^k (x_{\sigma(i)} y_{\sigma(i)})^{\ell - k}.$$

Lemma. *For $1 \leq i \leq r$ and $0 \leq k \leq \ell$ the element $z_{i,k}$ is central in $U_\epsilon^{\leq 0}$.*

Proof. It can be checked, using (7), that $z_{i,k}$ commutes with all matrix coefficients of the form $c^{V(\lambda)}_{\psi,v}$ where v is a lowest weight of $V(\lambda)$. By [12, Lemma 2.3(2)] $\mathcal{O}_\epsilon[B^+]$ is generated by such matrix coefficients, together with the inverse of $c^{V(\rho)}_{\phi^\rho, v_{-\rho}}$. □

A.4. Let $\tilde{I} = \{1 \leq i \leq r : \sigma(i) \neq i\}$ and let $n = |\tilde{I}|$. Let R be the algebra with generators X_j and Y_j for $j \in I$ satisfying the following relations

$$X_j X_{j'} = X_{j'} X_j, \quad Y_j Y_{j'} = Y_{j'} Y_j, \quad X_j Y_{j'} = \epsilon^{(\varpi_{j'}, w_0\varpi_j - \varpi_j)} Y_{j'} X_j. \tag{8}$$

Let R_0 be the subalgebra generated by X_j^ℓ and Y_j^ℓ, a polynomial ring in $2n$ variables. It is immediate that R is a free R_0-module of rank ℓ^{2n} with basis $\{X_{j_1}^{a_1} \dots X_{j_n}^{a_n} Y_{j_1}^{b_n} \dots Y_{j_n}^{b_n} : 0 \leq a_i, b_i < \ell\}$.

By (7) we have an algebra map

$$\psi : R \longrightarrow \mathcal{O}_\epsilon[B^+]$$

which sends X_j to x_j and Y_j to y_j. There is a commutative diagram,

$$\begin{array}{ccc} U_q^{\geq 0}(\mathfrak{sl}_2, i) & \longrightarrow & U_q^{\geq 0}(\mathfrak{g}) \\ \downarrow & & \downarrow \\ U_q(\mathfrak{sl}_2, i) & \longrightarrow & U_q(\mathfrak{g}) \end{array}$$

which, in particular, induces maps

$$\eta_i : \mathcal{O}_\epsilon[B^+] \longrightarrow \mathcal{O}_\epsilon[B^+(SL(2)), i]$$

and

$$\tau_i : \mathcal{O}_\epsilon[G] \longrightarrow \mathcal{O}_\epsilon[SL(2), i],$$

see [12, 2.4] for example. Let $w_0 = s_{i_1} \dots s_{i_N}$ be a fixed reduced expression for the longest word of the Weyl group. We define two algebra maps

$$\sigma : \mathcal{O}_\epsilon[B^+] \xrightarrow{\Delta^{(N-1)}} \mathcal{O}_\epsilon[B^+]^{\otimes N} \xrightarrow{\eta_{i_1}\otimes\cdots\otimes\eta_{i_N}} \bigotimes_{j=1}^N \mathcal{O}_\epsilon[B^+(SL(2)), i_j],$$

and

$$\tau : \mathcal{O}_\epsilon[G] \xrightarrow{\Delta^{(N-1)}} \mathcal{O}_\epsilon[G]^{\otimes N} \xrightarrow{\tau_{i_1}\otimes\cdots\otimes\tau_{i_N}} \bigotimes_{j=1}^N \mathcal{O}_\epsilon[SL(2), i_j].$$

The map σ is injective by [12, Theorem 3.2]. By construction we have another commutative diagram

$$\begin{tikzcd} \mathcal{O}_\epsilon[B^+] \arrow[r, "\eta"] & \bigotimes_{j=1}^N \mathcal{O}_\epsilon[B^+(SL(2)), i_j] \\ \mathcal{O}_\epsilon[G] \arrow[u, "\pi"] \arrow[r, "\tau"] & \bigotimes_{j=1}^N \mathcal{O}_\epsilon[SL(2), i_j] \arrow[u, "\pi_{i_1}\otimes\cdots\otimes\pi_{i_N}"'] \end{tikzcd} \tag{9}$$

A.5. For any algebra $\mathcal{O}_\epsilon[B^+(SL(2)), i]$ we can construct elements analogous to x_j and y_j, which we denote by $x(i)$ and $y(i)$.

Lemma. *Let $j \in J$. Then $\eta(x_j) = \otimes_i x(i)^{m_j(i)}$ and $\eta(y_j) = \otimes_i y(i)^{m'_j(i)}$.*

Proof. This first equality follows from [14, Proposition 3.1] together with (9). For the second part we note that in $\mathcal{O}_\epsilon[G]$ we have

$$\Delta(y_i) = \Delta_{\phi^{\varpi_i}, v_{-\varpi_i}} = \sum_s c_{\phi^{\varpi_i}, v_s} \otimes c_{\phi^s, v_{-\varpi_i}},$$

where $\{\phi^s\}$ and $\{v_s\}$ are dual bases for $V(-w_0\varpi_i)^*$ and $V(-w_0\varpi_i)$ respectively, chosen so that $\phi^1 = \phi^{\varpi_i}$ and $v_1 = v_{-\varpi_i}$. In $\mathcal{O}_\epsilon[B^+]$ we have $c_{\phi^{\varpi_i}, v_s} = 0$ unless $s = 1$. So we deduce that in $\mathcal{O}_\epsilon[B^+]$

$$\Delta^{(N-1)}(y_i) = y_i \otimes \ldots \otimes y_i.$$

Hence we need only describe $\tau_j(y_i)$ to complete the lemma. A simple calculation yields the following

$$\tau_i(y_j) = \begin{cases} y_j & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}$$

□

A.6. The above lemma provides us with a pair of linear maps

$$m : \mathbb{N}^n \longrightarrow \mathbb{N}^N, \quad m' : \mathbb{N}^n \longrightarrow \mathbb{N}^N,$$

which are determined on the components by m_j and m'_j respectively. These induce linear maps

$$\overline{m} : \left(\frac{\mathbb{N}}{\ell\mathbb{N}}\right)^n \longrightarrow \left(\frac{\mathbb{N}}{\ell\mathbb{N}}\right)^N, \quad \overline{m}' : \left(\frac{\mathbb{N}}{\ell\mathbb{N}}\right)^n \longrightarrow \left(\frac{\mathbb{N}}{\ell\mathbb{N}}\right)^N.$$

Lemma. *The maps $\overline{m}$ and $\overline{m}'$ are injective.*

Proof. For $\overline{m}$ this is proved in [14, Proposition 3.2 and Proposition 6.1]. That $\overline{m}'$ is injective follows from the explicit description given in the proof of Lemma A.5. □

A.7. We now have two algebra maps

$$\theta_1 : \mathcal{O}[B^+] \longrightarrow \bigotimes_{j=1}^{N} \mathcal{O}[B^+(SL(2)), i_j],$$

and

$$\theta_2 : R \longrightarrow \bigotimes O_\epsilon[B^+(SL(2)), i_j]$$

where θ_1 is the restriction of σ to $\mathcal{O}[B^+]$ and θ_2 is the composition $\sigma \circ \psi$.

Let I be a set of orbit representatives for the σ-action on $\tilde{I}$. In particular $|I| = r - s(w_0)$. Let $Z_{j,k} = (X_jY_j)^k(X_{\sigma(j)}Y_{\sigma(j)})^{\ell-k}$ for $j \in I$ and $0 \le k \le \ell$ and let $R' \subseteq R$ be the subalgebra generated by the elements $Z_{j,k}$. It follows from (8) that R' is commutative and it is straightforward to check that R' is free over the subring R'_0 generated by $Z_{j,0}$ and $Z_{j,\ell}$ with $j \in I$, a polynomial ring in n variables.

Lemma. *There exists an algebra map*

$$\theta : \mathcal{O}[B^+] \otimes_{R'_0} R' \longrightarrow \bigotimes_{j=1}^{N} \mathcal{O}_\epsilon[B^+(SL(2)), i_j], \tag{10}$$

such that θ is injective. In particular $\mathcal{O}[B^+] \otimes_{R'_0} R'$ is an integral domain of Krull dimension $\dim B^+$.

Proof. The map θ is obtained by combining θ_1 and θ_2. That this is well-defined follows immediately from construction. Let $t = r - s(w_0)$ and $\{j_1, \dots, j_t\} = I$. The left hand side of (10) is a free $\mathcal{O}[B^+]$-module with basis $\{Z_{j_1,k_1} \dots Z_{j_t,k_t} : 0 \le k_i < \ell\}$ whilst the right hand side is a free $\otimes_{j=1}^N \mathcal{O}[B^+(SL(2)), i_j]$-module with basis $\{\otimes_{j=1}^N x_j^{b_j} y_j^{c_j} : 0 \le b_j, c_j < \ell\}$. By Lemma A.6 θ_2 is injective with respect to these bases, proving the first claim. The second follows from the fact that the right hand side of (10) is a domain and that the left hand side is a finite extension of $\mathcal{O}[B^+]$. □

A.8. Now we have an algebra map

$$\phi : \mathcal{O}[B^+] \otimes_{R'_0} R' \longrightarrow \mathcal{O}_\epsilon[B^+],$$

whose image is generated by $\mathcal{O}[B^+]$ and the elements $z_{i,k}$.

Theorem. *The centre of $U_\epsilon^{\le 0}$ is isomorphic to $\mathcal{O}[B^+] \otimes_{R'_0} R'$ under the map ϕ.*

Proof. Let Z' be the image of ϕ. By Lemma A.3 Z' is central in $\mathcal{O}_\epsilon[B^+]$. Since Z' is an integral domain of Krull dimension $\dim B^+$ it follows from the second claim of Lemma A.7 that ϕ is an injection.

Let $P^{w_0} = \{\lambda \in P : w_0\lambda = \lambda\}$. Write $\lambda = \sum a_i\varpi_i$. It is clear that $w_0\lambda = \lambda$ if and only if $\sum_{i=1}^r (a_i + a_{\sigma(i)})\varpi_i = 0$. It follows that $P^{w_0} = \mathbb{Z}[\varpi_i - \varpi_{\sigma(i)} : i \in I]$. Now the quotient ring of Z' must equal the quotient ring of $Z(\mathcal{O}_\epsilon[B^+])$ thanks to the description given in [12, Theorem 4.5]. Since $Z(\mathcal{O}_\epsilon[B^+])$ is a finite extension of Z' it is therefore enough to show that Z' is integrally closed. But the arguments of [14, Section 7] can be applied verbatim, confirming this. □

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