Spectral gaps on loop spaces : A counterexample

Trous spectrals sur les espaces de lacets : Un contreexemple

Andreas Eberle[1]

Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
D–33501 Bielefeld
Germany
eberle@mathematik.uni–bielefeld.de

and

Univ. of CA, San Diego
Dept. of Mathematics
9500, Gilman Drive, Dept. 0112
La Jolla, CA 92093-0112, USA

[1]Supported by the European Union TMR–project "Stochastic Analysis", SFB 343, Bielefeld, and DFG

Running title : Spectral gaps on loop spaces : A counterexample

Mailing address :
Andreas Eberle
Faculty of Mathematics
University of Bielefeld
P.O. Box 10 01 31
33501 Bielefeld, Germany

Abstract.
It is shown that the Ornstein–Uhlenbeck operators on a certain class of loop spaces over simply connected compact Riemannian manifolds do not have a spectral gap above 0.

Résumé.
On donne un exemple d'un espace de lacets sur une variété riemannienne compacte et simplement connexe tel que l'opérateur d'Ornstein–Uhlenbeck correspondant n'a pas de trou spectral au-dessus de 0.

Spectral gaps on loop spaces : A counterexample

Andreas Eberle

December 6, 1999

Version française abrégée

Dans [8], S. Fang a démontré l'existence d'un trou spectral pour l'opérateur d'Ornstein–Uhlenbeck sur l'espace de chemins sur une variété riemannienne compacte. Malgré ce résultat positif, il est toujours inconnu, si, et sous quelles conditions à la variété riemannienne, il y a un résultat correspondant sur les espaces de lacets. Dans cette note, on présente une classe d'exemples montrant que même sur les espaces de lacets sur des variétés riemanniennes compactes et simplement connexes, l'opérateur d'Ornstein–Uhlenbeck n'a pas de trou spectral au–dessus de 0 en général.

En détail, soient $R,\ A > 0$ tels que $\sinh A < 1/R$. Soit M une variété riemannienne compacte et simplement connexe contenant un ouvert U isométrique à la surface de révolution dans $\mathbf{R}^3$ donnée par l'image de l'application $f : (-A, A) \times \mathbf{R} \to \mathbf{R}^3$ défini par (1). La surface est de courbure constante égale à -1. Soit $x_0 \in U$ fixe. Nous considérons l'espace de lacets $L_{x_0}(M) = \{\omega : [0,1] \to M\,;\, \omega(0) = \omega(1) = x_0\}$. La distribution P_{x_0} du pont brownien de x_0 à x_0 en temps 1 est une mesure de probabilité sur $L_{x_0}(M)$. D'une façon maintenant bien connu, on peut utiliser une version fixe du transport parallèle stochastique $U_t(\omega),\ t \in [0,1],\ \omega \in L_{x_0}(M)$, pour définir un mésurable "fibré tangent" $T_\omega L_{x_0}(M),\ \omega \in L_{x_0}(M)$, et une "métrique" $\langle\cdot\,,\cdot\rangle_\omega$ de type H^1 sur $TL_{x_0}(M)$, v. (2). Sur les fonctions cylindriques sur $L_{x_0}(M)$, le gradient D correspondant est donné par (3). Ce gradient est un analogue du gradient de Malliavin sur les espaces de lacets non–linéaires.

Considérons maintenant la forme bilinéaire $\mathcal{E}$ définie par (4) sur les fonctions cylindriques sur $L_{x_0}(M)$. Le théorème d'intégration par parties sur $L_{x_0}(M)$ implique que cette forme est fermable sur $L^2(L_{x_0}(M); P_{x_0})$. Soit $H^{1,2}(L_{x_0}(M); P_{x_0})$ le domaine de la fermeture, et soit $\mathcal{L} = D^*D$ l'opérateur auto–adjointe associé à la fermeture. Cet opérateur est appelé l'opérateur d'Ornstein–Uhlenbeck sur $L_{x_0}(M)$. S. Aida [1] a démontré que le noyau de $\mathcal{L}$ ne contient que des fonctions constantes si et seulement si M est simplement connexe. Voici maintenant l'énocé de notre résultat principal :

Théorème *Sous les conditions à M et à x_0 supposées au–dessus,* 0 *est un point d'accumulation dans le spectre de l'opérateur d'Ornstein-Uhlenbeck sur $L_{x_0}(M)$.*

L'opérateur $\mathcal{L}$ n'a donc pas de trou spectral au–dessus de 0, et, en particulier, il n'y a pas d'inégalité de Sobolev logarithmique pour le gradient D, la métrique $\langle\cdot\,,\cdot\rangle_\omega$, et la mesure P_{x_0}. En effet, on peut montrer qu'il y a un résultat analogue au théorème pour une grande classe de métriques sur $L_{x_0}(M)$. De plus, on peut appliquer le contreexemple à l'espace de lacets libres $C(S^1 \to M)$.

La démonstration du théorème est indiquée dans section 2 de la version anglaise. L'idée principale est de décomposer le sous–ensemble $\Omega = \{\omega \in L_{x_0}(M)\,;\, \omega(s) \in U\ \forall s \in [0,1]\,\}$ dans ses classes d'homotopie $\Omega_n,\ n \in \mathbf{Z}$, et de construire explicitement des fonctions $F_n \in H^{1,2}(L_{x_0}(M); P_{x_0})$, $n \in \mathbf{N}$, telles que supp $F_n \subset \Omega_n$ et (5) sont satisfait.

1. Introduction and main result

In spite of S. Fang's [8] positive result on the existence of a spectral gap for the Ornstein-Uhlenbeck operator on the path space over a compact Riemannian manifold, it is still not known if, and under which conditions, a corresponding result holds on loop spaces. The aim of this note is to present a prototypic class of examples showing that even on loop spaces over *compact simply connected* Riemannian manifolds, the Ornstein-Uhlenbeck operator does not have a spectral gap above 0 in general.

Let M be a two–dimensional simply connected compact Riemannian manifold, and fix $R,\ A > 0$ such that $\sinh A < 1/R$. We assume that M contains an open subset U that is isometric to the surface of revolution in $\mathbf{R}^3$ given as the image of the map $f : (-A, A) \times \mathbf{R} \to \mathbf{R}^3$,

$$f(s,\varphi) = (R\cosh s\ \cos\varphi,\ R\cosh s\ \sin\varphi,\ \int_0^s (1 - R^2 \sinh^2 t)^{1/2}\, dt\,) \tag{1}$$

The surface has constant curvature -1, i.e., it is a hyperbolic cylinder, cf. e.g. [11, p. 240]. Note that the non–trivial closed geodesics on U are *minimal*, and have lengths $2\pi Rn$, $n \in \mathbf{N}$. We fix a point $x_0 \in U$, and consider the based loop space $L_{x_0}(M)$ consisting of all continuous functions $\omega : [0,1] \to M$ such that $\omega(0) = \omega(1) = x_0$. The distribution P_{x_0} of the Brownian bridge from x_0 to x_0 in time 1 is a probability measure on the Borel σ–algebra of $L_{x_0}(M)$, cf. e.g. [4].

We now briefly recall the definition of a Malliavin type gradient and an Ornstein-Uhlenbeck type operator on $L_{x_0}(M)$, cf. [5] for details. Let $(t,\omega) \mapsto U_t(\omega)$, $t \in [0,1]$, $\omega \in L_{x_0}(M)$, be a fixed version of the stochastic parallel transport along the paths of the Brownian bridge. For $\omega \in L_{x_0}(M)$ let $T_\omega L_{x_0}(M)$ be the Hilbert space consisting of all vector fields $t \mapsto U_t(\omega) h_t$ along ω, where $h : [0,1] \to T_{x_0}M$ is an absolutely continous curve such that $h(0) = h(1) = 0$ and $\int_0^1 |h'(t)|^2_{T_{x_0}M} dt < \infty$. Let $\langle \cdot\,,\,\cdot \rangle_\omega$ be the inner product on $T_\omega L_{x_0}(M)$ defined by

$$\langle X,\, Y\rangle_\omega \;=\; \int_0^1 \left\langle \frac{d}{dt}(U_t(\omega)^{-1} X_t)\,,\, \frac{d}{dt}(U_t(\omega)^{-1} Y_t) \right\rangle_{T_{x_0}M} dt. \tag{2}$$

For a cylinder function $F(\omega) = f(\omega(s_1), \omega(s_2), \ldots, \omega(s_k))$, $k \in \mathbf{N}$, $f \in C^\infty(M^k)$, on $L_{x_0}(M)$, the gradient $DF : L_{x_0}(M) \to TL_{x_0}(M)$ is defined by $DF(\omega) \in T_\omega L_{x_0}(M)$ and $\langle DF(\omega),\, X\,\rangle_\omega = \partial_X F(\omega)$ for each $\omega \in L_{x_0}(M)$ and $X \in T_\omega L_{x_0}(M)$, where $\partial_X F$ denotes the directional derivative in direction X. Explicitly,

$$(DF(\omega))(t) \;=\; U_t(\omega) \sum_{i=1}^k U_{s_i}(\omega)^{-1}\, (\nabla_i f)(\omega(s_1), \ldots, \omega(s_k)) \cdot (s_i \wedge t - s_i t). \tag{3}$$

It has been shown in [5] that the quadratic form $\mathcal{E}$ given on smooth cylinder functions by

$$\mathcal{E}\,(F,\, G) \;=\; \int \langle (DF)(\omega),\, (DG)(\omega)\rangle_\omega\;\; P_{x_0}(d\omega) \tag{4}$$

is closable on $L^2(L_{x_0}(M); P_{x_0})$. The gradient operator D extends to functions in the domain $H^{1,2}(L_{x_0}(M); P_{x_0})$ of the form closure. The non–negative definite self–adjoint operator $\mathcal{L} = D^*D$ corresponding to the form closure is called the *Ornstein–Uhlenbeck operator on the loop space*. Clearly, the constant functions are in the kernel of $\mathcal{L}$, which is the lowest eigenspace. In fact, S. Aida [1] has shown for general compact Riemannian manifolds that the kernel only consists of constant functions (i.e., L^2 ergodicity holds) if (and only if) the manifold is simply connected. We now state the main result of this note :

Theorem *Under the assumptions on M and x_0 made above, 0 is an accumulation point of the spectrum of the Ornstein–Uhlenbeck operator on $L_{x_0}(M)$.*

Thus there is no spectral gap above 0 for the operator $\mathcal{L}$, and, in particular, a *logarithmic Sobolev inequality* (cf. [2]) w.r.t. the gradient D, the "metric" $\langle \cdot , \cdot \rangle_\omega$ and the measure P_{x_0} does not hold. Before giving an outline of the proof, we comment on possible generalizations of the counterexample, which will be discussed in detail in a forthcoming publication :

a) On the loop spaces considered, the absence of a spectral gap above 0 does not only occur for the metric $\langle \cdot , \cdot \rangle_\omega$ and the associated Ornstein–Uhlenbeck operator. In fact, the arguments outlined below go through for a **broad class of metrics** G on $L_{x_0}(M)$, and the corresponding gradients D_G **and diffusion operators** $D_G^* D_G$.

b) A similar result as stated above holds on the **free loop space** $L(M) = C(S^1 \to M)$. The proof is roughly the same, though some minor modifications are necessary.

c) For simplicity, we have made a rather specific assumption on the manifold M, which can be relaxed considerably.

Remark. The key ingredient in the assumption on M imposed above is the existence of a non–trivial *closed geodesic* on M that is a *local minimum for the length and energy functional* on $C^\infty(S^1 \to M)$. Intuitively, neighbourhoods of closed geodesics which are local energy minima are sets where the diffusion process generated by the Ornstein–Uhlenbeck operator (cf. [5, 9, 6]) spents a "relatively long time" if it is started there. In fact, for n getting large, it will be more and more "difficult" for the process to leave an appropriate neighbourhood of the closed geodesic obtained by winding around a fixed locally minimal closed geodesic n times. Therefore, the process does not decay to equilibrium sufficiently fast, which explains why its generator can not be expected to have a spectral gap above 0. Although the indicated heuristic consideration has played a rôle in constructing the counterexample, the proof does not make direct use of it.

2. Sketch of the proof of the theorem

Let Ω be the open set of all loops in $L_{x_0}(M)$ such that $\omega(s)$ is in U for all $s \in [0,1]$. Since U is homotopy equivalent to S^1, Ω is the disjoint union of its homotopy classes Ω_n, $n \in \mathbf{Z}$, where n denotes the winding number. We will construct functions $F_n \in H^{1,2}(L_{x_0}(M); P_{x_0})$, $n \in \mathbf{N}$, such that $\operatorname{supp} F_n \subset \Omega_n$, and

$$(5) \qquad \lim_{n\to\infty} \mathcal{E}(F_n, F_n) \Big/ \int F_n^2 \, dP_{x_0} \;=\; 0.$$

This implies the claimed assertion, because the functions F_n are orthogonal in $L^2(L_{x_0}(M); P_{x_0})$, and the kernel of $\mathcal{L}$ is one–dimensional by Aida's ergodicity result [1].

Let $\pi : \hat{U} \to U$ be the universal covering of U with induced Riemannian metric and corresponding distance function d. Note that $\hat{U}$ is isometric to a stripe $\{x \in H^2;\ \operatorname{dist}(x, \tilde{\sigma}(\mathbf{R})) < A\}$ in the hyperbolic plane H^2, where $\tilde{\sigma} : \mathbf{R} \to H^2$ is an arbitrary fixed geodesic. In particular, $\hat{U}$ is convex. Now fix a point $\hat{x}_0 \in \pi^{-1}(\{x_0\})$. Then each loop $\omega \in \Omega$ lifts to a unique continuous path $\hat{\omega} : [0,1] \to \hat{U}$ such that $\hat{\omega}(0) = \hat{x}_0$ and $\pi \circ \hat{\omega} = \omega$. For $n \in \mathbf{N}$ let $\hat{\Omega}_n = \{\hat{\omega};\ \omega \in \Omega_n\}$. There exists a point $\hat{x}_n \in \pi^{-1}(\{x_0\})$ such that $\hat{\Omega}_n$ consists of those paths $\hat{\omega}$ on $\hat{U}$ that start in $\hat{x}_0$ and end in $\hat{x}_n$. Let $\hat{\gamma}_n : [0,1] \to \hat{U}$ be the unique minimal geodesic from $\hat{x}_0$ to $\hat{x}_n$ parametrized proportional to arc length. For a point x in the closure of $\hat{U}$ let $r_n(x) = \operatorname{dist}(x, \hat{\gamma}_n([0,1]))$. Furthermore, let $R_n : \Omega_n \to [0,\infty)$ be given by

$$R_n(\omega) \;=\; \sup_{0 \le s \le 1} r_n(\hat{\omega}(s)) \,,$$

and let $B \in (0, A]$ be the distance in M of x_0 from ∂U. We define the function $F_n : L_{x_0}(M) \to [0, 1]$ by

$$(6) \qquad F_n(\omega) \;=\; \left(2 - \frac{3\,R_n(\omega)}{B}\right)^+ \wedge 1 \quad \text{for } \omega \in \Omega_n\,, \quad 0 \text{ else.}$$

In particular, $F_n = 1$ on $\{\omega \in \Omega_n;\, R_n(\omega) \le B/3\}$ and $F_n = 0$ on $\{\omega \in \Omega_n;\, R_n(\omega) \ge 2B/3\}$. Let $P^{(n)}$ denote the restriction of the conditional probability measure $P_{x_0}(\,\cdot\,|\Omega_n)$ to Ω_n.

Lemma *(i) For every $n \in \mathbf{N}$,* $\operatorname{supp} F_n \subset \Omega_n$.
(ii) The functions F_n, $n \in \mathbf{N}$, are in $H^{1,2}(L_{x_0}(M); P_{x_0})$. Moreover,

$$\lim_{n\to\infty} \int_{\Omega_n} \langle DF_n,\, DF_n\rangle_\omega\, P^{(n)}(d\omega) = 0, \quad \textit{and} \qquad \lim_{n\to\infty} P^{(n)}[F_n = 1] = 1.$$

Obviously, the lemma directly implies (5), and hence the assertion. We finally describe the key steps in its proof :

Key steps in the proof of the lemma. (i) Let $n \in \mathbf{N}$. Notice that $\operatorname{dist}(\hat{x}_0, \partial\hat{U}) = \operatorname{dist}(\hat{x}_n, \partial\hat{U}) = B$. By a short geometric consideration, this implies $\operatorname{dist}(\hat{\gamma}_n(s), \partial\hat{U}) \ge B$ for every $s \in [0,1]$, i.e., $r_n \ge B$ on $\partial\hat{U}$. Hence the closure of the set $\{\omega \in \Omega_n;\, R_n(\omega) < 2B/3\}$ in $L_{x_0}(M)$ is contained in Ω_n. (i) follows because F_n vanishes outside this set.

(ii) Using that $\operatorname{supp} F_n \subset \Omega_n$ by (i), a combination of Dirichlet form techniques can be applied to show that F_n is in $H^{1,2}(L_{x_0}(M); P_{x_0})$, DF_n vanishes P_{x_0}–a.e. both outside Ω_n and on $\{\omega \in \Omega_n;\, R_n(\omega) < B/3\}$, and

$$\langle DF_n, DF_n\rangle^{1/2} \;\le\; 3/B \quad P_{x_0}\text{–a.e. on } \Omega_n.$$

We refer to [7] for details. The claim is now a consequence of the following crucial estimate that holds for all $n \in \mathbf{N}$ with a finite constant C independent of n :

$$(7) \qquad P^{(n)}\,[\,R_n \ge B/3\,] \;\le\; C \cdot e^{-n^{1/5}}.$$

For the proof of this estimate, we refer again to the forthcoming publication [7]. It is based on the explicit formula for the heat kernel on H^2 (cf. e.g. [3]) and hyperbolic geometry. ■

References

[1] S. Aida, Differential calculus on path and loop spaces. II. Irredubility of Dirichlet forms on loop spaces, *Bull. Sci. Math.*, **122**, 1998, p. 635–666.

[2] M. Capitaine, E. P. Hsu, M. Ledoux, Martingale representation and a simple proof of logarithmic Sobolev inequalities on path spaces, *Electron. Comm. Probab.*, **2**, 1997, p. 71–81.

[3] E. B. Davies, *Heat kernels and spectral theory*, Cambridge University Press, Cambridge, 1990.

[4] B. Driver, A Cameron–Martin type quasi–invariance theorem for pinned Brownian motion on a compact Riemannian manifold, *Trans. Amer. Math. Soc.*, **342**, 1994, p. 375–395.

[5] B. Driver, M. Röckner, Construction of diffusions on path and loop spaces of compact Riemannian manifolds, *C. R. Acad. Sci. Paris Sér. I Math.*, **315**, 1992, p. 603–608.

[6] A. Eberle, Diffusions on path and loop spaces : Existence, finite–dimensional approximation and Hölder continuity, *Probab. Theory Related Fields*, **109**, 1997, p. 77–99.

[7] A. Eberle, in preparation.

[8] S. Fang, Inégalité du type de Poincaré sur l'espace des chemins riemanniens, *C. R. Acad. Sci. Paris Sér. I Math.*, **318**, 1994, p. 257–260.

[9] T. Kazumi, Le processus d'Ornstein-Uhlenbeck sur l'espace des chemins riemanniens et le problème des martingales, *J. Funct. Anal.*, **144**, 1997, p. 20–45.

[10] P. Malliavin, *Stochastic analysis*, Springer, Berlin, 1997.

[11] M. Spivak, *A comprehensive introduction to differential geometry*, Vol. III, Second edition, Publish or Perish, Inc., Wilmington, Del., 1979.