QF algebras and Gröbner bases of pure binomials

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Introduction. QF (Quasi-Frobenius) algebras were originally studied by R. Brauer [4], T. Nakayama [7] and C. Nesbitt [8],[4]. They are the same with selfinjective algebras and known to be a class of the most interesting non-semisimple algebras containning group algebras and exterior algebras. Commutative local QF algebras are also the same with zero dimensional Gorenstein rings. Cf.[6].

In a connection with vanishing problem of Hochschild's cohomology Q. Zeng [13] has proved that the local algebra $K[x_0, x_1, x_2]/((x_0, x_1, x_2)^4, x_0x_1 - x_2^3, x_1x_2 - x_0^3, x_2x_0 - x_1^3)$ over a field K is QF but has not any positive Z-grading. Cf. also [8].

On a way of generalization of his result we found that $K[x_0, x_1, x_2]/(x_0x_1 - x_2^3, x_1x_2 - x_0^3, x_2x_0 - x_1^3)$ is again artinian and not local, but a direct sum of a group algebra and a local QF-algebra. That is, the new residue class algebra is QF.

In this paper the author would like to mension that the result can be more generalized as follows: For any pair of positive integers n and t, let I be an ideal of a polynomial ring $K[x_0, x_1, \cdots, x_n]$ generated by $n+1$ pure binomials $\prod_{k \in N - \{i\}} x_k - x_i^t$ for $i = 0, 1, \cdots, n$ and $N = \{0, 1, \cdots, n\}$. Then $\Lambda = K[x_0, x_1, \cdots, x_n]/I$ is artinian if and only if $n \neq t$ and if $n \neq t, \Lambda$ is a direct sum of a group algebra and a local QF algebra.

We shall define simple transportations $\lambda_i, \mu_i, \sigma_i$ and $\tau_i, i = 0, 1, \cdots, n$, operating on lattices in $(n+1)$-dimensional Euclidean space. They play important role in our proofs of the existence of monomial idempoftents and the determination of Gröbner bases of definning ideals of local QF algebras. It is interesting that σ_i and τ_i are derived from Buchberger's Algorithm [5]. Cf.[1].

Our local QF algebras are generally not positive Z-graded, while in many papers [3],[10] and [11], etc, local QF algebras were assumed to be positively Z-graded at the beginning. Among our local QF algebras, however, it seems to exist many interesting QF algebras. For example, the associated graded algebras for the case $t = 2$ seem to be a new commutative version of exterior algebras from their Hilbert series.

Further, in the case $n = t$ it does not happen the direct sum decomposition but the local (not finite dimensional K-) algebras are known to be 1-dimensional Gorenstein rings. So our proposed family of rings for any pair of n and t composes a small genealogy of direct sums of Gorenstein rings at most Krull dimension one.

1 Existence of monomial idempotents

Throughout this paper we consider a polynomial ring $K[x_0, x_1, \cdots, x_n]$ of variables $x_0, x_1, \ldots, x_n$ over a field K and an ideal I generated by $n+1$ pure binomials $\prod_{k\in N-\{i\}} x_k - x_i^t$ for $i = 0, 1, \cdots, n$ and $N = \{0, 1, \cdots, n\}$, and a positive integer t. Let us denote $K[x_0, x_1, \cdots, x_n]/I$ by Λ.

For $f \in K[x_0, x_1, \cdots, x_n]$ we denote $f + I \in K[x_0, x_1, \cdots, x_n]/I$ by $\bar{f}$. Then our binomial relations $\prod_{k\in N-\{i\}} x_k - x_i^t$ for $i = 0, 1, \cdots, n$ and $N = \{0, 1, \cdots, n\}$ induce the equalities $\prod_{k\in N-\{i\}} \bar{x}_k = \bar{x}_i^t$ for $i = 0, 1, \cdots, n$.

We shall introduce the following denomination $(a_0, a_1, \cdots, a_n)$ in order to express $\bar{x}_0^{a_0}\bar{x}_1^{a_1}\cdots\bar{x}_n^{a_n} \in \Lambda$. Here we note $(0, 0, \cdots, 0) = \bar{x}_0^0\bar{x}_1^0\cdots\bar{x}_n^0 = \bar{1}$.

Then it follows $(a_0, a_1, \cdots, a_n)^m = (ma_0, ma_1, \cdots, ma_n)$ and $(a_0, a_1, \cdots, a_n)+(b_0, b_1, \cdots, b_n)$ $= \bar{x}_0^{a_0}\bar{x}_1^{a_1}\cdots\bar{x}_n^{a_n} + \bar{x}_0^{b_0}\bar{x}_1^{b_1}\cdots\bar{x}_n^{b_n}$.
We shall call $(a_0, a_1, \cdots, a_n)$ the exponential expression of $\bar{x}_0^{a_0}\bar{x}_1^{a_1}\cdots\bar{x}_n^{a_n} \in \Lambda$ and say $a_0 + a_1 + \cdots + a_n$ the level of $(a_0, a_1, \cdots, a_n)$.
Now our first result is

Proposition 1.1 *If $n < t$, there is a monomial $e = x_0^{s(t-n)}x_1^{s(t-n)}\cdots x_n^{s(t-n)}$ such that $e^2 \equiv e$ mod I, where s is a positive integer satisfying the inequality $n \le s(t-n) < t$. That is, $\bar{x}_0^{s(t-n)}\bar{x}_1^{s(t-n)}\cdots\bar{x}_n^{s(t-n)}$ is an idempotent of Λ.*

Proof: From the equation $\prod_{k\in N-\{i\}} \bar{x}_k = \bar{x}_i^t$ it follows $(a_0, a_1, \cdots, a_i, \cdots, a_n) = (a_0 + 1, \cdots, a_{i-1}+1, a_i-t, a_{i+1}+1, \cdots, a_n+1)$ provided $a_i > t$, where the levels go down from the left to the right in the equality. Thus to any exponential expression $(a_0, a_1, \cdots, a_i, \cdots, a_n)$ we may assume that $t > a_i \ge 0$ for $i = 0, 1, \cdots, n$. It follows $\dim_K \Lambda \le t^n$ because the corresponding monomials generate K-vector space Λ.

For the sake of convenience let us consider a transformation λ_i on $\{Z_+ \cup \{0\}\}^{n+1}$ defined by $\lambda_i((a_0, a_1, \cdots, a_i, \cdots, a_n)) = (a_0+1, \cdots, a_{i-1}+1, a_i-t, a_{i+1}+1, \cdots, a_n+1)$ for any $a_i > t$. Then it is easy to check that $\lambda_n \cdots \lambda_1\lambda_0$ $((a_0, a_1, \cdots, a_i, \cdots, a_n)) = (a_0 - (t-n), a_1 - (t-n), \cdots, a_n - (t-n))$, provided $a_i \geq t$ for $i = 0, 1, \cdots, n$.

Let s be a positive integer satisfying the inequality $n \leq s(t-n) < t$. Then since $t \leq (s+1)(t-n)$ it holds that

$$\begin{aligned} &[\lambda_n \cdots \lambda_1\lambda_0]^s((s(t-n), s(t-n), \cdots, s(t-n))^2)) \\ &= [\lambda_n \cdots \lambda_1\lambda_0]^s((2s(t-n), 2s(t-n), \cdots, 2s(t-n))) \\ &= (s(t-n), s(t-n), \cdots, s(t-n)). \end{aligned}$$

This completes the proof.

We note that the inequality $n \leq s(t-n) < t$ assures the uniqueness of $s(t-n)$.

Throughout this paper we shall call an idempotent $e + I$ of Λ as in Proposition 1.1 a monomial idempotent. Thus Proposition 1.1 shows the existence of a monomial idempotent e of Λ.

In case of $t < n$ we have the following similar

Proposition 1.2. *If $t < n$, there is a monomial $e = x_0^{s(n-t)} x_1^{s(n-t)} \cdots x_n^{s(n-t)}$ such that $e^2 \equiv e$ mod I, where s is a positive integer satisfying the inequality $t \leq s(n-t) < n$.*

Proof: Similarly as in the proof of Proposition 1.1, from $\prod_{k \in N-\{i\}} \bar{x}_k = \bar{x}_i^t$ it follows $(a_0, a_1, \cdots, a_i, \cdots, a_n) = (a_0 - 1, \cdots, a_{i-1} - 1, a_i + t, a_{i+1} - 1, \cdots, a_n - 1)$ for $a_j > 0, j \neq i$, where the levels go down from the left to the right in the equality.

Again for the sake of convenience let us consider a transformation μ_i on $\{Z_+ \cup \{0\}\}^{n+1}$ defined by

$\mu_i((a_0, a_1, \cdots, a_i, \cdots, a_n)) = (a_0 - 1, \cdots, a_{i-1} - 1, a_i + t, a_{i+1} - 1, \cdots, a_n - 1)$ for all $a_j > 0, j \neq i$.

Then it is easy to check that $\mu_n \cdots \mu_1\mu_0$ $((a_0, a_1, \cdots, a_i, \cdots, a_n)) = (a_0 - (n-t), a_1 - (n-t), \cdots, a_n - (n-t))$, provided $a_i \geq i$ for $i = 0, 1, \cdots, n$.

Let s be a positive integer satisfying the inequality $t \leq s(n-t) < n$. then since $n \leq (s+1)(n-t) \leq 2s(n-t)$ it holds that

$$\begin{aligned} &[\mu_n \cdots \mu_1\mu_0]^s((s(n-t), s(n-t), \cdots, s(n-t))^2)) \\ &= [\mu_n \cdots \mu_1\mu_0]^s((2s(n-t), 2s(n-t), \cdots, 2s(n-t))) \\ &= (s(n-t), s(n-t), \cdots, s(n-t)). \end{aligned}$$

This completes the proof.

2 Decomposition theorem

Under the same situation with §1 let us denote $K[x_0, x_1, \cdots, x_n, x_0^{-1}, x_1^{-1}, \cdots, x_n^{-1}]$ $(\subset K(x_0, x_1, \cdots, x_n))$ by Γ and ΓI by J respectively. Then we can define an algebra homomorphism $\theta : \Lambda \to \Gamma/J$ by $\theta(f + I) = f + J$ for $f \in K[x_0, x_1, \cdots, x_n]$.

Lemma 2.1. *If there is a monomial* $e = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$ *such that* $a_i \geq 1, i = 0, 1, \cdots, n$ *and* $e^2 \equiv e$ *mod* I*, then* $\theta : \Lambda \to \Gamma/J$ *is a split epimorphism.*

Proof: Since $\frac{1}{e} \in \Gamma$, $\frac{1}{e}(e - e^2) = 1 - e \equiv 0 \bmod J$. Hence $x_j^{-1}(1 - e) \equiv 0 \bmod J$. Therefore $x_j^{-1}e \equiv x_j^{-1} \bmod J$, where $x_j^{-1}e \in K[x_0, x_1, \cdots, x_n], x_j^{-1} \in K[x_0, x_1, \cdots, x_n, x_0^{-1}x_1^{-1}, \cdots, x_n^{-1}]$. This implies that $\theta(x_j^{-1}e + I) = x_j^{-1} + J$ and hence θ is an epimorphism. Further if we put $\Lambda_1 = \{ef + I | f \in K[x_0, x_1, \cdots, x_n]\}$, the restriction $(\theta|\Lambda_1) : \Lambda_1 \to \Gamma/J$ is an epimorphism. Because $(1 - e)f \equiv 0 \bmod J$, and hence $f \equiv ef \bmod J$.

Assume $\theta(ef + I) = 0 + J$. Then $ef \equiv 0 \bmod J$. This implies $ef \in J$. But since $e = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$, $a_i \geq 1, i = 0, 1, \cdots, n$, there exists a positive integer r such that $e^r f \in I$. Hence $ef \equiv e^r f \equiv 0 \bmod I$. This implies $(\theta|\Lambda_1)$ is a monomorphism.

By Proposition 1.1, 1.2 and Lemma 2.1 we have a direct sum decomposition of $\Lambda : \Lambda_0 \oplus \Lambda_1$ if $n \neq t$, where $\Lambda_1 = \{ef + I | f \in K[x_0, x_1, \cdots, x_n]\} \simeq \Gamma/J$, $\Lambda_0 = \{(1 - e)f + I | f \in K[x_0, x_1, \cdots, x_n]\}$ and $e = x_0^{s|n-t|} x_1^{s|n-t|} \cdots x_n^{s|n-t|}$.

Proposition 2.2. *If* $n \neq t$*, then* Λ_1 *is isomorphic to a group algebra* KG *of a group* $G \simeq Z_{t+1} \times Z_{t+1} \times \cdots \times Z_{t+1} \times Z_{(t+1)|n-t|}$*, where* Z_{t+1} *and* $Z_{(t+1)|n-t|}$ *are cyclic groups of order* $t+1$ *and* $(t+1)|n-t|$ *respectively and* $Z_{t+1} \times Z_{t+1} \times \cdots \times Z_{t+1}$ *is a direct product of* $(n-1)$ *copies of* Z_{t+1}*.*

Proof: We shall try at first to extend the denomination of exponential expression for monomials of Λ to ones of Γ/J allowing negative coordinates as follows: For $r_i \in Z$, $(r_0, r_1, \cdots, r_n) \equiv x_0^{r_0} x_1^{r_1} \cdots x_n^{r_n} \bmod J$.
Since $\{(\prod_{k \in N - \{i\}} x_k) x_i^{-t} - 1 \mid i = 0, 1, \cdots, n\}$ is a system of generators of J, we shall consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & -t \\ -t & 1 & \cdots & 1 & 1 & 1 \\ 1 & -t & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & -t & 1 & 1 \\ 1 & 1 & \cdots & 1 & -t & 1 \end{pmatrix},$$

where the i-th row follows from the extended exponential expression of $1 \equiv (\prod_{k\in N-\{i\}} x_k)x_i^{-t}$ mod J which corresponds to an element of the above system of generators of J. We know that each elementary transformation of rows and columns of A corresponds a transformation of system of generators of J. By a calculation we obtain the matrix

$$B = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & (1+t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & (1+t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (t+1) & 0 \\ 0 & 0 & 0 & \cdots & 0 & (1+t)(n-t) \end{pmatrix},$$

after successive elementary transformations of rows and columns in A. Thus we can take $\{y_0^1 - 1, y_1^{1+t} - 1, \cdots, y_{n-1}^{1+t} - 1, y_n^{(1+t)|n-t|} - 1\}$ as a system of generators of J for a new indeterminates $y_0, y_1, \cdots, y_n, y_0^{-1}, y_1^{-1}, \cdots, y_n^{-1}$ of $K[x_0, x_1, \cdots, x_n, x_0^{-1}, x_1^{-1}, \cdots, x_n^{-1}]$. Now it is clear that $\Lambda_1 \simeq K[x_0, x_1, \cdots, x_n, x_0^{-1}, x_1^{-1}, \cdots, x_n^{-1}]/J \simeq$ $K[y_0, y_1, \cdots, y_n, y_0^{-1}, y_1^{-1}, \cdots, y_n^{-1}]/J$ which is isomorphic to a group algebra over an abelian group $G \simeq Z_{t+1} \times Z_{t+1} \times \cdots \times Z_{t+1} \times Z_{(t+1)|n-t|}$, where Z_{t+1} and $Z_{(t+1)|n-t|}$ are cyclic groups of order $t+1$ and $(t+1)|n-t|$ respectively, and $Z_{t+1} \times Z_{t+1} \times \cdots \times Z_{t+1}$ is a direct product of $(n-1)$ copies of Z_{t+1}. This completes the proof.

Corollary 2.3. *If $n = t$, then Λ_1 is not artinian.*

Further we have

Lemma 2.4. *If $n \neq t$, $\Lambda_0 = (\bar{1} - \bar{e})\Lambda$ is a local artinian algebra.*

Proof: For any $j, x_j^{t+1} \equiv x_0 x_1 \cdots x_n$ mod I. Since $(\bar{1} - \bar{e}) \in \Lambda$ is an idempotent, $\{(1-e)x_j\}^{(t+1)} \equiv (1-e)x_0 x_1 \cdots x_n$ mod I. Hence $\{(1-e)x_j\}^{(t+1)s|t-n|} \equiv (1-e)e \equiv 0$ mod I. Thus $(\bar{1} - \bar{e})\ \bar{x_j}$, $j = 0, 1, \cdots, n$ are nilpotent. This completes the proof.

Now we have one of the main results.

Theorem 2.5. *If $t \neq n$, Λ is isomorphic to a direct sum of finite dimensional local algebra* $\Lambda_0 = \Lambda(\bar{1} - \bar{x}_0^{s|t-n|}\bar{x}_1^{s|t-n|}, \cdots, \bar{x}_n^{s|t-n|})$ *and a group algebra KG, where G is a direct product of $(n-1)$ copies of a cyclic group Z_{t+1} and a cyclic group $Z_{(t+1)|t-n|}$.*

3 Injectivity of Λ_0 and Gröbner bases

In this section we shall prove that the local algebra Λ_0 in Theorem 2.5 is injective. Since $\bar{e}$ and $(\bar{1} - \bar{e})$ are orthogonal to each other, $\Lambda_0 = \Lambda(\bar{1} - \bar{e}) \simeq \Lambda/\Lambda\, \bar{e}$.
Hence Λ_0 is isomorphic to a residue class algebra of $K[x_0, x_1, \cdots, x_n]$ by the ideal $I_0 = (x_0^{s|t-n|}x_1^{s|t-n|} \cdots x_n^{s|t-n|}, \prod_{k \in N-\{i\}} x_k - x_i^t \mid i = 0, 1, \cdots, n)$, because $e \equiv x_0^{s|t-n|}x_1^{s|t-n|} \cdots x_n^{s|t-n|}$ mod I and $I = (\prod_{k \in N-\{i\}} x_k - x_i^t \mid i = 0, 1, \cdots, n)$.

In order to determine a Gröbner bases of I_0 we shall introduce the following degree lexicographical order for the set of all power products (= monomials of $K[x_0, x_1, \cdots, x_n]$ with the coefficent 1) with $x_0 < x_1 < \cdots < x_n$: $x_0^{a_0}x_1^{a_1} \cdots x_n^{a_n} < x_0^{b_0}x_1^{b_1} \cdots x_n^{b_n}$ if and only if $a_0 + a_1 + \cdots + a_n < b_0 + b_1 + \cdots + b_n$ or $a_0 + a_1 + \cdots + a_n = b_0 + b_1 + \cdots + b_n$ and $a_{i+1} = b_{i+1}, a_{i+2} = b_{i+2}, \cdots, a_n = b_n$, but $a_i < b_i$ for some i.

For the sake of reader's convenience we shall introduce here several notations and results on Gröbner bases which will be used in after proofs.

Let us denote the leading term of a polynomial f by $\mathrm{Lt}(f)$, i.e., when $\mathrm{Lt}(f)$ is expressed as a product of coefficent $\mathrm{Lc}(f)$ and power product $\mathrm{Lp}(f)$, the order of $\mathrm{Lp}(f)$ is the largest among power products of terms of f with respect the degree lexicographical order.

Then a Gröbner bases $G = \{g_1, g_2, \cdots, g_m\}$ of an ideal Q of $K[x_0, x_1, \cdots, x_n]$ is a set of polynomials in Q such that $\{$ $\mathrm{Lt}(g_1)$,$\mathrm{Lt}(g_2)$,$\cdots$,$\mathrm{Lt}(g_m)$ $\}$ generates the ideal generated by all the leading power products of polynomials in Q. Hence if $f \in Q$ then there is some i such that $\mathrm{Lt}(g_i)$ divides $\mathrm{Lt}(f)$.

Let S be a set of polynomials and g an element of S. If $\mathrm{Lp}(g)$ divides a term X of f, then "$f \xrightarrow{g}_+ h$" means $h = f - \frac{X}{\mathrm{Lt}(g)}g$ and we say that f reduces h modulo g. Further if there exists a series of polynomials $g_1, g_2, \cdots, g_l \in S$ for some l such that "$f \xrightarrow{g_1}_+ h_1$", "$h_1 \xrightarrow{g_2}_+ h_2$", $\cdots$, "$h_{l-1} \xrightarrow{g_l}_+ h$" , then we say that f reduces h modulo S and use the notation "$f \xrightarrow{S}_+ h$".

For polynomials f and g, put $S(f, g) = \dfrac{C}{\mathrm{Lt}(f)}f - \dfrac{C}{\mathrm{Lt}(g)}g$, where C is the least common multiple of $\mathrm{Lt}(f)$ and $\mathrm{Lt}(g)$. Then Buchberger's theorem is stated as follows:

A set G of polynomials in Q is a Gröbner bases of Q if and only if for any pair of $g_i \in G$ and $g_j \in G$, $S(g_i, g_j) \xrightarrow{G}_+ 0$.

There is the following algorithm to obtain a Gröbner bases:

If a set S_0 of polynomials in an ideal Q is not a Gröbner bases of Q, then take a pair of $g_i \in Q$ and $g_j \in Q$ such that $S(g_i, g_j)$ is not reduced to zero modulo S_0 (the above Buchberger's theorem assures us the existence of such pair (g_i, g_j)) *and construct a set $S_1 = S_0 \cup \{S(g_i, g_j)\}$. If S_1 is again not a Gröbner bases of Q, construct a set S_2 by the similar way in which we take S_1 in place of S_0. Repeating such constructions we obtain finally a Gröbner bases of Q.*

This is the Buchberger's algorithm [5]. For the further detailes of the Buchberger's algorithm we shall refer to [1].

Here we would like to mension that in order that the reader can check our calculations, we do not abbreviate each step of Buchberger's algorithm, even if it is very simple and reader may feel to check it tediously.

From now on, following the Buchberger's algorithm we shall determine a Gröbner bases of I_0 for cases $n < t$ and $t < n$ separately.
In any case, however, we shall try at first to obtain Gröbner basis of I.

We shall start to consider the case $n < t$.

Let us denote $x_{i+1}x_{i+2}\cdots x_n x_0 x_1 \cdots x_{i-1} - x_i^t$ by f_i, provided when $i = n$, we put $i+1 = 0$, and $\{f_0, f_1, \cdots, f_n\}$ by F respectively. Then with respect to the degree lexicographical order, the leading term $\mathrm{Lt}(f_i)$ of f_i is x_i^t. Then from $(\mathrm{Lt}\ (f_i), \mathrm{Lt}(f_j)) = 1$ for $i \neq j$ it follows that the $\mathrm{S}(f_i, f_j) \xrightarrow{F}_+ 0$, and Buchberger's theorem induces

Proposition 3.1. *F is itself a Gröbner bases of I (= the ideal generated by F).*

Put $g = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n}$. If $a_i \leq t$, then $\mathrm{S}(f_i, g) =$
$x_0^{a_0} x_1^{a_1} \cdots x_{i-1}^{a_{i-1}} x_{i+1}^{a_{i+1}} \cdots x_n^{a_n} (x_{i+1}x_{i+2} \cdots x_n x_0 x_1 \cdots x_{i-1} - x_i^t) + x_i^{t-a_i} x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} =$
$x_0^{a_0+1} x_1^{a_1+1} \cdots x_{i-1}^{a_{i-1}+1} x_i^0 x_{i+1}^{a_{i+1}+1} \cdots x_n^{a_n+1}$. On the other hand, if $t < a_i$, then $S(f_i, g) =$
$x_0^{a_0} x_1^{a_1} \cdots x_i^{a_i - t} \cdots x_n^{a_n} (x_{i+1}x_{i+2} \cdots x_n x_0 \cdots x_{i-1} - x_i^t) + x_{0]}^{a_0} x_1^{a_1} \cdots x_n^{a_n} =$
$x_0^{a_0+1} x_1^{a_1+1} \cdots x_{i-1}^{a_{i-1}+1} x_i^{a_i - t} x_{i+1}^{a_{i+1}+1} \cdots x_n^{a_n+1}$

Now we shall introduce a transportation σ_i operating on $\{Z_+ \cup \{0\}\}^{n+1}$ such that σ_i (exponential expression of g) = exponential expression of $S(f_i, g)$.
Thus if we put $\sigma_i(x_0^{a_0} \cdots x_i^{a_i} \cdots x_n^{a_n}) = (x_0^{b_0} \cdots x_i^{b_i} \cdots x_n^{b_n})$, it holds that

$$b_i = \begin{cases} a_i - t & \text{if } a_i > t \\ a_i = 0 & \text{if } a_i \leq t \end{cases}$$

$$b_j = \quad a_j + 1 \quad \text{if} \quad j \neq i.$$

Then we have

Lemma 3.2. *Let s be an integer satisfying the inequality $n \leq s(t-n) < t$. Then* $[\sigma_n \cdots \sigma_1\sigma_0]^k((s(t-n), s(t-n), \cdots, s(t-n))) =$
$(n, n-1, \cdots, (s-k)(t-n)+1, (s-k)(t-n), (s-k)(t-n), \cdots, (s-k)(t-n))$
for $1 \leq k \leq s$.

Proof: We shall prove this Lemma by induction on k. Put $r = t - s(t-n)$. Then by the assumption $0 < r$. Put $[\sigma_i \cdots \sigma_1\sigma_0]((s(t-n), s(t-n), \cdots, s(t-n))) = (a_0, a_1, \cdots, a_j, \cdots, a_n)$ for $0 \leq i < r$, then $a_{i+1} \leq t$. Hence by the definitions of σ_{i+1} for all i such that $0 \leq i < r$, we have $[\sigma_r \cdots \sigma_1\sigma_0]((s(t-n), s(t-n), \cdots, s(t-n))) =$
$((r, r-1, \cdots, 1, 0, s(t-n)+r+1, \cdots, s(t-n)+r+1)$.

Consequently $[\sigma_{r+1}\sigma_r \cdots \sigma_1\sigma_0]((s(t-n), s(t-n), \cdots, s(t-n))) =$
$((r+1, r, \cdots, 2, 1, 1, s(t-n)+r+2, \cdots, s(t-n)+r+2)$, because $t < s(t-n)+r+1$. Further, since $n-r = (s-1)(t-n)$, we have finally $[\sigma_n \cdots \sigma_1\sigma_0]((s(t-n), s(t-n), \cdots, s(t-n))) = (n, n-1, \cdot, (s-1)(t-n)+1, (s-1)(t-n), (s-1)(t-n), \cdots, (s-1)(t-k))$ and this proves that Lemma 3.2 is true for $k = 1$.

Now assume Lemma 3.2 is true for $k = l$. Put $j = n - (s-l)(t-n)$. Similarly as in the proof of the case $k = 1$, applying first the definition of σ_i for the case $a_i \leq t$ repeatedly we have $[\sigma_{j+(t-n)-1} \cdots \sigma_1\sigma_0][\sigma_n \cdots \sigma_1\sigma_0]^l((s(t-n), s(t-n), \cdots, s(t-n))) =$
$(j+(t-n)-1, \cdots, 2, 1, 0, n+(t-n), n+(t-n), \cdots, n+(t-n))$.

Hence it follows $[\sigma_{j+(t-n)} \cdots \sigma_1\sigma_0][\sigma_n \cdots \sigma_1\sigma_0]^l((s(t-n), s(t-n), \cdots, s(t-n))) =$
$(j+(t-n), \cdots, 3, 2, 1, 0, t+1, t+1, \cdots, t+1)$.

However $j+(t-n) = n - \{s-(l+1)\}(t-n)$. Hence by applying the definition of σ_i for the case $t < a_i$ repeatedly, we have finally $[\sigma_n \cdots \sigma_1\sigma_0][\sigma_n \cdots \sigma_1\sigma_0]^l((s(t-n), s(t-n), \cdots, s(t-n))) = (n, n-1, \cdots, \{s-(l+1)\}(t-1)+1, \{s-(l+1)\}(t-1), \{s-(l+1)\}(t-1), \cdots, \{s-(l+1)\}(t-1))$. This shows that Lemma 3.2 is true for $k = l+1$. This completes the proof.

By Lemma 3.2 we have

Corollary 3.3. *A monomial $x_0^n x_1^{n-1} \cdots x_{n-1}^1 x_n^0$ appears after s times of S-polynomial making processes $S(f_n, S(f_{n-1}, \cdots, S(f_0, -)\cdots))$ to $(x_0x_1\cdots x_n)^{s(t-n)}$.*

Hence $x_0^n x_1^{n-1} \cdots x_{n-1}^1 x_n^0$ can be considered as an element of Gröbner bases of I_0.

Further we have

Corollary 3.4. $H = \{x_{i_1}^n x_{i_2}^{n-1} \cdots x_{i_n} | i_j = 0, 1, \cdots, n and i_j \neq i_k if j \neq k\}$ *is a subset of Gröbner bases of I_0.*

Proof. Let $(c_0, c_1, \cdots, c_n)$ be an exponential expression of a monomial $x_{i_1}^n x_{i_2}^{n-1} \cdots x_{i_n}$. Assume $c_n \neq 0$ and denote c_n by r. Then, since $\{c_0, c_1, \cdots, c_{n-1}, c_n\} = \{0, 1, \cdots, n-1, n\}$, there is $k_1\,(k_1 \neq n)$ such that $c_{k_1} = 0$. Put ${}^1c_i = c_i - 1$ for all $i \neq k_1$ and ${}^1c_{k_1} = n$. Then ${}^1c_n = r-1$ and it holds that $\sigma_{k_1}({}^1c_0, {}^1c_1, \cdots, {}^1c_{n-1}, {}^1c_n) = (c_0, c_1, \cdots, c_{n-1}, c_n)$, since $n < t$. Assume further ${}^1c_n = r - 1 \neq 0$, then similarly there is $k_2\,(k_2 \neq n)$ such that ${}^1c_{k_2} = 0$ and by putting ${}^2c_i = {}^1c_i - 1$ for all $i \neq k_2$ we have $\sigma_{k_2}({}^2c_0, {}^2c_1, \cdots, {}^2c_{n-1}, {}^2c_n) = ({}^1c_0, {}^1c_1, \cdots, {}^1c_{n-1}, {}^1c_n)$ and ${}^2c_n = r-2$.
Therefore by repeating the similar processes we arrive at an exponential expression of a monomial $(a_0, a_1, \cdots, a_{n-1}, 0)$ such that $(c_0, c_1, \cdots, c_{n-1}, c_n)$ can be obtained by applying a suitable sequence of σ_k's (= S-polynomial making processes S(f_k,-)) to $(a_0, a_1, \cdots, a_{n-1}, 0)$ and $\{a_0, a_1, \cdots, a_{n-1}\} = \{1, 2, \cdots, n\}$.

So by Corollary 3.3 it is enough to prove that there is a sequence of σ_k's such that $(a_0, a_1, \cdots, a_{n-1}, 0)$ is obtained by applying them to $(n, n-1, \cdots, 1, 0)$.
Now Put $\theta_i = \sigma_j, \theta_j = \sigma_i$ for $0 \leq i < j \leq n$ and $\theta_k = \sigma_k$ for $k \neq i, j$ and $0 \leq k \leq n$. Then since $n < t$ it holds that $[\theta_n \theta_{n-1} \cdots \theta_j \cdots \theta_i \cdots \theta_0 \sigma_i]((n, n-1, \cdots, n-i, \cdots, n-j, \cdots, 1, 0)) = (n, n-1, \cdots, n-j, \cdots, n-i, \cdots, 1, 0)$, where it happens transposition of i-th coordinate and j-th coordinate in the above exponential expressions.

As any permutation is expressed as a product of transpositions the last equality is enough to complement the remainning proof.

Now by caluculations we can prove that for any $i, S(x_{i_1}^n x_{i_2}^{n-1} \cdots x_{i_n}, f_i) \xrightarrow{H}_+ 0$.

Hence by Buchberger's theorem we conclude
Proposition 3.5. *If $n < t$, then $F \cup H$ is a Gröbner bases of I_0.*

For $f \in K[x_0, x_1, \cdots, x_n]$ let us denote $f + I_0 \in$ by $\widetilde{f}$. Now we can prove

Proposition 3.6. *If $n < t$, then the socle of Λ_0 is a simple Λ_0- module generated by $(\widetilde{x}_0 \widetilde{x}_1 \cdots \widetilde{x}_n)^{n-1}$ and Λ_0 is injective.*

Proof: A bases of the K-vector space $\Lambda_0 (= K[x_0, x_1, \cdots, x_n]/I_0)$ can be chosen as the cosets containning the power products which are not divided by all the leading terms of polynomials in $F \cup H$. Cf.[1, Proposition 2.1.6]
Now let $(c_0, c_1, \cdots, c_n)$ be an exponential expression of a monomial X and assume $c_0 \geq c_1 \geq \cdots \geq c_n$. If $n-2 \geq c_0$, then $\widetilde{x}_0\ \widetilde{X} \neq \widetilde{0}$ because $x_0 X$ is not divided all the leading terms of polynomials in $F \cup H$. Hence if $\widetilde{X} \in \mathrm{Soc}(\Lambda_0)$,$c_0 \geq n-1$. As I_0 is defined by the relation which is symmetry for variables $x_0, x_1, \cdots, x_n$, we can assume $c_i \geq n-1$ for $i = 0, 1, \cdots, n$.

However $\widetilde{X}=\widetilde{0}$ if $c_j = n$ for any j. Consequently we have $c_0 = c_1 = \cdots = c_n = n-1$ if and only if $\widetilde{X}\in \mathrm{Soc}(\Lambda_0)$.
It follows also that $\widetilde{x}_{i_0}^{c_0}\widetilde{x}_{i_1}^{c_1}\cdots\widetilde{x}_{i_n}^{c_n} \neq \widetilde{0}$ if $c_j \leq n-1$ for $j = 0, 1, \cdots, n$.

By considering generators of I_0, i.e., especially elements of F and H, for $t-1 = t_0 \geq t_1 \geq \cdots \geq t_{n-2} \geq t_{n-1} \geq t_n \geq 0$ and $c_0 \neq t_0$, $\widetilde{x}_{i_0}(\widetilde{x}_{i_0}^{t_0}\widetilde{x}_{i_1}^{t_1}\cdots\widetilde{x}_{i_n}^{t_n}) = \widetilde{x}_{i_0}(\widetilde{x}_{i_0}^{c_0}\widetilde{x}_{i_1}^{c_1}\cdots\widetilde{x}_{i_n}^{c_n})$ if and only if there are integers k and l such that $2 \leq k \leq l \leq (n-1), t_k < t-k, t_l < n-l$ and $n-j \leq t_j$ for all $0 < j < k$. So it holds $t_0 = t-1, t_1 = t-1, t_2 \geq t-2, \cdots, t_{k-1} \geq t-(k-1), t-k > t_k = c_k - k, t-k > t_{k-1} = c_{k+1} - k, \cdots, n-l > t_l = c_l - k, \cdots, n-l > t_n = c_n - k$.
In this case for $c_0 = k-2, c_1 \geq k-2, c_2 \geq k-3, \cdots, c_{k-1} \geq k-k = 0, t > c_k \geq k, \cdots, n+k-l > c_l \geq k, \cdots, n+k-l > c_n \geq k$ there corresponds conversely $t_0 = t-1 = c_0-k+t+1, t_1 = c_1 - k + t + 1, t_2 = c_2 - k + t + 1, \cdots, t_{k-1} = c_{k-1} - k + t + 1, t_k = c_k - k, \cdots, t_n = c_n - k$.

We remark here that by Proposition 3.5 the above residue classes $\widetilde{x}_{i_0}^{t_0}\widetilde{x}_{i_1}^{t_1}\cdots\widetilde{x}_{i_n}^{t_n}$ and $\widetilde{x}_{i_0}^{c_0}\widetilde{x}_{i_1}^{c_1}\cdots\widetilde{x}_{i_n}^{c_n}$ can be chosen as elements of K-bases of Λ_0 and $\widetilde{x}_{i_k}(\widetilde{x}_{i_0}^{t_0}\widetilde{x}_{i_1}^{t_1}\cdots\widetilde{x}_{i_n}^{t_n}) \neq \widetilde{0}$. However there is no element $\widetilde{x}_{i_0}^{c_0'}\widetilde{x}_{i_1}^{c_1'}\cdots\widetilde{x}_{i_n}^{c_n'}$ such that $\widetilde{x}_{i_k}(\widetilde{x}_{i_0}^{t_0}\widetilde{x}_{i_1}^{t_1}\cdots\widetilde{x}_{i_n}^{t_n}) = \widetilde{x}_{i_k}(\widetilde{x}_{i_0}^{c_0'}\widetilde{x}_{i_1}^{c_1'}\cdots\widetilde{x}_{i_n}^{c_n'})$.

Now let us denote $\{((i_0, i_1, \cdots, i_n),(t_0, t_1, \cdots, t_n)) \mid t_0 = t-1, t_1 = t-1, t_2 \geq t-2, \cdots, t_{k-1} \geq t-(k-1), t-k > t_k, t-k > t_{k-1}, \cdots, n-l > t_l, \cdots, n-l > t_n \text{for } n-1 \geq k \geq 2\}$ by $T_{k,l}$, the corresponding c_j by $\phi\ (t_j)$ for $j = 0, 1, \cdots, n$ and $\{((i_0, i_1, \cdots, i_n), (\phi(t_0), \phi(t_1), \cdots, \phi(t_n)))\}$ by $C_{k,l}$ respectively.

Further denote $\{((i_0, i_1, \cdots, i_n),(d_0, d_1, \cdots, d_n)) \mid \widetilde{x}_{i_0}^{d_0}\widetilde{x}_{i_1}^{d_1}\cdots\widetilde{x}_{i_n}^{d_n}$ is an element of a K-bases of Λ_0 $\}$ by B. Then an element $\widetilde{Y}$ of Λ_0 is written by

$$\sum_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))\in B} a_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))}\widetilde{x}_{i_0}^{d_0}\widetilde{x}_{i_1}^{d_1}\cdots\widetilde{x}_{i_n}^{d_n}, \text{ where } a_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))} \in K.$$

Assume $\widetilde{Y}\in \mathrm{Soc}(\Lambda_0)$. Then $\widetilde{x}_{i_0}\ \widetilde{Y} = \widetilde{0} =$

$$\sum_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))\in B} a_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))}\widetilde{x}_{i_0}^{d_0+1}\widetilde{x}_{i_1}^{d_1}\cdots\widetilde{x}_{i_n}^{d_n} =$$

$$\sum_{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n))\in B\cap T_{k,l}} (a_{((i_0,i_1,\cdots,i_n),(\phi(t_0),\phi(t_1),\cdots,\phi(t_n)))} + a_{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n))})\widetilde{x}_{i_0}^{t_0+1}\widetilde{x}_{i_1}^{t_1}\cdots$$

$$\widetilde{x}_{i_n}^{t_n} + \sum_{((i_0,i_1,\cdots,i_n),(t_0',t_1',\cdots,t_n'))\in B-(T_{k,l}\cup C_{k,l})} a_{((i_0,i_1,\cdots,i_n),(t_0',t_1',\cdots,t_n'))}\widetilde{x}_{i_0}^{t_0'+1}\widetilde{x}_{i_1}^{t_1'}\cdots\widetilde{x}_{i_n}^{t_n'}.$$

Therefore $a_{((i_0,i_1,\cdots,i_n),(\phi(t_0),\phi(t_1),\cdots,\phi(t_n)))} = -a_{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n))}$ for $((i_0, i_1, \cdots, i_n), (t_0, t_1, \cdots, t_n)) \in B \cap T_{k,l}$.
However by the preceded remark $a_{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n))} = 0$ follows from $\widetilde{x}_{i_k}\widetilde{Y} = \widetilde{0}$.

And $a_{((i_0,i_1,\cdots,i_n),(t_0',t_1',\cdots,t_n'))} = 0$ for $((i_0, i_1, \cdots, i_n), (t_0', t_1', \cdots, t_n')) \in B - (T_{k,l} \cup C_{k,l})$ if $\widetilde{x}_{i_0}^{t_0'+1}\widetilde{x}_{i_1}^{t_1'}\cdots\widetilde{x}_{i_n}^{t_n'} \neq \widetilde{0}$.

Consequently $a_{((i_0,i_1,\cdots,i_n)(d_0,d_1,\cdots,d_n))} = 0$ for $((i_0,i_1,\cdots,i_n)(d_0,d_1,\cdots,d_n)) \in B$ if $\widetilde{x}_{i_0}^{d_0+1}\widetilde{x}_{i_1}^{d_1}\cdots\widetilde{x}_{i_n}^{d_n} \neq \widetilde{0}$. As $(\widetilde{x}_{i_0},\widetilde{x}_{i_1},\cdots,\widetilde{x}_{i_n})$ is any permutation of $(\widetilde{x}_0,\widetilde{x}_1,\cdots,\widetilde{x}_n)$, from the beginning part of this proof we can conclude that $(\widetilde{x}_0\widetilde{x}_1\cdots\widetilde{x}_n)^{n-1}$ is only a generator of $\mathrm{Soc}(\Lambda_0)$ as a K-vector space. As $K \simeq \Lambda_0/\operatorname{Rad}(\Lambda_0)$, $\mathrm{Soc}(\Lambda_0)$ is simple Λ_0-module. Hence Λ_0 is injective, as Λ_0 is local. Cf.[2]

Now we shall begin to consider the case $t < n$.

In this case, with respect to the degree lexicographical order, the leading term $\mathrm{Lt}(f_i)$ of f_i $(= x_{i+1}x_{i+2}\cdots x_nx_0x_1\cdots x_{i-1} - x_i^t)$ is $x_{i+1}x_{i+2}\cdots x_nx_0x_1\cdots x_{i-1}$ and $S(f_i,f_j) = x_j^{t+1} - x_i^{t+1}$.

Denote $x_j^{t+1} - x_i^{t+1}$ by $d_{i,j}$ and $\{d_{i,j} | i < j \text{ and } i = 0,1,\cdots,n-1, j = 1,2,\cdots,n\}$ by D respectively. Then $\mathrm{Lt}(d_{i,j}) = x_j^{t+1}$. Hence for $i < j,\ k < l$, $S(d_{i,j},d_{k,l}) \xrightarrow{D}_+ 0$.
Further for $j = k, i < j$, $S(f_k,d_{i,j}) \xrightarrow{F\cup D}_+ 0$, since $(\mathrm{Lt}(f_k),\mathrm{Lt}(d_{i,j})) = 1$. Also for $i = k, i < j, S(f_k,d_{i,j}) \xrightarrow{F}_+ 0$.

On the other hand, for $0 \le i < j \le n$ and $i \neq k, j \neq k$, $S(f_k,d_{i,j}) = x_i^{t+1}(\prod_{l\in N-\{j,k\}} x_l) - x_j^tx_k^t$, where $N = \{0,1,\cdots,n\}$ is not reduced to 0 by $F \cup D$.

Now denote $S(f_k,d_{i,j}) = x_i^{t+1}(\prod_{l\in N-\{j,k\}} x_l) - x_j^tx_k^t$ by $g_{k;i,j}$ and $\{g_{k;i,j} | k,i,j = 0,1,\cdots,n,$ such that $i < j$ and $i \neq k, j \neq k$ $\}$ by G_1 respectively.
Then $\mathrm{Lt}(g_{k;i,j}) = x_i^{t+1}(\prod_{l\in N-\{j,k\}} x_l)$ and it holds that $S(g_{p;q,r},g_{k;i,j}) \xrightarrow{D}_+ 0$ for any pair of $\{p;q,r\}$ and $\{k;i,j\}$. Further for any pair of l and $\{k;i,j\}$ we have $S(f_l,g_{k;i,j}) \xrightarrow{D}_+ 0$.

However in the case where $i < j$, $p < q$ and $\{i,p\} \cap \{j\} \cap \{q\} \cap \{k\} = \phi$, $S(g_{k;i,j},d_{p,q}) = x_i^{t+1}x_p^{t+1}(\prod_{l\in N-\{j,k,q\}} x_l) - x_j^tx_k^tx_q^t$ is not reduced to 0 by $F \cup D \cup G_1$.
Because $x_i^{t+1}x_p^{t+1}(\prod_{l\in N-\{j,k,q\}} x_l)$ is not divided by $\mathrm{Lt}(f_u)(= \prod_{l\in N-\{u\}} x_l)$ and $\mathrm{Lt}(g_{r;p;q})(= x_p^{t+1}(\prod_{s\in N-\{q,r\}} x_s))$. If $x_i^{t+1}x_p^{t+1}(\prod_{l\in N-\{j,k,q\}} x_l)$ is divided by $\mathrm{Lt}(d_{u,v})(= x_v^{t+1})$, $v = i$ or p. Then by putting u = 0 we have $x_i^{t+1}x_p^{t+1}(\prod_{l\in N-\{j,k,q\}} x_l) \xrightarrow{D}_+ x_0^{t+1}x_p^{t+1}(\prod_{l\in N-\{j,k,q\}} x_l)$ or $x_i^{t+1}x_0^{t+1}(\prod_{l\in N-\{j,k,q\}} x_l) \xrightarrow{D}_+ x_0^{t+1}x_0^{t+1}(\prod_{l\in N-\{j,k,q\}} x_l)$. But $x_0^{t+1}x_0^{t+1}(\prod_{l\in N-\{j,k,q\}} x_l)$ is not divided by $\mathrm{Lt}(d_{u,v})$. on the other hand $x_j^tx_k^tx_q^t$ is clearly not divided by $\mathrm{Lt}(f_u)$, $\mathrm{Lt}(d_{u,v})$ and $\mathrm{Lt}(g_{r;p,q})$.

Now we shall define inductively $g_{k;p_1,p_2,\cdots,p_r,q_1,q_2,\cdots,q_r}$ by putting $g_{k;p_1,p_2,\cdots,p_r,q_1,q_2,\cdots,q_r} = S(g_{k;p_1,p_2,\cdots,p_{r-1},q_1,q_2,\cdots,q_{r-1}},d_{p_r,q_r})$. Then $g_{k;p_1,p_2,\cdots,p_r,q_1,q_2,\cdots,q_r} = \prod_{p\in P} x_p^{t+1}\prod_{r\in N-\{Q\cup\{k\}\}} x_r - \prod_{q\in Q\cup\{k\}} x_q^t$, where $P = \{p_1,p_2,\cdots,p_r\} \subset N = \{0,1,\cdots,n\}$, $Q = \{q_1,q_2,\cdots,q_r\} \subset N$ such that $p_1 < q_1, p_2 < q_2, \cdots, p_r < q_r$ and $\{p_1,p_2,\cdots,p_r\} \cap \{q_1\} \cap \{q_2\} \cap \cdots \cap \{q_r\} \cap \{k\} = \phi$.
Then it holds similarly that

$S(g_{k;p_1,p_2,\cdots,p_r,q_1,q_2,\cdots,q_r}, g_{k';p'_1,p'_2,\cdots,p'_{r'},q'_1,q'_2,\cdots,q'_{r'}}) \xrightarrow{D}_+ 0$,if $r' \le r$ and $\{p_1, p_2, \cdots, p_r\} \cap \{p'_1, p'_2, \cdots, p'_{r'}\} \cap \{q_1\} \cap \{q_2\} \cap \cdots \cap \{q_r\} \cap \{q'_1\} \cap \{q'_2\} \cap \cdots \{q'_{r'}\} \cap \{k\} \cap \{k'\} = \phi$, and also $S(f_l, g_{k;p_1,p_2,\cdots,p_{r-1},q_1,q_2,\cdots,q_r}) \xrightarrow{D}_+ 0$.

Let us denote $\{g_{k;p_1,p_2,\cdots,p_r,q_1,q_2,\cdots,q_r} | p_1 < q_1, p_2 < q_2, \cdots, p_r < q_r$ and $\{p_1, p_2, \cdots, p_r\} \cap \{q_1\} \cap \{q_2\} \cap \cdots \cap \{q_r\} \cap \{k\} = \phi\}$ by G_r. Then by the similar arguments as in the case of $g_{k;i,p,j,q}$, we can prove that $g_{k;p_1,p_2,\cdots,p_{r-1},q_1,q_2,\cdots,q_r}$ is not reduced to 0 by $F \cup D \cup G_1 \cup G_2 \cdots \cup G_{r-1}$.

Again let us denote $G_1 \cup G_2 \cdots \cup G_{n-1}$ by G. Here we want to remark that $G_{n-1} = \{g_{k;0,\cdots,0;1,2,\cdots,k-1,k+1,\cdots,n} = x_0^{(n-1)(t+1)} x_0 - x_1 x_2 \cdots x_n\}$ and $\mathrm{Lt}(g_{k;0,\cdots,0;1,2,\cdots,k-1,k+1,\cdots,n}) = x_0^{(n-1)(t+1)+1}$.

Now by the above preceded reductions and Buchberger's theorem we have

Proposition 3.7. *If $t < n$, the set $F \cup D \cup G$ is a Gröbner basis of I, where $F = \{x_{i+1}x_{i+2}\cdots x_n x_0 \cdots x_{i-1} - x_i^t) | \, i \in N = \{0, 1, \cdots, n\}\}, D = \{x_j^{t+1} - x_i^{t+1} | \, i, j \in N = \{0, 1, \cdots, n\}$ and $i < j\}$, $G = \{(\prod_{p \in P} x_p^{t+1})(\prod_{r \in N-\{Q \cup \{k\}\}} x_r) - (\prod_{q \in Q \cup \{k\}} x_q^t) | \, P = \{p_1, p_2, \cdots, p_s\} \subset N = \{0, 1, \cdots, n\}, Q = \{q_1, q_2, \cdots, q_s\} \subset N$ such that $P \cap \{q_1\} \cap \{q_2\} \cap \cdots \cap \{q_s\} \cap \{k\} = \phi$, $p_l < q_l$ for $l \le s$ and $s \le n-1$.*

Similarly as before put $f_i = x_{i+1}x_{i+2}\cdots x_n x_0 \cdots x_{i-1} - x_i^t$ and $g = x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_n}$.
As $n \ge t$, $\mathrm{Lt}(f_i) = x_{i+1}x_{i+2}\cdots x_n x_0 \cdots x_{i-1}$. Hence $S(f_i, g) =$
$-x_0^{a_0-1} x_1^{a_1-1} \cdots x_{i-1}^{a_{i-1}-1} x_n^{a_n-1}(x_0 x_1 \cdots x_{i-1} x_{i+1} \cdots x_n - x_i^t) +$
$x_0^{a_0} x_1^{a_1} \cdots x_{n-1}^{a_n} = x_0^{a_0-1} x_1^{a_1-1} \cdots x_{i-1}^{a_{i-1}-1} x_i^{a_i+t} x_{i+1}^{a_{i+1}-1} \cdots x_n^{a_n-1}$
if $a_j \ge 1, j = 0, 1, \cdots, i-1, i+1, \cdots, n$. On the other hand, in the above equality we should replace $a_j - 1$ by 0 if $a_j = 0$ for $j \ne i$.

Thus for the case where $n \ge t$ we shall introduce a transportation τ_i operating on $\{Z_+ \cup \{0\}\}^{n+1}$ such that τ_i (exponential expression of g) = exponential expresion of $S(f_i, g)$.

Hence if we put $\tau_i(x_0^{a_0} \cdots x_i^{a_i} \cdots x_n^{a_n}) = (x_0^{b_0} \cdots x_i^{b_i} \cdots x_n^{b_n})$, it holds

$$b_i \quad = \quad a_i + t \qquad ,$$

$$b_j = \begin{cases} a_j - 1 & \text{if } a_j > 0 \text{ and } j \ne i, \\ 0 & \text{if } a_j = 0 \text{ and } j \ne i. \end{cases}$$

Then we have

Lemma 3.8. *Let s be an integer satisfying the ineqality $t \le s(n-t) < n$. It holds that $[\tau_n \cdots \tau_1 \tau_0]^k (s(n-t), s(n-t), \cdots, s(n-t)) =$*

$((s-k)(n-t),(s-k)(n-t),\cdots,(s-k)(n-t),(s-k)(n-t)+1,(s-k)(n-t)+2,\cdots,t-1,t)$ *for* $0 \le k \le s$.

Proof: As the proof is done by induction and similar to the proof of Lemma 3.2, we shall omit the proof.

Corollary 3.9. *A monomial* $x^1_{n-(t-1)}\cdots x^{t-1}_{n-1}x^t_n$ *apears after* s *times of S-polynomial making process* $S(f_n, S(f_{n-1},\cdots,S(f_0,-)\cdots))$ *to* $(x_0x_1\cdots x_n)^{s(n-t)}$.

So $x^1_{n-(t-1)}\cdots x^{t-1}_{n-1}x^t_n$ can be considered as an element of Gröbner basis of I_0.

Proposition 3.10. $\{x^1_{i_1}x^2_{i_2}\cdots x^t_{i_t} | 0 \le i_j \le n,\ j = 1,2,\cdots,t\}$ is a subset of Gröbner basis of I_0.

Proof. Let $(c_0,c_1,\cdots,c_n)$ be an exponential expression of a monomial $x^1_{i_1}x^2_{i_2}\cdots x_{i_t}$. Assume $t > c_0 \neq 0$ and denote c_0 by r. Then, since $\{c_0,c_1,\cdots,c_n\} = \{0,\cdots,0,1,2,\cdots,t\}$, there are $l_1,l_2,\cdots,l_{n-t}$ $(l_1,l_2,\cdots,l_{n-t}\neq 0)$ and m_1 $(0 < m_1 \le n)$ such that $c_{l_1},c_{l_2},\cdots,c_{l_{n-t}} = 0$ and $c_{m_1} = t$. Then put ${}^1c_i = c_i + 1$ for $i \notin \{l_2,\cdots,l_{n-t},m_1\}, {}^1c_{l_2},\cdots,{}^1c_{l_{n-t}} = 0$ and ${}^1c_{m_1} = 0$. It holds that $\tau_{m_1}({}^1c_0,{}^1c_1,\cdots,{}^1c_n) = (c_0,c_1,\cdots,c_n)$ and $\{{}^1c_0,{}^1c_1,\cdots,{}^1c_n\} = \{0,\cdots,0,1,2,\cdots,t\}$. If ${}^1c_0 = r+1 \neq t$, continue similar prcesses until the first coefficient of exponential expression of the last obtained monomial is t.
Now by the above argument we may assume from the first step that $c_0 = t$. Then we can put ${}^1c_0 = 0$ and ${}^1c_i = c_i + 1$ for $i \notin \{l_2,\cdots,l_{n-t}\}, {}^1c_{l_2},\cdots,{}^1c_{l_{n-t}} = 0$ so as hold $\tau_0({}^1c_0,{}^1c_1,\cdots,{}^1c_n) = (c_0,c_1,\cdots,c_n)$. Therefore by repeating the similar processes we arrive at an exponential expression of a monomial $(0,\cdots,0,a_1,a_2,\cdots,a_t)$ such that $(c_0,c_1,\cdots,c_n)$ can be obtained by applying a suitable sequence of τ_k's (= S-polynomial making processes S(f_k,-)) to $(0,\cdots,0,a_1,a_2,\cdots,a_t)$, where $\{0,\cdots,0,a_1,a_2,\cdots,a_t\} = \{0,\cdots,0,1,2,\cdots,t\}$. So by Corollary 3.9 it is enough to prove that there is a sequence of τ_k's such that $(0,\cdots,0,a_1,a_2,\cdots,a_t)$ is obtained by applying them to $(0,\cdots,0,1,2,\cdots,t)$.

It holds ,however, for $0 \le i < j \le t$ that $(0,\cdots,0,1,\cdots,j,\cdots,i,\cdots,t) = [\tau_n\tau_{n-1}\cdots \tau_{n-t+j+1}\tau_{n-t+i-1}\tau_{n-t+j-1}\cdots\tau_{n-t+i+1}\tau_{n-t+j-1}\tau_{n-t+i-1}\cdots\tau_{n-t+1}\tau_{n-t}\tau_n\tau_{n-1}\cdots \tau_{n-t+j+1}\tau_0\tau_{n-t+j-1}\cdots\tau_{n-t+1}\tau_{n-t}](0,\cdots,0,1,\cdots,i,\cdots,j,\cdots,t)$, where it happens transposition of i-th coordinate and j-th coordinate in the above exponential expressions.

As any permutation is expressed as a product of transpositions the last equality is enough to complement the remainning proof.

Denote $\{x^1_{i_1}x^2_{i_2}\cdots x^t_{i_t} | 0 < i_j < n, j = 1,2,\cdots,t\}$ by H. Then by calculations we have that $S(f_i, x^1_{i_1}x^2_{i_2}\cdots x^t_{i_t}) \xrightarrow{D\cup H}_+ 0$, $S(d_{i,j}, x^1_{i_1}x^2_{i_2}\cdots x^t_{i_t}) \xrightarrow{D\cup H}_+ 0$ and

$S(g_{k;p_1,p_2,\cdots,p_{r-1},q_1,q_2,\cdots,q_r}, x^1_{i_1}x^2_{i_2}\cdots x^t_{i_t}) \xrightarrow{D\cup H}_+ 0.$

Consequently by Buchberger's theorem we can conclude

Proposition 3.11. *If $t < n$, then $F \cup D \cup G \cup H$ is a Gröbner basis of I_0.*

Denoting $f + I_0 \in K[x_0, x_1, \cdots, x_n]/I_0$ by $\widetilde{f}$, we have

Proposition 3.12. *If $t < n$, then the socle of Λ_0 is a simple Λ_0- module generated by $(\widetilde{x}_0\widetilde{x}_1 \cdots \widetilde{x}_n)^{t-1}$ and Λ_0 is injective.*

Proof: Since $\widetilde{x}_0\widetilde{x}_1 \cdots \widetilde{x}_n = \widetilde{x}_0^{t+1}$, $(\widetilde{x}_0\widetilde{x}_1 \cdots \widetilde{x}_n)^{t-1} = (\widetilde{x}_0)^{(t+1)(t-1)}$.

Now all the leading terms of Gröbner bases of I_0, i.e., all the leading terms of polynomials in $F \cup D \cup G \cup H$ did not divide $x_0^{(t+1)(t-1)}$. Because by Propositions 3.7 and 3.9 the leading terms of polynomials in $F \cup D \cup G \cup H$ are $\prod_{j\in N-\{i\}} x_j$ for $i \in N = \{0, 1, \cdots, n\}, x_j^{t+1}$ for $j(\neq 0) \in N$, $(\prod_{p\in P} x_p^{t+1})(\prod_{r\in N-\{Q\cup\{k\}\}} x_r)$ for $P = \{p_1, p_2, \cdots, p_s\} \subset N, Q = \{q_1, q_2, \cdots, q_s\} \subset N$ such that $p_i < q_i \in P, P \cap \{q_1\} \cap \{q_2\} \cap \cdots \cap \{q_s\} \cap \{k\} = \phi$, and $x^1_{i_1}x^2_{i_2}\cdots x^t_{i_t}$ for $0 \le i_j \le n$, $j = 1, 2, \cdots, t$ such that $i_j \neq i_k$ for $j \neq k$ respectively, and then it is clear all the leading terms of polynomials in $F \cup D \cup H \cup (G - G_{n-1})$ did not divide $x_0^{(t+1)(t-1)}$, where $G_{n-1} = \{g_{k;0,\cdots,0;1,2,\cdots,(k-1),(k+1),\cdots,n} = x_0^{(n-1)(t+1)}x_0 - (x_1x_2\cdots x_n)^t\}$. And further $x_0^{(t+1)(n-1)}x_0$ (=Lt$(g_{k;0,\cdots,0;1,2,\cdots,(k-1),(k+1),\cdots,n})$) dose not divide $x_0^{(t+1)(t-1)}$ because $t < n$. Consequently by the equality quoted at the beginning of this proof we conclude $(\widetilde{x}_0\widetilde{x}_1 \cdots \widetilde{x}_n)^{t-1} \neq \widetilde{0}$.

On the other hand $(\widetilde{x}_0\widetilde{x}_1 \cdots \widetilde{x}_n)^{t-1}\ \widetilde{x}_0 = \widetilde{x}_0^{t-1}\widetilde{x}_1^{t-1} \cdots \widetilde{x}_n^{t-1}\ \widetilde{x}_i = \widetilde{0}$ for all $i\,(0 \le i \le n)$ for $x_0^{t-1}x_1^{t-1}\cdots x_n^{t-1}x_i$ can be divided by some $x^1_{i_1}x^2_{i_2}\cdots x^t_{i_t} (\in I_0)$. Hence $\widetilde{x}_0^{t-1}\widetilde{x}_1^{t-1} \cdots \widetilde{x}_n^{t-1} \in$ Soc(Λ_0) .

Since a bases of the K-vector space Λ_0 $(= K[x_0, x_1, \cdots, x_n]/I_0)$ can be chosen as a set of cosets containning the power products and $\widetilde{x}_0^{t-1}\widetilde{x}_1^{t-1} \cdots \widetilde{x}_n^{t-1} \neq \widetilde{0}$, $\{\widetilde{x}_0^{t_0}\widetilde{x}_1^{t_1} \cdots \widetilde{x}_n^{t_n} \mid 0 \le t_k \le t-1, k = 0, 1, \cdots, n\}$ is a bases of K-vector space Λ_0.

However for $t-1 = t_0 \ge t_1 \ge \cdots \ge t_{n-2} \ge t_{n-1} \ge t_n \ge 0$ and $c_0 \neq t_0$, $\widetilde{x}_{i_0}(\widetilde{x}_{i_0}^{t_0}\widetilde{x}_{i_1}^{t_1} \cdots \widetilde{x}_{i_n}^{t_n}) = \widetilde{x}_{i_0}(\widetilde{x}_{i_0}^{c_0}\widetilde{x}_{i_1}^{c_1} \cdots \widetilde{x}_{i_n}^{c_n})$ if and only if there are a integers k such that $2 \le k \le (t-1), t_k < t-k$ and $n - j \le t_j$ for all $0 < j < k$. So it holds $t_0 = t-1, t_1 = t-1, t_2 \ge t-2, \cdots, t_{k-1} \ge t-(k-1), t-k > t_k = c_k - k, t-k > t_{k-1} = c_{k+1} - k, \cdots, t-k > t_n = c_n - k$.
In this case to $c_0 = k-2, c_1 \ge k-2, c_2 \ge k-3, \cdots, c_{k-1} \ge k-k = 0, t > c_k \ge k, t > c_{k+1} \ge k, \cdots, t > c_n \ge k$ there corresponds conversely $t_0 = t-1 = c_0 - k + t + 1, t_1 = c_1 - k + t + 1, t_2 = c_2 - k + t + 1, \cdots, t_{k-1} = c_{k-1} - k + t + 1, t_k = c_k - k, \cdots, t_n = c_n - k$.

We remark here that the above residue classes $\widetilde{x}_{i_0}^{t_0}\widetilde{x}_{i_1}^{t_1} \cdots \widetilde{x}_{i_n}^{t_n}$ and $\widetilde{x}_{i_0}^{c_0}\widetilde{x}_{i_1}^{c_1} \cdots \widetilde{x}_{i_n}^{c_n}$ can be chosen as elements of K-bases of Λ_0 and $\widetilde{x}_{i_k}(\widetilde{x}_{i_0}^{t_0}\widetilde{x}_{i_1}^{t_1} \cdots \widetilde{x}_{i_n}^{t_n}) \neq \widetilde{0}$. However there is no element $\widetilde{x}_{i_0}^{c_0'}\widetilde{x}_{i_1}^{c_1'} \cdots \widetilde{x}_{i_n}^{c_n'}$ such that $\widetilde{x}_{i_k}(\widetilde{x}_{i_0}^{t_0}\widetilde{x}_{i_1}^{t_1} \cdots \widetilde{x}_{i_n}^{t_n}) = \widetilde{x}_{i_k}(\widetilde{x}_{i_0}^{c_0'}\widetilde{x}_{i_1}^{c_1'} \cdots \widetilde{x}_{i_n}^{c_n'})$.

Now let us denote $\{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n)) \mid t_0 = t-1, t_1 = t-1, t_2 \geq t-2,\cdots,t_{k-1} \geq t-(k-1), t-k > t_k, t-k > t_{k-1},\cdots,t-k > t_n.$ for $t-1 \geq k \geq 2\}$ by T_k, the corresponding c_j by $\phi\ (t_j)$ for $j = 0,1,\cdots,n$ and $\{((i_0,i_1,\cdots,i_n),(\phi(t_0),\phi(t_1),\cdots,\phi(t_n)))\}$ by C_k respectively.

Further denote $\{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n)) \mid \tilde{x}_{i_0}^{d_0}\tilde{x}_{i_1}^{d_1}\cdots\tilde{x}_{i_n}^{d_n}$ is an element of a K-bases of Λ_0 $\}$ by B. Then an element $\tilde{Y}$ of Λ_0 is written by

$\sum_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))\in B} a_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))}\tilde{x}_{i_0}^{d_0}\tilde{x}_{i_1}^{d_1}\cdots\tilde{x}_{i_n}^{d_n}$, where $a_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))} \in K$.

Assume $\tilde{Y}\in \mathrm{Soc}(\Lambda_0)$. Then $\tilde{x}_{i_0}\ \tilde{Y} = \tilde{0} =$

$$\sum_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))\in B} a_{((i_0,i_1,\cdots,i_n),(d_0,d_1,\cdots,d_n))}\tilde{x}_{i_0}^{d_0+1}\tilde{x}_{i_1}^{d_1}\cdots\tilde{x}_{i_n}^{d_n} =$$

$$\sum_{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n))\in B\cap T_{k,l}} (a_{((i_0,i_1,\cdots,i_n),(\phi(t_0),\phi(t_1),\cdots,\phi(t_n)))} + a_{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n))})\tilde{x}_{i_0}^{t_0+1}\tilde{x}_{i_1}^{t_1}\cdots$$

$$\tilde{x}_{i_n}^{t_n} + \sum_{((i_0,i_1,\cdots,i_n),(t_0',t_1',\cdots,t_n'))\in B-(T_k\cup C_k)} a_{((i_0,i_1,\cdots,i_n),(t_0',t_1',\cdots,t_n'))}\tilde{x}_{i_0}^{t_0'+1}\tilde{x}_{i_1}^{t_1'}\cdots\tilde{x}_{i_n}^{t_n'}.$$

Therefore $a_{((i_0,i_1,\cdots,i_n),(\phi(t_0),\phi(t_1),\cdots,\phi(t_n)))} = -a_{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n))}$ for $((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n)) \in B\cap T_k$.
However by the preceded remark $a_{((i_0,i_1,\cdots,i_n),(t_0,t_1,\cdots,t_n))} = 0$ follows from $\tilde{x}_{i_k}\tilde{Y} = \tilde{0}$.

And $a_{((i_0,i_1,\cdots,i_n),(t_0',t_1',\cdots,t_n'))} = 0$ for $((i_0,i_1,\cdots,i_n),(t_0',t_1',\cdots,t_n')) \in B-(T_k\cup C_k)$ if $\tilde{x}_{i_0}^{t_0'+1}\tilde{x}_{i_1}^{t_1'}\cdots\tilde{x}_{i_n}^{t_n'} \neq \tilde{0}$.

Consequently $a_{((i_0,i_1,\cdots,i_n)(d_0,d_1,\cdots,d_n))} = 0$ for $((i_0,i_1,\cdots,i_n)(d_0,d_1,\cdots,d_n)) \in B$ if $\tilde{x}_{i_0}^{d_0+1}\tilde{x}_{i_1}^{d_1}\cdots\tilde{x}_{i_n}^{d_n} \neq \tilde{0}$. As $(\tilde{x}_{i_0},\tilde{x}_{i_1},\cdots,\tilde{x}_{i_n})$ is any permutation of $(\tilde{x}_0,\tilde{x}_1,\cdots,\tilde{x}_n)$, from the beginning part of this proof we can conclude that $(\tilde{x}_0\tilde{x}_1\cdots\tilde{x}_n)^{n-1}$ is only a generator of $\mathrm{Soc}(\Lambda_0)$ as a K-vector space.

Therefore as $K \simeq \Lambda_0/\ \mathrm{Rad}(\Lambda_0)$, $\mathrm{Soc}(\Lambda_0)$ is simple Λ_0-module. Hence Λ_0 is injective, because Λ_0 is local. Cf.[2]

Now we have the following final result.

Theorem 3.12. *If $n \neq t$, then the local algebra Λ_0 is injective.*

4 Appendix

The preceding sections we did not mension about the ground field K. It is well known, however, that the group algebra Λ_1 is not semisimple provided the characteristic p of K divides $(t+1)s|t-n|$ and the other case Λ_1 is a direct sum of extended fields of K. On the other hand, local QF-ness of Λ_0 does not depend on the characteristic of K.

Next we shall consider Λ_0 for the case $t = 2 < n$. Since $I = (x_{i+1}x_{i+2}\cdots x_nx_0x_1\cdots x_{i-1} - x_i^2|\ i = 0, 1, \cdots, n)$, homogeneous elements $\widetilde{x}_i^{\ 2}$ of degree 2 overlap homogeneous elements $\widetilde{x}_i\widetilde{x}_{i+1}\cdots\widetilde{x}_n\widetilde{x}_0\widetilde{x}_1\cdots\widetilde{x}_{i-1}$ of degree n if we take $\widetilde{x}_i$ as homogeneous elements of degree 1. So in this grading Λ_0 is not positively Z-graded. It follows from Proposition 3.11 that a Gröbner bases consist of $F = \{x_{i+1}x_{i+2}\cdots x_nx_0x_1\cdots x_{i-1} - x_i^2 \mid i = 0, 1, \cdots, n\}$, $H = \{x_i^1x_j^2 \mid\ i \neq j$ and $i, j = 0, 1, \cdots, n\}$ and $D = \{x_j^3 - x_i^3 \mid i \neq j$ and $i, j = 0, 1, \cdots, n\}$, and $G = \{\prod_{p\in P} x_p^3 \prod_{r\in N-\{Q\cup\{k\}\}} x_r - \prod_{q\in Q\cup\{k\}} x_q^2 \mid P = \{p_1, p_2, \cdots, p_s\} \subset N = \{0, 1, \cdots, n\}, Q = \{q_1, q_2, \cdots, q_s\} \subset N$ such that $p_l < q_l$ for $l \leq s$ and $P \cap \{q_1\} \cap \{q_1\} \cap \cdots \cap \{q_s\} \cap \{k\} = \phi$. However among them we can exclude G by H. Therefore since rad (Λ_1) is generated by $\{\widetilde{x}_i \mid i = 0, 1, \cdots, n\}$ we have $\{\widetilde{x}_{i_1}\widetilde{x}_{i_2}\cdots\widetilde{x}_{i_r} \mid 0 \leq i_1 < i_2 < \cdots < i_r \leq n\}$ as representatives of K-bases of rad $(\Lambda_0)^r$/rad $(\Lambda_0)^{r+1}$. This implies that Hilbert series of the associated graded algebra of Λ_0 is $\begin{pmatrix} n \\ 0 \end{pmatrix}, \begin{pmatrix} n \\ 1 \end{pmatrix}, \cdots, \begin{pmatrix} n \\ r \end{pmatrix}, \cdots, \begin{pmatrix} n \\ n \end{pmatrix}$. Hence Grad$(\Lambda_0)$ seems to be a new commutative version of an exterior algebra.

Now we shall consider Λ for the case $n = t$. From the arguments of §1 and §2 it does not happen the direct sum decomposition of Λ into a group algebra Λ_1 and a local algebra Λ_0. However Λ is a positively Z-graded with respect to homogeneous elements $\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_n$ of degree 1, because both $\bar{x}_i^{\ n}$ and $\bar{x}_{i+1}\bar{x}_{i+2}\cdots\bar{x}_n\bar{x}_0\bar{x}_1\bar{x}_2\cdots\bar{x}_{i-1}$ have the same homogeneous degree n. Further we can prove

Proposition 4.1. *If $n = t$, then Λ is 1-dimensional Gorenstein.*

Proof: Let us denote $\bar{x}_0\bar{x}_1\cdots\bar{x}_n$ by u. As was just remarked above Λ is positively Z-graded and $K[u]$ is a graded subalgebra of Λ having a homogenious element u of degree n.

Hence u is a regular element in $K[u]$ and u is a transcendental element over K. Further $\bar{x}_i$, $i = 0, 1, \cdots, n$, satisfy the equation $X^{n+1} - u = 0 \in K[u][X]$. Hence by Noether's normalization theorem, the Krull dimension of $\Lambda = 1$.

Consider a residue class algebra of Λ by an ideal (u), where u is a regular element. Then $\Lambda/(u) \simeq K[v_0, v_1, \cdots, v_n]/({v_0}^{n+1}, {v_1}^{n+1}, \cdots, {v_n}^{n+1})$, where $K[v_0, v_1, \cdots, v_n]$ is a polynomial ring over K with indeterminants $v_i, i = 0, 1, \cdots, n$.

Now ${v_0}^{n+1}, {v_1}^{n+1}, \cdots, {v_n}^{n+1}$ are a regular sequence and hence $\Lambda/(u)$ is a complete intersection. Therefore $\Lambda/(u)$ is 0-dimensional Gorenstein. It concludes Λ is a 1-dimensional Gorenstein. Cf.[2], [6].

Residue class ring Λ of polynomial ring by ideal generated by pure binomials associated with toric variety, the general theory of Gröbner bases and polytopes[12], however, seems to be not effective in our case, because Λ is 0-dimensional.

Notes: The investigation of this subject begun from the time when I was invited to Bielefeld University by Professor C.M.Ringel (and Prof. M. Auslander too stayed in Bielefeld at the same time). Here the author would like to express his thanks to Professor Ringel for this publication.

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