THE VARIETY OF PAIRS OF NILPOTENT MATRICES ANNIHILATING EACH OTHER

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Abstract. We solve a well-known problem of Kraft, the classification of irreducible components of the variety

$$\{(A,B) \in M_n(k) \times M_n(k) \mid AB = BA = A^n = B^n = 0\}.$$

1. Introduction

Let k be an algebraically closed field, and let $M_n(k)$ be the set of $n \times n$ matrices with entries in k. The study of varieties given by matrices or pairs of matrices, which satisfy certain relations, is a classical subject. One fundamental question is the decomposition of these varieties into irreducible components. On the other hand, one is interested in classical groups operating on matrix varieties. For example, the variety

$$\{A \in M_n(k) \mid A^n = 0\}$$

of nilpotent $n \times n$-matrices was studied by Gerstenhaber and Hesselink, see [7] and [8]. They determined the orbit closures with respect to classical groups acting via conjugation. This variety is irreducible of dimension $n^2 - n$. Next, we look at the variety

$$\{(A,B) \in M_n(k) \times M_n(k) \mid AB = BA = 0\}.$$

The group $GL_n(k) \times GL_n(k)$ is acting by simultaneous conjugation, that is $(g_1, g_2) \cdot (A,B) = (g_1 A g_2^{-1}, g_2 B g_1^{-1})$. By $\mathrm{rk}(M)$ we denote the rank of a matrix M. One easily checks that there are only finitely many orbits with respect to this operation. Determining the orbit closures is also straightforward. This yields that the irreducible components are

$$\{(A,B) \in M_n(k) \times M_n(k) \mid AB = BA = 0, \mathrm{rk}(A) \leq n - i, \mathrm{rk}(B) \leq i\}$$

for $0 \leq i \leq n$. Each component has dimension n^2.

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Now, we take the product of the variety of nilpotent matrices with itself, and intersect it with the variety of pairs of matrices annihilating each other. In other words, we are interested in the variety

$$\mathrm{GP}(n) = \{(A, B) \in M_n(k) \times M_n(k) \mid AB = BA = A^n = B^n = 0\}.$$

The reason for denoting this variety by $\mathrm{GP}(n)$ is the following: One can identify $\mathrm{GP}(n)$ with the variety of n-dimensional $k[x, y]/(xy, x^n, y^n)$-modules. The indecomposable finite-dimensional modules over $k[x, y]/(xy, x^n, y^n)$ were classified by Gelfand and Ponomarev in [6], and their classification is used throughout this article.

It is a well-known problem of Kraft to classify the irreducible components of $\mathrm{GP}(n)$, see [9], p. 203, Problem 3. Our main result is the following.

Theorem 1.1. *For $n \geq 2$ the irreducible components of $\mathrm{GP}(n)$ are*

$$\{(A, B) \in \mathrm{GP}(n) \mid \mathrm{rk}(A) \leq n - i, \mathrm{rk}(B) \leq i\}$$

for $1 \leq i \leq n - 1$. Each component has dimension $n^2 - n + 1$.

In Section 2 we use Gelfand and Ponomarev's classification of indecomposable $k[x, y]/(xy, x^n, y^n)$-modules to show that the set

$$\{(A, B) \in \mathrm{GP}(n) \mid \mathrm{rk}(A) + \mathrm{rk}(B) = n\}$$

is dense in $\mathrm{GP}(n)$.

Recently, Richmond introduced a new stratification of varieties of modules over finite-dimensional algebras, see [13]. In Section 3 we show that this stratification if finite for $\mathrm{GP}(n)$. Using our results from Section 2 and a theorem of Richmond, we can control the closures of the strata yielding our main result.

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2. Gelfand-Ponomarev modules

2.1. Let A be a finite-dimensional k-algebra. Denote by $\mathrm{mod}_A(n)$ the variety of n-dimensional A-modules. Recall that these are the k-algebra homomorphisms $A \longrightarrow M_n(k)$. For more details we refer to [5], [11] or the survey articles [2] and [9]. Throughout, we use only the Zariski topology.

Let $\Gamma = \Gamma_q = k[x, y]/(xy, x^q, y^q)$ with $q \geq n$ and let M be in $\mathrm{mod}_\Gamma(n)$. The map $M \mapsto (Mx, My)$ yields an isomorphism $\mathrm{mod}_\Gamma(n) \longrightarrow \mathrm{GP}(n)$. In the following we do not distinguish between these two varieties. Sometimes we regard their points as modules and sometimes as pairs of matrices.

Let $G = GL_n(k)$. The group G operates on $\mathrm{mod}_\Gamma(n)$ by simultaneous conjugation, that is for an element g in G and (A, B) in $\mathrm{mod}_\Gamma(n)$ we have $g \cdot (A, B) = (gAg^{-1}, gBg^{-1})$. We denote the G-orbit of $M = (A, B)$ by $\mathcal{O}(M)$. Recall that $\dim \mathcal{O}(M) = n^2 - \dim \mathrm{End}(M)$. If the closure of an orbit $\mathcal{O}(M)$ contains an orbit $\mathcal{O}(N)$, then we write $M \leq_{\deg} N$. The G-orbits are in one to one correspondence with the isomorphism classes of n-dimensional Γ-modules. The classification of indecomposable Γ-modules is due to Gelfand and Ponomarev, see [6]. For convenience we recall their result, using different notation which is more appropriate to our situation.

A *word* w of length $l(w) = m \geq 1$ is a sequence $w = w_1 \cdots w_m$ with w_i in $\{x, y\}$ for all i.. The words $w_i \cdots w_j$ with $1 \leq i \leq j \leq m$ are called *subwords* of w. A word w which does not contain x^q or y^q as a subword is called a *string*. Let $w = w_1 \cdots w_m$ be a string. We define a Γ-module $M(w) = (Mx, My)$ as follows. We fix a basis $\{z_1, \cdots, z_{m+1}\}$ and define two linear maps $Mx, My : k^{m+1} \longrightarrow k^{m+1}$ by

$$Mx(z_i) = \begin{cases} z_{i-1} & : \quad w_{i-1} = x, 2 \leq i \leq m+1; \\ 0 & : \quad \text{otherwise} \end{cases}$$

and

$$My(z_i) = \begin{cases} z_{i+1} & : \quad w_i = y, 1 \leq i \leq m; \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Additionally, we define a word e of length 0. We also call e a string and let $M(e)$ be the (unique) simple Γ-module. Define $we = ew = w$ and $w^0 = e$ for all words w. Modules of the form $M(w)$ for some string w are called *string modules*.

A word which contains x and y as subwords is called *mixed*. A word w is *periodic* if it is mixed and $w = w_1 w_1$ for some word w_1. A mixed string w is called a *band* if it is not periodic. Let $w = w_1 \cdots w_m$ be a mixed string, let $l \geq 1$ and for $1 \leq i \leq l$ let $\lambda_i \in k^*$. We define a module $M(w; \lambda_1, \cdots, \lambda_l) = (Mx, My)$ as follows. Fix a basis $\{z_i^{(j)} \mid 1 \leq i \leq m, 1 \leq j \leq l\}$, and let

$$Mx(z_i^{(j)}) = \begin{cases} z_{i-1}^{(j)} & : \quad w_{i-1} = x, 2 \leq i \leq m, 1 \leq j \leq l; \\ \lambda_1 z_m^{(1)} & : \quad w_m = x, i = 1, j = 1; \\ \lambda_j z_m^{(j)} + z_m^{(j-1)} & : \quad w_m = x, i = 1, 2 \leq j \leq l; \\ 0 & : \quad \text{otherwise} \end{cases}$$

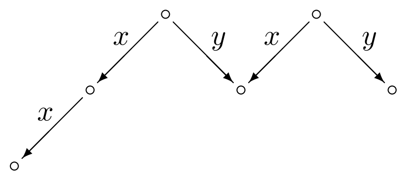
and

$$My(z_i^{(j)}) = \begin{cases} z_{i+1}^{(j)} & : \quad w_i = y, 1 \leq i \leq m-1, 1 \leq j \leq l; \\ \lambda_1 z_1^{(1)} & : \quad w_m = y, i = m, j = 1; \\ \lambda_j z_1^{(j)} + z_1^{(j-1)} & : \quad w_m = y, i = m, 2 \leq j \leq l; \\ 0 & : \quad \text{otherwise.} \end{cases}$$

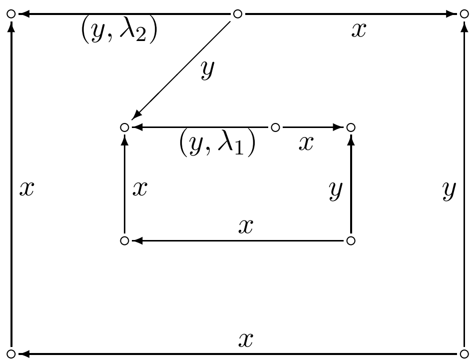
Observe that a module of the form $M(w; \lambda_1, \cdots, \lambda_l) = (Mx, My)$ always has the property $\mathrm{rk}(Mx) + \mathrm{rk}(My) = \dim M(w; \lambda_1, \cdots, \lambda_l)$.

If w is a band and $\lambda_i = \lambda$ for all i, then $M(w; \lambda_1, \cdots, \lambda_l)$ is called a *band module*. In this case, we also write $M(w, \lambda, l)$ instead of $M(w; \lambda_1, \cdots, \lambda_l)$.

It is useful to visualize modules $M(w)$ or $M(w; \lambda_1, \cdots, \lambda_l)$ as weighted graphs. Namely, let $M = (Mx, My)$ be in $\mathrm{mod}_\Gamma(n)$ with basis $\{b_1, \cdots, b_n\}$ and $Mx(b_i) = \sum_{j=1}^n \lambda_{ij} b_j$ and $My(b_i) = \sum_{j=1}^n \mu_{ij} b_j$ for $1 \leq i \leq n$. We define a graph $\mathcal{G}(M)$ as follows. The vertices of $\mathcal{G}(M)$ are elements of $\{1, \cdots, n\}$, and for each $\lambda_{ij} \neq 0$ or $\mu_{ij} \neq 0$ we draw an arrow $i \longrightarrow j$ with weight (x, λ_{ij}) or (y, μ_{ij}), respectively. If $\lambda_{ij} = 1$ or $\mu_{ij} = 1$, then we just draw an arrow with weight x or y, respectively. For example, let $w = xxyxy$. Then $\mathcal{G}(M(w))$ looks as follows:



The graph $\mathcal{G}(M(w; \lambda_1, \lambda_2))$ has the following form:



Let w_1 and w_2 be mixed strings. We write $w_1 \sim w_2$ if $w_1 = u_1 u_2$ and $w_2 = u_2 u_1$ for some strings u_1 and u_2. Let $\mathcal{B}$ be the set of all bands. Note that $\sim$ defines an equivalence relation on $\mathcal{B}$. Let $\mathcal{B}_\sim$ be a set of representatives of equivalence classes of bands, and let $\mathcal{S}$ be the set of all strings. The following theorem is proved in [6]. As a reference we also recommend the generalizations in [3] or [16].

Theorem 2.1 (Gelfand, Ponomarev)**.** *The* Γ*-modules*

$$\{M(w) \mid w \in \mathcal{S}\} \cup \{M(w, \lambda, l) \mid w \in \mathcal{B}_\sim, l \geq 1, \lambda \in k^*\}$$

are a complete set of representatives of isomorphism classes of indecomposable finite-dimensional Γ*-modules.*

Recall that according to the Theorem of Krull-Remak-Schmidt, each finite-dimensional module over an algebra is a direct sum of indecomposable modules, and these are uniquely determined up to isomorphism.

2.2. The following lemma is a direct consequence of the definition of string and band modules and of the theorem of Gelfand and Ponomarev.

Lemma 2.2. *Let M be in $\mathrm{mod}_\Gamma(n)$ and let $s = n - \mathrm{rk}(Mx) - \mathrm{rk}(My)$. Then each direct sum decomposition of M into indecomposables contains exactly s modules which are isomorphic to string modules.*

Let M be an n-dimensional Γ-module, which is isomorphic to

$$\bigoplus_{i=1}^{l} M(u_i) \oplus \bigoplus_{j=1}^{m} M(w_j; \lambda_{1j}, \cdots, \lambda_{l_j j}),$$

and let $p = \sum_{j=1}^{m} l_j$. The map

$$\eta_M : G \times (k^*)^p \longrightarrow \mathrm{mod}_\Gamma(n)$$

with

$$(g, (\mu_{ij})_{1\le j\le m, 1\le i\le l_j}) \mapsto g \cdot \left(\bigoplus_{i=1}^{l} M(u_i) \oplus \bigoplus_{j=1}^{m} M(w_j; \mu_{1j}, \cdots, \mu_{l_j j}) \right)$$

is a morphism of quasi-affine varieties. We denote the image of η_M by $\mathcal{F}(M)$. By Chevalley's Theorem we know that $\mathcal{F}(M)$ is constructible. We say that $\mathcal{F}(M)$ is a *p-parametric family.*

Lemma 2.3. *If a family $\mathcal{F}(M)$ is p-parametric, then $\mathcal{F}(M)$ has dimension $p + \dim \mathcal{O}(\eta_M((\mu_{ij})_{ij}))$ where $(\mu_{ij})_{ij} \in (k^*)^p$ with pairwise different μ_{ij}'s.*

Proof. In [10] the dimensions of homomorphism spaces between string modules and band modules are computed. A special case can be found in [4]. Note that string modules are a special kind of tree modules in the sense of [4]. It follows that the orbits $\mathcal{O}(\eta_M((\mu_{ij})_{ij}))$ with pairwise different μ_{ij}'s are G-orbits of maximal dimension in $\mathcal{F}(M)$. The set of these orbits is obviously dense. This yields the result. □

Lemma 2.4. *Let w be a mixed string. If $\lambda_r \neq \lambda_{r+1}$, then $M(w; \lambda_1, \cdots, \lambda_l)$ is isomorphic to $M(w; \lambda_1, \cdots, \lambda_r) \oplus M(w; \lambda_{r+1}, \cdots, \lambda_l)$.*

Proof. Let $\{z_i^{(j)} \mid 1 \le i \le m, 1 \le j \le l\}$ be the basis of $M(w; \lambda_1, \cdots, \lambda_l)$ as defined above. Assume $\lambda_r \neq \lambda_{r+1}$ and define $\mu = (\lambda_{r+1} - \lambda_r)^{-1}$. Without loss of generality, we assume $\lambda_i = \lambda_r$ for all $r+1 \le i \le l$. The vectors $\{z_i^{(s)} + \mu z_i^{(s-1)} \mid 1 \le i \le m, r+1 \le s \le l\}$ are then a basis of a submodule isomorphic to $M(w; \lambda_{r+1}, \cdots, \lambda_l)$. On the other hand, $\{z_i^{(s)} \mid 1 \le i \le$

$m, 1 \leq s \leq r\}$ generates a submodule isomorphic to $M(w; \lambda_1, \cdots, \lambda_r)$. These submodules have trivial intersection. This implies the result. $\square$

Corollary 2.5. *If w is a mixed string, then $M(w; \lambda_1, \cdots, \lambda_l)$ is contained in the closure of $\mathcal{F}(\bigoplus_{i=1}^{l} M(w; \lambda_i))$.*

Lemma 2.6. *Let $w = w_1 \cdots w_m$ be a mixed string with $w_{j-1} \neq w_j = w_{j+1}$ for some j. Then the module $M(e) \oplus M(w_1 \cdots w_j w_{j+2} \cdots w_m; \mu)$ is contained in the closure of $\mathcal{F}(M(w; \mu))$.*

Proof. Without loss of generality, assume $w_j = w_{j+1} = x$. For $\mu_1, \mu_2 \in k$ we define an m-dimensional Γ-module $M(\mu_1, \mu_2) = (Mx, My)$ as follows. Let $\{z_1, \cdots, z_m\}$ be the basis of $M(\mu_1, \mu_2)$. Let

$$Mx(z_i) = \begin{cases} z_{i-1} & : \quad w_{i-1} = x, 2 \leq i \leq m, i \neq j+1, j+2; \\ z_m & : \quad w_m = x, i = 1; \\ \mu_1 z_{j+1} + \mu_2 z_j & : \quad i = j+2; \\ \mu_1 z_j & : \quad i = j+1; \\ 0 & : \quad \text{otherwise} \end{cases}$$

and

$$My(z_i) = \begin{cases} z_{i+1} & : \quad w_i = y, 1 \leq i \leq m-1; \\ z_1 & : \quad w_m = y, i = m; \\ 0 & : \quad \text{otherwise.} \end{cases}$$

If μ_1 is not zero, then one easily sees that $M(\mu_1, \mu_2)$ is isomorphic to $M(w; \mu_1^2)$. If μ_2 is not zero, but $\mu_1 = 0$, then $M(\mu_1, \mu_2)$ is isomorphic to $M(e) \oplus M(w_1 \cdots w_j w_{j+2} \cdots w_m; \mu_2)$. This finishes the proof. $\square$

Lemma 2.7. *For $n \leq q-1$ let M be in $\mathrm{mod}_\Gamma(n)$ with $\mathrm{rk}(Mx) \leq n-s$ and $\mathrm{rk}(My) \leq s$ for some $1 \leq s \leq n-1$. Then M is contained in the closure of some family $\mathcal{F}(N = \bigoplus_{i=1}^{m} M(w_i; \lambda_i))$ with $\mathrm{rk}(Nx) = n-s$ and $\mathrm{rk}(Ny) = s$.*

Proof. If $\mathrm{rk}(Mx) = n-s$ and $\mathrm{rk}(My) = s$, then the statement follows from Lemma 2.2 and Corollary 2.5. Without loss of generality, assume $\mathrm{rk}(Mx) < n-s$. Thus M is isomorphic to $\bigoplus_{i=1}^{m} M(w_i) \oplus C$, where $m \geq 1$ and C is zero or a direct sum of band modules. If $\bigoplus_{i=1}^{m} M(w_i) = M(e)$, then our lemma follows from Lemma 2.6. Thus assume that the length l of $w_1 \cdots w_m$ is at least two. Let $d_1, \cdots, d_m$ be words of length one and define $w = w_1 d_1 w_2 d_2 \cdots d_{m-1} w_m d_m = f_1 \cdots f_{l+m}$. Observe that this is a string since $n \leq q-1$. We choose the d_i's such that $w \neq x^{l+m}$ and $w \neq y^{l+m}$. Now we define a Γ-module $N(\lambda) = (Nx, Ny)$ as follows. As a basis we fix $\{z_1, \cdots, z_{l+m}\}$. For $1 \leq r \leq m-1$ define $i_r = r + \sum_{i=1}^{r} l(w_i)$. Let

$$Nx(z_i) = \begin{cases} z_{i-1} & : \quad f_{i-1} = x, 2 \leq i \leq l+m, i \neq i_r \text{ for all } r; \\ \lambda z_{i-1} & : \quad f_{i-1} = x, 2 \leq i \leq l+m, i = i_r \text{ for some } r; \\ \lambda z_{l+m} & : \quad d_m = x, i = 1; \\ 0 & : \quad \text{otherwise} \end{cases}$$

and

$$Ny(z_i) = \begin{cases} z_{i+1} & : \quad f_i = y, 1 \le i \le l+m-1, i \ne i_r \text{ for all } r; \\ \lambda z_{i+1} & : \quad f_i = y, 1 \le i \le l+m-1, i = i_r \text{ for some } r; \\ \lambda z_1 & : \quad d_m = y, i = l+m; \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Note that $N(0)$ is isomorphic to $\bigoplus_{i=1}^m M(w_i)$. If $\lambda \ne 0$, then $N(\lambda)$ is isomorphic to $M(w;\lambda^m)$. The closure of $\mathcal{F}(M(w;\mu))$ contains obviously $N(0)$. Note that we can choose the d_i's such that the rank conditions in our claim are satisfied. This finishes the proof. □

As a main result of this section, we get the following.

Proposition 2.8. *For* $1 \le i \le n-1$ *the set*

$$\{(A,B) \in \mathrm{GP}(n) \mid \mathrm{rk}(A) \le n-i, \mathrm{rk}(B) \le i\}$$

is contained in the closure of

$$\{(A,B) \in \mathrm{GP}(n) \mid \mathrm{rk}(A) = n-i, \mathrm{rk}(B) = i\}.$$

3. Richmond's stratification

3.1. Let A be a finite-dimensional k-algebra of dimension d. For $n \ge 1$ let $\mathcal{S}_A(n)$ be a set of representatives of isomorphism classes of submodules of A^n which have dimension $n(d-1)$. For each L in $\mathcal{S}_A(n)$ let $\mathrm{mod}_A(n)_L$ be the points M in $\mathrm{mod}_A(n)$ such that there exists a short exact sequence

$$0 \longrightarrow L \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

of A-modules. Such a set is called a *stratum*. Note that $\mathrm{mod}_A(n)$ is the disjoint union of the $\mathrm{mod}_A(n)_L$'s where L runs through $\mathcal{S}_A(n)$. The following two theorems can be found in [13], Theorems 2.6.2, 2.7.3 and 2.7.4.

Theorem 3.1 (Richmond)**.** *The set* $\mathrm{mod}_A(n)_L$ *is irreducible and locally closed in* $\mathrm{mod}_A(n)$ *and has dimension* $\dim \mathrm{Hom}(L, A^n) - \dim \mathrm{End}(L)$.

Theorem 3.2 (Richmond)**.** *Let* L *and* L' *be in* $\mathcal{S}_A(n)$. *If* $\mathrm{mod}_A(n)_{L'}$ *is contained in the closure of* $\mathrm{mod}_A(n)_L$, *then* $L \le_{\deg} L'$. *On the other hand, if* $L \le_{\deg} L'$ *and* $\dim \mathrm{Hom}(L, A) = \dim \mathrm{Hom}(L', A)$, *then* $\mathrm{mod}_A(n)_{L'}$ *is contained in the closure of* $\mathrm{mod}_A(n)_L$.

In case $\mathcal{S}_A(n)$ is finite, then the closures of the strata, which are not contained in the closures of other strata, are exactly the irreducible components of $\mathrm{mod}_A(n)$. Algebras where $\mathcal{S}_A(n)$ is finite for every n are called *subfinite*, see [13].

3.2. Let $M = \bigoplus_{i=1}^{l} M(w_i)$ be a direct sum of string modules. We call M *biserial* if $w_i = x^{a_i}y^{b_i}$ for all i and some $a_i, b_i \geq 0$.

For the rest of this section, let $n \geq 2$ and

$$\Gamma = \Gamma_{n+1} = k[x,y]/(xy, x^{n+1}, y^{n+1}).$$

Note that the free Γ-module of rank 1, thus Γ itself, is isomorphic to $M(x^n y^n)$.

Lemma 3.3. *The algebra Γ is subfinite, and each submodule of any free module Γ^m is isomorphic to a biserial module.*

Proof. The algebra Γ belongs to the class of monomial algebras as defined in [17]. If U is an m-dimensional submodule of a projective module over Γ, then $Ux(k^m) \cap Uy(k^m) = 0$, see [17], Lemma 3. It follows from the description of indecomposable Γ-modules that the biserial modules are the only modules which have this property. □

Lemma 3.4. *If M is in $\mathrm{mod}_\Gamma(n)_L$, then L is isomorphic to a direct sum of n indecomposable modules if and only if M is isomorphic to a direct sum of band modules.*

Proof. Let $w = w_1 \cdots w_m = x^{a_1}y^{b_1} \cdots x^{a_r}y^{b_r}$ with $a_{i+1}, b_i \geq 1$ for $1 \leq i \leq r-1$ and $a_1, b_r \geq 0$. The dimension of the top of $M(w)$ is r. A straightforward calculation shows that the kernel of the projective cover of $M(w)$ is isomorphic to $M(x^{n-a_1-1}) \oplus M(y^{n-b_l-1}) \oplus \bigoplus_{i=1}^{r-1} M(x^{n-a_{i+1}}y^{n-b_i})$. Thus it has $r+1$ indecomposable direct summands. Next we assume that w is mixed. Without loss of generality, let $a_i, b_i \geq 1$ for all $1 \leq i \leq r$. The kernel U of the projective cover of $M(w; \lambda_1, \cdots, \lambda_l)$ is isomorphic to $M(x^{n-a_1}y^{n-b_l})^l \oplus \bigoplus_{i=1}^{l-1} M(x^{n-a_{i+1}}y^{n-b_i})^l$. Thus the dimension of the top of $M(w; \lambda_1, \cdots, \lambda_l)$ and the number of indecomposable direct summands of U coincide. This finishes the proof. □

Thus if a stratum $\mathrm{mod}_\Gamma(n)_L$ contains a module M, which is a direct sum of band modules, then it contains only modules which are isomorphic to direct sums of band modules. Such strata are called *string-free*. A module L in $\mathcal{S}_\Gamma(n)$ is called *homogeneous* if $\mathrm{mod}_\Gamma(n)_L$ is string-free. The next lemma is a consequence of Proposition 2.8.

Lemma 3.5. *The variety $\mathrm{mod}_\Gamma(n)$ is the union of the closures of the string-free strata.*

Let $d = 2n+1$, the dimension of Γ.

Lemma 3.6. *The modules $\Gamma^{n-p} \oplus \bigoplus_{i=1}^{p} M(x^{a_i}y^{b_i})$ with $p \geq 1$, $1 \leq a_i, b_i \leq n-1$ and $\sum_{i=1}^{p}(a_i + b_i) = (2p-1)n$ are exactly the homogeneous modules in $\mathcal{S}_\Gamma(n)$.*

Proof. Note that the dimension of a module $M(x^a y^b)$ is $a+b+1$. Observe also that a module $M(x^a y^b)$ with $a = n$ and $b < n$, or $a < n$ and $b = n$ can not be a submodule of some Γ^m. The condition $\sum_{i=1}^{p}(a_i + b_i) = (2p-1)n$ ensures that the dimension of $\Gamma^{n-p} \oplus \bigoplus_{i=1}^{p} M(x^{a_i} y^{b_i})$ is $n(d-1)$. The statement then follows from the proof of Lemma 3.4. □

For a homogeneous module $L = \Gamma^{n-p} \oplus \bigoplus_{i=1}^{p} M(x^{a_i} y^{b_i})$ in $\mathcal{S}_\Gamma(n)$ we have $\sum_{i=1}^{p} a_i = (p-1)(n-1) + a$ for some $1 \leq a \leq n-1$. We call (p, a) the *type* of L.

The modules in $\mathcal{S}_\Gamma(n)$ of the form

$$\Gamma^{n-p} \oplus M(x^{n-1} y^{n-1})^{p-2} \oplus M(x^a y^{n-1}) \oplus M(x^{n-1} y^b)$$

with $p \geq 2$ are denoted by $L_{(p,a)}$. Note that $L_{(p,a)}$ has type (p, a) and observe that b is uniquely determined by p and a. If a module L in $\mathcal{S}_\Gamma(n)$ is of type $(1, a)$, then define $L_{(1,a)} = L$. Observe that this is well-defined.

Lemma 3.7. *If a homogeneous module L in $\mathcal{S}_\Gamma(n)$ is of type (p, a), then* $\dim \operatorname{Hom}(L, \Gamma) = n(d-1) + p$.

Proof. If $X = M(x^a y^b)$ and $Y = M(x^i y^j)$ with $a, b, i, j \geq 1$, then

$$\dim \operatorname{Hom}(X, Y) = \begin{cases} i+j+1 & : \quad a \geq i, b \geq j; \\ a+b+1 & : \quad a < i, b < j; \\ i+b+1 & : \quad a > i, b < j. \end{cases}$$

This can be checked directly and follows also from [4] and [10]. Then the lemma follows from the definition of $\mathcal{S}_\Gamma(n)$ and the definition of the type of L. □

Next we study degenerations between modules in $\mathcal{S}_\Gamma(n)$.

Lemma 3.8. *If $0 \leq a \leq i \leq n$ and $0 \leq b \leq j \leq n$, then*

$$M(x^i y^b) \oplus M(x^a y^j) \leq_{\deg} M(x^i y^j) \oplus M(x^a y^b).$$

Proof. One can construct a short exact sequence

$$0 \longrightarrow M(x^i y^j) \longrightarrow M(x^i y^b) \oplus M(x^a y^j) \longrightarrow M(x^a y^b) \longrightarrow 0.$$

Thus the statement follows by [1], Lemma 1.1. □

Lemma 3.9. *If $1 \leq a \leq i \leq n-1$ and $0 \leq b, j \leq n$, then*

$$M(x^{i+1} y^j) \oplus M(x^{a-1} y^b) \leq_{\deg} M(x^i y^j) \oplus M(x^a y^b).$$

Proof. Again, it is straightforward to construct a short exact sequence

$$0 \longrightarrow M(x^a y^b) \longrightarrow M(x^{i+1} y^j) \oplus M(x^{a-1} y^b) \longrightarrow M(x^i y^j) \longrightarrow 0.$$

□

Lemma 3.10. *If L in $\mathcal{S}_\Gamma(n)$ is homogeneous of type (p,a), then we get $L_{(p,a)} \leq_{\deg} L$.*

Proof. If $p = 1$, then there is nothing to prove. Thus assume $p \geq 2$. Let $L = \Gamma^{n-p} \oplus \bigoplus_{i=1}^{p} M(x^{a_i}y^{b_i})$ be in $\mathcal{S}_\Gamma(n)$. Assume that L is homogeneous of type (p,a) and let $m_x = \min\{a_i \mid 1 \leq i \leq p\}$ and $m_y = \min\{b_i \mid 1 \leq i \leq p\}$. First assume that $a_r = m_x$ and $b_s = m_y$ for some $r \neq s$. Applying Lemma 3.9 several times we get $L_{(p,a)} \leq_{\deg} L$. Next assume $a_r = m_x$ and $b_r = m_y$ for some r. Again, we use Lemma 3.9 and get

$$\Gamma^{n-p} \oplus M(x^{n-1}y^{n-1})^{p-1} \oplus M(x^a y^b) \leq_{\deg} L.$$

Then the statement follows from applying Lemma 3.8 once. □

Corollary 3.11. *If L in $\mathcal{S}_\Gamma(n)$ is homogeneous of type (p,a), then the stratum $\mathrm{mod}_\Gamma(n)_L$ is contained in the closure of $\mathrm{mod}_\Gamma(n)_{L_{(p,a)}}$.*

Proof. Let L be homogeneous of type (p,a). By the previous lemma we get $L_{(p,a)} \leq_{\deg} L$. It follows from Lemma 3.7 that the dimension of $\mathrm{Hom}(L_{(p,a)},\Gamma)$ and $\mathrm{Hom}(L,\Gamma)$ coincide. Now we apply the second part of Richmond's Theorem 3.2. □

Next we describe the generic structure of the $\mathrm{mod}_\Gamma(n)_{L_{(p,a)}}$'s. For $L_{(p,a)}$ in $\mathcal{S}_\Gamma(n)$ with $p \geq 2$ let $\mathcal{G}_{(p,a)} = \mathcal{F}(X)$ where

$$X = \bigoplus_{i=1}^{p-2} M(xy;\lambda_i) \oplus M(x^{n-a}y;\lambda_{p-1}) \oplus M(xy^{n-b};\lambda_p).$$

For $L_{(1,a)} \in \mathcal{S}_\Gamma(n)$ let $\mathcal{G}_{(1,a)} = \mathcal{F}(M(x^{n-a}y^{n-b};\lambda))$. The next lemma is clear.

Lemma 3.12. *For $1 \leq i \leq n-1$ we have*

$$\mathcal{G}_{(1,i)} = \{(A,B) \in \mathrm{mod}_\Gamma(n) \mid \mathrm{rk}(A^{n-i}) = \mathrm{rk}(B^i) = 1\} = \mathrm{mod}_\Gamma(n)_{L_{(1,i)}}.$$

Lemma 3.13. *The set $\mathcal{G}_{(p,a)}$ is dense in $\mathrm{mod}_\Gamma(n)_{L_{(p,a)}}$.*

Proof. Let

$$L = L_{(p,a)} = \Gamma^{n-p} \oplus M(x^{n-1}y^{n-1})^{p-2} \oplus M(x^a y^{n-1}) \oplus M(x^{n-1}y^b)$$

where $p \geq 2$. This implies $a = n - i + p - 1$ and $b = i + p - 1$ for some $1 \leq i \leq n-1$. From Lemma 3.7 we get

$$\dim \mathrm{Hom}(L,\Gamma) = n(d-1) + p = 2n^2 + p.$$

Next we use the dimension formulas occuring in the proof of Lemma 3.7 and get

$$\dim \mathrm{End}(L) = 2n^3 - n^2 + n + np + p^2 - 2p.$$

Theorem 3.1 yields

$$\dim \operatorname{mod}_{\Gamma}(n)_L = n^2 - n - p^2 + 2p.$$

Now let

$$X = \bigoplus_{i=1}^{p-2} M(xy; \lambda_i) \oplus M(x^{i-p+1}y; \lambda_{p-1}) \oplus M(xy^{n-i-p+1})$$

with pairwise different λ_i's. Observe that X lies in $\operatorname{mod}_{\Gamma}(n)_L$. Using [10] we get

$$\dim \operatorname{End}(X) = n + p^2 - p.$$

Thus by Lemma 2.3 we have

$$\dim \mathcal{G}_{(p,a)} = n^2 - n - p^2 + 2p.$$

This implies that $\mathcal{G}_{(p,a)}$ is dense in $\operatorname{mod}_{\Gamma}(n)_L$. Finally, observe that the case $p = 1$ is trivial, since $\mathcal{G}_{(1,n-i)} = \operatorname{mod}_{\Gamma}(n)_{L_{(1,i)}}$ for all $1 \leq i \leq n-1$. One easily checks that the dimension of $\operatorname{mod}_{\Gamma}(n)_{L_{(1,i)}}$ is $n^2 - n + 1$. □

Lemma 3.14. *If $a + i, b + j \leq n$, then*

$$M(x^{a+i}y^{b+j}; -\lambda_1\lambda_2) \leq_{\deg} M(x^a y^b; \lambda_1) \oplus M(x^i y^j; \lambda_2).$$

Proof. Again, one constructs a short exact sequence

$$0 \longrightarrow M(x^a y^b; \lambda_1) \longrightarrow M(x^{a+i}y^{b+j}; -\lambda_1\lambda_2) \longrightarrow M(x^i y^j; \lambda_2) \longrightarrow 0.$$

□

We finally get our main result. For $1 \leq i \leq n-1$ let $\mathcal{C}_i$ be the closure of

$$\{(A, B) \in \mathrm{GP}(n) \mid \operatorname{rk}(A^{n-i}) = \operatorname{rk}(B^i) = 1\}.$$

Corollary 3.15. *The sets $\mathcal{C}_1, \cdots, \mathcal{C}_{n-1}$ are the irreducible components of $\mathrm{GP}(n)$. Each component has dimension $n^2 - n + 1$.*

Proof. From Lemma 3.5 we know that the union of the string-free strata is dense in $\operatorname{mod}_{\Gamma}(n)$. Corollary 3.11 yields that the union of the $\operatorname{mod}_{\Gamma}(n)_{L_{(p,a)}}$'s is dense. For each such $\operatorname{mod}_{\Gamma}(n)_{L_{(p,a)}}$ we defined a subset $\mathcal{G}_{(p,a)}$. By Lemma 3.13 we know that $\mathcal{G}_{(p,a)}$ is dense in $\operatorname{mod}_{\Gamma}(n)_{L_{(p,a)}}$. Finally, Lemma 3.14 yields that for $p \geq 2$ the set $\mathcal{G}_{(p,a)}$ is contained in the closure of $\mathcal{G}_{(p-1,a-1)}$. Thus the union of the strata $\operatorname{mod}_{\Gamma}(n)_{L_{(1,i)}}$ for $1 \leq i \leq n-1$ is dense in $\operatorname{mod}_{\Gamma}(n)$. Since these $n-1$ strata all have dimension n^2-n+1, we get that their closures are the irreducible components of $\operatorname{mod}_{\Gamma}(n)$. □

Lemma 3.16. *Let (A, B) be in* $\mathrm{GP}(n)$. *If* $\mathrm{rk}(A) > n - i$ *or* $\mathrm{rk}(B) > i$, *then* (A, B) *is not contained in* $\mathcal{C}_i$.

Proof. Let $1 \leq i \leq n-1$. Since the function $M \mapsto \mathrm{rk}(M)$ is lower semi-continuous, we know that

$$\mathcal{U}_i = \{(A, B) \in \mathcal{C}_i \mid \mathrm{rk}(A) > n - i\} \cup \{(A, B) \in \mathcal{C}_i \mid \mathrm{rk}(B) > i\}$$

is open in $\mathcal{C}_i$. On the other hand, we know that $\mathcal{G}_{(1,i)} = \mathrm{mod}_\Gamma(n)_{L_{(1,i)}}$ is non empty and open in $\mathcal{C}_i$. These sets have an empty intersection. Thus the irreducibility of $\mathcal{C}_i$ implies that $\mathcal{U}_i$ is empty. □

Corollary 3.17. *For* $1 \leq i \leq n-1$ *we have*

$$\mathcal{C}_i = \{(A, B) \in \mathrm{GP}(n) \mid \mathrm{rk}(A) \leq n - i, \mathrm{rk}(B) \leq i\}.$$

If $\mathrm{mod}_A(n)$ is a variety of modules, then the union of G-orbits of maximal dimension is called the *open sheet* of $\mathrm{mod}_A(n)$.

Corollary 3.18. *The open sheet of* $\mathrm{GP}(n)$ *is dense in* $\mathrm{GP}(n)$.

Proof. Let M be in $\mathcal{G}_{(1,i)}$ for some i. Recall that $\dim \mathcal{O}(M) = n^2 - \dim \mathrm{End}(M)$. Using [10] one shows that $\mathcal{O}(M)$ belongs to the open sheet of $\mathrm{GP}(n)$. □

4. Some remarks

4.1. Let $\Gamma_2 = k[x, y]/(xy, x^2, y^2)$. The irreducible components of $\mathrm{mod}_{\Gamma_2}(n)$ for any n are computed in [11]. The variety

$$\mathrm{mod}_{\Gamma_2}(6) = \{(A, B) \in M_6(k) \times M_6(k) \mid AB = BA = A^2 = B^2 = 0\}$$

has exactly three irreducible components. The closures of $\mathcal{O}(M(xy) \oplus M(xy))$ and $\mathcal{O}(M((yx) \oplus M(yx))$ are both 24-dimensional irreducible components. Note that the module $M(xy)$ is projective and $M(yx)$ is injective. Additionally, the closure of the family $\mathcal{F} = \mathcal{F}(M(xy, \lambda_1) \oplus M(xy, \lambda_2) \oplus M(xy, \lambda_3))$ is an irreducible component of dimension 27. Thus the variety $\mathrm{mod}_{\Gamma_2}(6)$ has irreducible components of different dimensions. Observe also that $\mathcal{F}$ contains the set

$$\{(A, B) \in \mathrm{mod}_{\Gamma_2}(6) \mid \mathrm{rk}(A) = 3, \mathrm{rk}(B) = 3\},$$

but the closure of $\mathcal{F}$ does not contain

$$\{(A, B) \in \mathrm{mod}_{\Gamma_2}(6) \mid \mathrm{rk}(A) \leq 3, \mathrm{rk}(B) \leq 3\}.$$

4.2. Gerstenhaber proved in [7] that the *commuting variety*

$$\{(A, B) \in M_n(k) \times M_n(k) \mid AB = BA\}$$

is irreducible, see also [12]. One might ask for the irreducible components of the variety

$$\{(A, B) \in M_n(k) \times M_n(k) \mid A^n = B^n = 0, AB = BA\}.$$

In this case, the corresponding algebra $k[x, y]/(x^n, y^n)$ is not subfinite. Thus one can not use Richmond's approach.

4.3. Richmond's Theorem 3.2 gives some criteria when a stratum is contained in the closure of another one. It is important to find a necessary and sufficient condition. Maybe this would lead to a much shorter proof of our main result.

4.4. Let Q be a quiver and ρ a set of relations in Q, for details see [15]. The associated algebra kQ/I (where kQ is the path algebra of Q, and I is the ideal generated by ρ) is called a *string algebra* if the following hold:

1. Any vertex of Q is the starting point of at most two arrows and also the endpoint of at most two arrows;
2. The elements in ρ are zero relations;
3. Given an arrow β, there is at most one arrow α with $\alpha\beta \notin I$ and at most one arrow γ with $\beta\gamma \notin I$;
4. There exists some n such that each path of length greater than n contains an element of ρ as a subpath.

Note that the algebra $\Gamma_q = k[x, y]/(xy, x^q, y^q)$ is a string algebra. In this paper we restricted our attention to the special case Γ_q, but one can prove that all string algebras are subfinite. Thus one can use Richmond's stratification in this much more general setting. The indecomposable modules over string algebras are also string modules or band modules. This generalization of [6] was first proved in [16] and later revisited in [3], see also [14]. The varieties of modules over string algebras provide many other interesting examples of varieties, and it should be an important task to study them intensively.

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