On Nonlinear Stability of Strong Shock Profiles in Viscous Conservation Laws*

Mingyou Huang†

Abstract

The paper mainly concerns stability problem of strong shock profiles in viscous conservation laws. An example of 2x2 system in isentropic gas dynamics is discussed. First by using a simple geometric argument we prove that this system has two kinds of strong shock profiles and based on results of G. Kreiss-H.O. Kreiss and of K. Zumbrun-P. Howard,we show that these shock profiles are nonlinearly stable if and only if they are linearly asymptotic stable. In addition, we identify that in the scale of standard Lax shock profiles,the criteria in [4] and in [9] for nonlinear stability are equivalent.

Keywords: Viscous conservation laws, traveling wave, viscous shock profile, existence, nonlinear stability.

AMS Classification: Primary 35K55, Secondary 76N10, 35K40, 35L65.

1 Introduction

Due to its close relation with classical shock theory, people are interested in systems of viscous conservation laws

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, t \geq 0 \\ &u = (u_1, u_2, \cdots, u_n), f(u) : \mathbb{R}^n \to \mathbb{R}^n \end{aligned} \tag{1.1}$$

*Supported by the Research Fund of the Doctoral Program of Higher Education, P. R. China and partially by Volkswagen Stiftung, Germany.

†Inst. of Math., Jilin Univ., Changchun 130023, China. E-mail: hmy@mail.jlu.edu.cn

and their traveling wave solutions. Here it is assumed that $A(u) = (\partial f)(u)$ has real, distinct eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u)$$

the corresponding eigenvectors are denoted by $\{r_j(u)\}$

$$A(u)r_j(u) = \lambda_j(u)r_j(u), \quad j = 1, 2, \cdots, n.$$

If $\{\lambda_j(u), r_j(u)\}$ satisfies

$$\nabla_u \lambda_j(u) \cdot r_j(u) \neq 0$$

then the j-th family of $A(u)$ is called genuinely nonlinear.

Let u_L, $u_R \in \mathbb{R}$ be two distinct constant states, which satisfy the Rankine-Hugoniot relation

$$f(u_R) - f(u_L) - s(u_R - u_L) = 0 \tag{1.2}$$

and Lax's entropy condition:

$$\lambda_j(u_R) < s < \lambda_j(u_L) \text{ and } \lambda_{j-1}(u_L) < s < \lambda_{j+1}(u_R) \tag{1.3}$$

for some $1 \leq j \leq n$ and $s \in \mathbb{R}$, we look for progressing wave solution to (1.1) of the form

$$u(x, t) = \phi(\xi), \quad \xi = x - st$$

which satisfies

$$\lim_{\xi \to -\infty} \phi(\xi) = \phi_L, \quad \lim_{\xi \to +\infty} \phi(\xi) = \phi_R, \tag{1.4}$$

where $\phi_L = u_L$, $\phi_R = u_R$. Such solution necessarily is a traveling wave of equation (1.1) with velocity s, and it is also referred to be a "viscous j-shock profile".

In fact, to find the "viscous j-shock profile" consists in solving the following system of nonlinear ordinary differential equations

$$\frac{d^2\phi}{d\xi^2} - \frac{df(\phi)}{d\xi} + s\frac{d\phi}{d\xi} = 0, \quad -\infty < \xi < +\infty$$

subject to boundary conditions (1.4). Thus the problem can be solved by seeking connecting orbit of a first order system of nonlinear ODEs

$$\frac{d\phi}{d\xi} = f(\phi) - f(\phi_L) - s(\phi - \phi_L) =: g(\phi) \tag{1.5}$$

which connects two steady states ϕ_L and ϕ_R of this system. Here $(\partial g)(\phi) = A(\phi) - sI$, which has eigenvalues

$$\mu_j(\phi) = \lambda_j(\phi) - s, \quad j = 1, 2, \cdots, n.$$

The entropy condition (1.3) reads

$$\begin{aligned} &\mu_l(\phi_L) > 0, \quad l = j, \cdots, n, \quad \mu_{j-1}(\phi_L) < 0; \\ &\mu_l(\phi_R) < 0, \quad l = 1, \cdots, j, \quad \mu_{j+1}(\phi_R) > 0 \end{aligned} \tag{1.6}$$

Let $M_s(\phi_R)$ and $M_u(\phi_L)$ be the stable manifold of ϕ_R and the unstable manifold of ϕ_L respectively. The condition (1.6) implies

$$\dim M_s(\phi_R) + \dim M_u(\phi_L) = n + 1.$$

When $M_s(\phi_R)$ and $M_u(\phi_L)$ intersect, i.e. $M_s(\phi_R) \cap M_u(\phi_L) \neq \emptyset$, then system (1.5) necessarily has a connecting orbit between ϕ_L and ϕ_R, and the system of viscous conservation laws (1.1) will possess a viscous j-shock profile.

The best known results on the existence of viscous shock profiles of general conservation laws are confined to weak shock waves, i.e. when u_L and u_R are close enough (see [3]), their proof used the centre manifold theory, but certainly P. D. Lax in his paper [5] had laid down the base for the arguments in [3]. A question that should be asked is: is it possible at least for certain systems of dimension $n > 1$ to show the existence of viscous shock profiles without the requirement of $|u_R - u_L|$ being sufficiently small. C. Conley and J. Smoller in [1] found that in some circumstances, 2×2 system may have strong shock waves.

Another problem that has been intensively studied in the recent decade is the nonlinear stability of viscous shock profiles (see [2],[6]-[8]). Most of these studies mainly concern with weak shock profiles. However, we learned from a recent paper of G. Kreiss and H.-O. Kreiss ([4]) that some algebraic criteria for nonlinear stability are available, which, as declared in their paper, can be applied to viscous shock profiles of arbitrary strength. However there is no example in their paper. We also learned of results recently obtained by K. Zumbrun and P. Howard in [9], where a criterion of nonlinear stability for wider class of viscous shock profiles was given in terms of Evens function. Notice that a major condition in Proposition 10.3 of [9] is not correct, where the set $\{r_j^\pm : a_j^\pm \lessgtr 0\}$ is necessarily to be changed into $\{r_j^\pm : a_j^\pm \gtrless 0\}$ (outgoing eigenvectors).

In this paper, we shall analyze an example of 2×2 system in isentropic gas dynamics. This system has been used as a model example in the study of admissibility of viscous matrixes (see [1]) for conservation laws, but to our knowledge till now this example has not been investigated in the context of nonlinear stability. Firstly in section 2 using a simple geometric method we prove that this system has two kinds of strong viscous shock waves. In section 3, we investigate the nonlinear stability of these shock profiles in terms of the criteria in [4] and [9]. For our example it will be shown that if a viscous shock profile is linearly asymptotic stable,then it must be nonlinearly stable with respect to zero-mass perturbations. Since linear stability is implied by nonlinear stability, we then find that for this example the concepts of linear and nonlinear stability for shock profiles are equivalent. In addition, we find that when restrict to standard Lax shock profiles of genuinely nonlinear conservation laws, the criterion (D) in [9], which implies nonlinear Lp-stability, is equivalent to the algebraic criterion in [4].

2 A 2×2 system and the existence of shock profiles

Consider one dimensional isentropic gas flow, which under certain conditions can be described by the following 2×2 system of viscous conservation laws

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial u}{\partial x} = \dfrac{\partial^2 v}{\partial x^2} \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial p(v)}{\partial x} = \dfrac{\partial^2 u}{\partial x^2} \end{cases} \qquad -\infty < x < +\infty, t > 0 \tag{2.1}$$

where $v(x,t)$ represents the specified volume of the gas, $u(x,t)$ the velocity of the flow, and $p(v)$ is a function representing the relation between pressure and volume which depends on the gas property. Usually $p(v)$ satisfies

$$p'(v) < 0, \quad p''(v) > 0. \tag{2.2}$$

Henceforth we assume that $p(v)$ is a smooth function satisfying (2.2).

In system (2.1), $f = (-u, p(v))^T$ and

$$A = \partial f = \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix}$$

which has real, distinct eigenvalues

$$\lambda_1(v) = -\sqrt{-p'(v)} < \lambda_2(v) = \sqrt{-p'(v)},$$

and the corresponding eigenvectors are

$$r_1(v) = (1, \sqrt{-p'(v)})^T, \text{ and } r_2(v) = (1, -\sqrt{-p'(v)})^T.$$

From (2.2) we see that both families $(\lambda_i(v), r_i(v))$, $i = 1, 2$ are genuinely nonlinear.

Given constant states (v_L, u_L) and (v_R, u_R) satisfying the R-H relation (1.2) and the entropy condition (1.3), we look for shock profiles of system (2.1). This consists in finding connecting orbits of the following first order system of nonlinear ODEs

$$\begin{cases} \dfrac{dv}{d\xi} = -(u - u_L) - s(v - v_L) =: g_1(v, u) \\ \dfrac{du}{d\xi} = p(v) - p(v_L) - s(u - u_L) =: g_2(v, u) \end{cases} \tag{2.3}$$

which has equilibrium points (v_L, u_L) and (v_R, u_R).

Existence of the viscous 2-shock profile

In this case, the entropy condition (1.3) reads

$$\begin{array}{l} -\sqrt{-p'(v_L)} < s < \sqrt{-p'(v_L)} \\ -\sqrt{-p'(v_R)} < \sqrt{-p'(v_R)} < s \end{array} \tag{2.4}$$

which implies (since $p'' > 0$) $s > 0$ and $v_L < v_R$. Further, by the R-H relation

$$\begin{cases} -(u_R - u_L) - s(v_R - v_L) = 0 \\ p(v_R) - p(v_L) - s(u_R - u_L) = 0 \end{cases} \tag{2.5}$$

we have $u_R < u_L$.

The remaining arguments will use a geometric method and take advantage of invariant manifolds. In the phase space with coordinates(v, u) the curve

$$g_1(v, u) = 0, \ \ i.e. \ u = u_L - \frac{1}{s}(v - v_L)$$

is a straight line and the curve

$$g_2(v, u) = 0, \ \ i.e. \ u = u_L + \frac{1}{s}(p(v) - p(v_L))$$

is concave. They intersect at (v_L, u_L), (v_R, u_R). Since $\partial g = A - sI$, we see that (v_L, u_L) is a hyperbolic saddle of (2.3) and (v_R, u_R) is a stable node of (2.3). Let G be the bounded region in the (v, u)-plane confined by curves $g_1(v, u) = 0$ and $g_2(v, u) = 0$, see Figure 2.1.

From the properties of $p(v)$ and (2.5), we find

$$\begin{matrix} g_1(v, u) \geq 0 \\ g_2(v, u) \leq 0 \end{matrix} \quad when \ (v, u) \in \bar{G}. \tag{2.6}$$

Then we claim that at every point

$$p \in \partial G \backslash \{(v_L, u_L), (v_R, u_R)\}$$

the right-hand vector of (2.3) always points into the interior of G as shown in Figure 2.1. In addition, by (2.5) we see that the tangent $\frac{du}{dv} = -\sqrt{-p'(v_L)}$ of the unstable manifold at $\phi_L = (v_L, u_L)$ satisfies

$$\frac{1}{s}p'(v_L) < -\sqrt{-p'(v_L)} < -s,$$

so it is also clear that the unstable manifold of ϕ_L points into the interior of G.

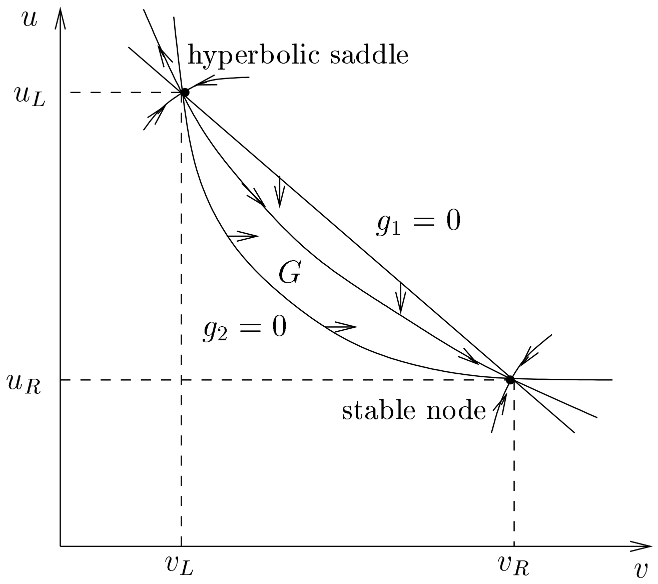

Figure 2.1

Evidently, system (2.3) has no other equilibrium point except (v_L, u_L) and (v_R, u_R).

Based on these facts, we can conclude that system (2.3) possesses a unique orbit which connects the two equilibria (v_L, u_L) and (v_R, u_R) in $\bar{G}$, correspondingly equation (2.1) has a viscous 2-shock profile.

Existence of the viscous 1-shock profile

Here the entropy condition (1.3) reads

$$-\sqrt{-p'(v_R)} < s < \sqrt{-p'(v_R)}, \quad s < -\sqrt{-p'(v_L)} \tag{2.7}$$

which implies

$$s < 0, \quad v_R < v_L, \quad u_R < u_L \text{ (by R-H relation).}$$

In (v, u) plane, now the shape of region G confined by the curves

$$g_1(v, u) = 0 \quad \text{and} \quad g_2(v, u) = 0$$

is shown in Figure 2.2, and it turns out that

$$\begin{matrix} g_1(v, u) \le 0 \\ g_2(v, u) \le 0 \end{matrix} \quad \text{when } (v, u) \in \bar{G}. \tag{2.8}$$

From condition (2.7), we see that (v_L, u_L) is an unstable node and (v_R, u_R) a hyperbolic saddle of system (2.3).

(2.8) tells us that at every point

$$P \in \partial G \backslash \{(v_L, u_L), (v_R, u_R)\}$$

the right-hand vector of (2.3) always points to the exterior of G. Then by considering negative trajectories we can conclude that there exists an orbit of system (2.3) which connects the equilibria (v_L, u_L) and (v_R, u_R).

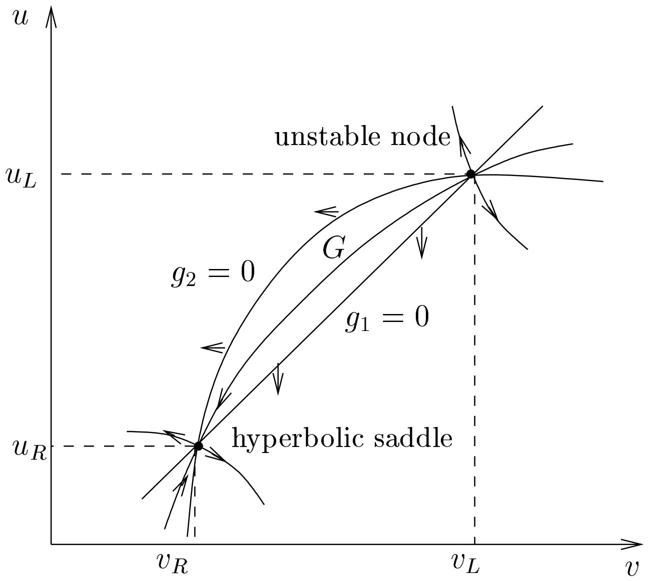

Figure 2.2

Summarizing on analysis above we have proved the following theorem

Theorem 2.1 *Assume that the given states (v_L, u_L), (v_R, u_R) and the parameter $s \in \mathbb{R}$ satisfy the Rankine-Hugoniot relation (2.5) and the entropy condition (2.4) ((2.7)), then system (2.1) possesses a viscous 2-shock profile (viscous 1-shock profile) propagating with velocity s.*

3 Nonlinear stability of shock profiles

To start, we recall the concept of nonlinear stability. Let $U(x - st) = U(\xi)$ be a smooth shock profile of the viscous conservation laws (1.1), and

$$\lim_{\xi \to -\infty} U(\xi) = u_L, \quad \lim_{\xi \to +\infty} U(\xi) = u_R$$

where $u_L, u_R \in \mathbb{R}^n$, $s \in \mathbb{R}$ satisfy the R-H relation (1.2) and the entropy condition (1.3).

Given initial data of type

$$v(x) = U(x) + w_x(x) \tag{3.1}$$

we consider the solution of system (1.1) with initial data (3.1) and denote it by

$$u(x,t) = U(x - st) + v(x,t) \tag{3.2}$$

here $v(x,0) = w_x(x)$ satisfies

$$\int_{-\infty}^{+\infty} v(x,0)dx = 0.$$

and is referred to "zero-mass perturbation". We assume that $v(x,0)$ are smooth and small perturbations and that the corresponding solutions (3.2) exist for all $t > 0$. Then we say the shock profile $U(\xi)$ is nonlinearly stable if the maximum norm

$$|v(\cdot,t)|_\infty \to 0, \text{ as } t \to +\infty \quad \text{when } v(x,0) \text{ is small enough.}$$

In addition, consider the following linear equation

$$\frac{\partial v}{\partial t} + [(\partial f(U) - sI)v]_\xi = \frac{\partial^2 v}{\partial \xi^2} \tag{3.3}$$

and the associated eigenvalue problem

$$\frac{\partial^2 v}{\partial \xi^2} - [(\partial f(U) - sI)v]_\xi = \mu v, \quad \|v\|^2 =: \int_{-\infty}^{+\infty} |v|^2 d\xi < \infty. \tag{3.4}$$

We say that $U(\xi)$ is linearly asymptotic stable if (3.4) has no eigenvalues with $\mathrm{Re}\mu \geq 0$, $\mu \neq 0$.

In [4], combined with some smoothness assumptions of coefficients and data, a criterion to justify nonlinear stability of viscous shock profiles was created, which consists of four "structural conditions":

(i) (3.4) has no eigenvalues with $\mathrm{Re}\mu \geq 0$, $\mu \neq 0$.

(ii) equation

$$\frac{\partial v}{\partial \xi} - [\partial f(U) - sI]\, v = 0 \tag{3.5}$$

has a nontrivial solution $v_0 = U_\xi$, and the dimension of the eigenspace is exactly 1.

(iii) $\mu = 0$ is not a generalized eigenvalue, i.e. equation

$$\frac{\partial v}{\partial \xi} - [\partial f(U) - sI]\, v = D, \quad D = \text{constant}$$

has no bounded solution v with $D \neq 0$.

(iv) equation

$$\frac{\partial^2 v}{\partial \xi^2} - [(\partial f(U) - sI)v]_\xi = \alpha v_0, \ \ -\infty < x < +\infty, \ \alpha \text{ constant}$$

($v_0 = U_\xi$, see (ii)) has no bounded solution with $\alpha \neq 0$.

The main result in [4, Th. 1.9] can be formulated as follows: under certain smooth assumptions (see [4], (1.3)$\sim$(1.6)), if conditions (i)$\sim$(iv) are satisfied, then the shock profile $U(\xi)$ is nonlinearly stable.

It has also been identified in [4] that assumption (iii) is equivalent to the "algebraic condition":

$$\text{the columns of } M_1 = \left[\begin{array}{cc} S_R^{II}, & S_L^I \end{array} \right] \text{ are linearly independent} \tag{3.6}$$

and that assumption (iv) is equivalent to

$$\text{the } n \times n \text{ matrix } M = \left[S_R^{II}, S_L^I, u_R - u_L\right] \text{ is nonsingular} \tag{3.7}$$

where S_R^{II} consists of the eigenvectors of $\partial f(u_R) - sI$ corresponding to positive eigenvalues, and S_L^I consists of the eigenvectors of $\partial f(u_L) - sI$ corresponding to negative eigenvalues.

Now we concentrate on the 2×2 system (2.1) of Section 2. Let $U(\xi) = (v(\xi), u(\xi))$ be the shock profile of (2.1) determined in Th. 2.1. Since

$$\partial f(U(\xi)) - sI = \left[\begin{array}{cc} -s & -1 \\ p'(v(\xi)) & -s \end{array} \right]$$

and the eigenvalues and eigenvectors of this matrix are

$$\lambda_1(v) = -\sqrt{-p'(v)}, \quad \lambda_2(v) = \sqrt{-p'(v)}$$

$$r_1(v) = [1, \sqrt{-p'(v)}]^T, \quad r_2(v) = [1, -\sqrt{-p'(v)}]^T,$$

we see that the matrix in (3.6) is

$M_1 = [1, -\sqrt{-p'(v_R)}]^T$ for the viscous 1-shock profile

$M_1 = [1, \sqrt{-p'(v_L)}]^T$ for the viscous 2-shock profile,

hence condition (3.6), i.e. condition (iii) evidently holds. Besides, the matrix in (3.7) is

$$M = \begin{bmatrix} 1 & v_R - v_L \\ -\sqrt{-p'(v_R)} & u_R - u_L \end{bmatrix} \quad \text{for the viscous 1-shock profile}$$

$$M = \begin{bmatrix} 1 & v_R - v_L \\ \sqrt{-p'(v_L)} & u_R - u_L \end{bmatrix} \quad \text{for the viscous 2-shock profile.}$$

In both cases, it is easy to verify $\det M \neq 0$ by using the R-H relation (2.5) and the entropy condition (2.4) or (2.7), so condition (iv) holds as well.

Further, we discuss condition (ii). Assume on the contrary, equation (3.5) has two independent bounded solutions $\phi_1(\xi)$, $\phi_2(\xi)$, then by using Liouville's theorem, we have

$$\begin{aligned} W(\phi_1, \phi_2) &= W_0 \exp\{\textstyle\int_{\xi_0}^{\xi} \mathrm{trace}[\partial f(U(\xi)) - sI] d\xi\} \\ &= W_0 \exp\{-2s(\xi - \xi_0)\} \end{aligned} \tag{3.8}$$

where W represents the Wronski-determinant, $W_0 = W(\phi_1(\xi_0), \phi_2(\xi_0))$. Since $s \neq 0$ and $W_0 \neq 0$ for properly chosen ξ_0, by taking limit ($\xi \to -\infty$ or $\xi \to +\infty$), a contradiction will be generated for the left-hand side of (3.8) is bounded as assumed. Therefore, condition (ii) is also valid for the system (2.1).

Finally, condition (i) is nothing else,but a condition of linear asymptotic stability,and nonlinear stability implies linear asymptotic stability. Then applying the criterion and theorem in [4, Th. 1.9] leads to the following conclusion

Theorem 3.1 *Assume the function $p(v)$ in (2.1) is smooth and satisfies condition (2.2), then the viscous shock profiles of system (2.1) will be nonlinearly stable if and only if they are linearly asymptotic stable.*

The theory in [9] also can be applied to system (2.1).

A sufficient condition provided in [9] for nonlinear stability of a given stationary wave $\bar{u}(x)$, reads

$$\text{(D)}: \quad D_L(\lambda) \text{ has precisely } l \text{ zeroes in } \{\mathrm{Re}\lambda \geq 0\}$$

where $D_L(\lambda)$ is the Evens function associated with the linearized operator

$$Lv := v_{xx} - (f'(\bar{u})v)_x \tag{3.9}$$

and l is the dimension of the stationary manifold $\{\bar{u}^\delta, \delta \in \mathbb{R}^l\}$ to be the set of all solutions connecting the same limit points $u_\pm = \bar{u}(\pm\infty)$ with $\bar{u}^0 = \bar{u}$. It has been shown (Proposition 11.1, [9]) that for pure and overcompressive shock waves, condition (D) impies nonlinear orbital L^p-stability ($p > 1$) with respect to perturbation in

$$\mathcal{A}_\xi := \{v(x) : |v(x)| \leq \xi(1 + |x|^{-3/2})\}$$

for ξ sufficiently small. It also has been proved (Lemma 9.3, [9]) that condition (D) is equivalent to

$$\sigma(L)\backslash\{0\} \subset \{\mathrm{Re}\lambda < 0\} \tag{3.10}$$

together with a "transversality condition"

$$\left(\frac{d}{d\lambda}\right)^l D_L(0) \neq 0. \tag{3.11}$$

Further, by Proposition 10.3 in [9], condition (3.11) is equivalent to that the set of vectors

$$\{r_j^\pm : a_j^\pm \gtrless 0\} \cup \{\int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta_j} dx, 1 \leq j \leq l\} \tag{3.12}$$

is a basis of $\mathbb{R}^n$, where $a_j^\pm$, $r_j^\pm$ are the eigenvalues and eigenvectors of $A^\pm = f'(u_\pm)$.

These results can be shifted to traveling waves with speed $s \neq 0$ by normalization $x \to \xi = x - st$, $f(u) \to f(u) - su$.

Remark 3.2 *The formulation of condition (3.12) in [9] (Proposition 11.1, (i), (ii)) is not correct, where the eigenvector set $\{r_j^\pm : a_j^\pm \lessgtr 0\}$ (incoming eigenvectors) is necessarily to be changed into $\{r_j^\pm : a_j^\pm \gtrless 0\}$ (outgoing eigenvectors). Condition similar to (3.12) has been used in many studies (see [6], [7] and [4]).*

We see that for the shock waves (1-shock or 2-shock) of system (2.1), $\bar{u}^\delta(x) = \bar{u}(x-\delta)$, $\delta \in \mathbb{R}$, i.e. $l = 1$ and

$$\int_{-\infty}^{+\infty} \frac{\partial \bar{u}^\delta}{\partial \delta} dx = u_+ - u_-.$$

Thus condition (3.12) is reduced to condition (3.7), which is valid as shown in the previous part. Hence we have verified that for system (2.1) the condition (D) is fulfilled if and only if (3.10) holds. Here, (3.10) is the same condition as (i) for linear asymptotic stability. To some extent, the criterion in [9] is designed for application to a wider class of viscous shock waves including undercompressive,overcompressive and the standard(pure) Lax shock (see Definition 10.1-10.2 in [9]). However, it would be of concern to compare the criterion in [9] and in [4] when confined to the most interesting cases of standard Lax shock waves of genuinely nonlinear hyperbolic conservation laws.

Obviously, condition (3.7) implies (3.6), so condition (iv) contains (iii) in the criterion of [4]. By Definition 10.1 and 10.2 in [9], for standard Lax shock waves it turns out that

$$\dim \operatorname{Ker}(L) = l = 1$$

where $\operatorname{Ker}(L)$ and $\sigma(L)$ are referred in $L_2(\mathbb{R})$. Therefore, the condition (ii) of [4] acturely is implicitly contained in the definition of "standard shock wave" of [9] and by the Lax entropy condition. Finally, since (3.10) and (3.12) are equivalent to condition (i) and (iv) respectively, then we prove the following conclusion.

Theorem 3.3 *Assume B(viscosity matrix) $\equiv I$. Then in the scale of standard Lax shock waves, the criterion (D) in [9] is equivalent to the algebraic criterion ((i)-(iv)) in [4].*

Finally, whether system (2.1) is linearly stable might depend on the form of the function $p(v)$, and we leave this as an open question. Also, one may get an answer to this problem by using numerical method when $p(v)$ is specified.

Acknowledgment This paper was finished while the author was visiting Bielefeld University and Cologne University. He would like to thank Prof. W.-J. Beyn and Prof. T. Küpper for their encouragement and valuable discussions.

References

[1] Conley, C. and Smoller,J.A., Viscosity matrices for two-dimensional hyperbolic systems. Comm. Pure Appl. Math. 23, 1970, pp.876-884.

[2] Goodman, J., Nonlinear asymptotic stability of viscous shock profiles for conservation laws, Arch. Rat. Mech. Anal. 95, 1986, pp. 325-344.

[3] Kopell, N. and Howard, L., Bifurcation and trajectories joining critical points, Adv. Math. 18, 1975, pp. 306-358.

[4] Kreiss, G and Kreiss, H.-O., Stability of systems of viscous conservation laws, Comm. Pure Appl. Math. 51, 1998, pp. 1397-1424.

[5] Lax, P. D., Hyperbolic systems of conservation laws and the theory of shock waves. Society for Industrial and Applied Mathematics, Pennsylvania, 1973.

[6] Liu, T. P., Nonlinear stability of shock waves for viscous conservation laws, Mem. Amer. Math. Soc., 328, 1995, pp. 1-108.

[7] Majda, A. and Pego, R., Stable viscosity matrices for systems of conservation laws, J. Diff. Eqs.,56,1985,pp.229-262.

[8] Szepessy, A. and Xin, Z. P., Nonlinear stability of viscous shock waves, Arch. Rat. Mech. Anal. 122, 1993, pp.53-103.

[9] Zumbrun, K. and Howard, P., Pointwise semigroup methods and stability of viscous shock waves, Indiana Univ. Math. J., 47 (1998), No. 3, pp.741-874.