REPRESENTATION-DIRECTED DIAMONDS

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1. Introduction and Main Result

Following [Ri2] a (right) module D over an associative ring A said to be a *diamond* provided it has a simple essential submodule and a superfluous maximal submodule. Obviously any diamond is indecomposable. If A happens to be a finite-dimensional algebra over a field k, then a module D is a diamond if and only D is a finite-dimensional module with a simple socle and a simple top. (Recall, that the top of a module is the factor module by the Jacobson radical.) Since any indecomposable module of length 2 is a diamond, a finite-dimensional algebras A usually will have infinitely many isomorphism classes of diamonds. On the other hand, an algebra A of finite representation type (i.e. A has only finitely many indecomposable modules up to isomorphism) can have only finitely many isomorphism classes of diamonds. At least if the field k is algebraically closed, the algebras of finite representation type are well studied. We refer to [GR] for an exhaustive introduction into this theory.

Using the covering theory developed in [BG], the study of modules over finite-dimensional algebras A of finite representation type over an algebraically closed field k is reduced to the case that A is representation-directed. In particular, any diamond over an algebra of finite representation type is obtained from a diamond over a representation-directed algebra by application of the push down functor associated with the universal Galois covering. Recall, that following [Ri1] an algebra A is said to be *representation-directed* if there does not exist a sequence $X_0, \dots , X_n$ of indecomposable finite-dimensional A-modules with $n > 0$ and $X_0 \cong X_n$ such that for each $i = 1, \dots , n$ there exists a non-zero non-isomorphism $X_{i-1} \to X_i$.

Since factor algebras of representation-directed algebras are again representation-directed, for finding all diamonds over representation-directed algebras it suffices to look at all representation-directed algebras having faithful indecomposable modules and to check which of these faithful modules are diamonds. Fortunately, all representation-directed algebras over an algebraically closed field having an indecomposable faithful module are classified. They appear in 24 families (see [Bo1]) together with many exceptions in low dimensions (see [Dr1]) which were found by a computer program and are accessible via a data basis in the CREP system (see [DN]). Hence, it remains to find out which of the algebras appearing in the families and in the data base have a faithful indecomposable module which is a diamond. It is the aim of this note to present a convenient criterion when this happens:

Theorem. *Let A be representation-directed algebra over an algebraically closed. Then A is obtained from a representation directed algebra having a faithful diamond by reorientation of arms if and only if the vector $\varepsilon = (1, \dots , 1)$ is the only sincere positive 1-root of the Tits form of A.*

We will explain all the notation in the next section. But let us stress that the list of the 24 families (see [Bo1],[Ri1]) as well as the data base of the exceptional algebras (see [Dr1]) for each appearing algebra provides the maximal (with respect to the natural product order on $\mathbb{Z}^r$) positive roots of the associated Tits form. Thus our result really makes the classification of diamonds over representation-finite algebras into an easy exercise. In fact, it is observed already in [Ri2] that among the 24 families one encounters precisely (Bo1), (Bo15), (Bo16), (Bo17), (Bo19), (Bo20), (Bo21) (for the labels used for the families see [Ri1]). For reasons of space we refrain from presenting the explicite list of the 157 exceptional algebras (up to isomorphism and duality) having a faithful diamond but refer to [Dr1] or better to the date base in CREP where everybody can extract the list easily as we did.

2. Representation-Directed Algebras

We refer to [Ri1] for the basic notation. For the study of diamonds we may assume without loss of generality that our given algebra A is basic and connected. It is well-known (see [Ga2]) that any basic finite-dimensional algebra A up to isomorphism an be written as $k\vec{\Delta}/I$ where $\vec{\Delta}$ is a finite quiver and I is an admissible ideal of the path algebra $k\vec{\Delta}$. The quiver $\vec{\Delta}$ is a combinatorial invariant of the algebra A. Since an arrow $\alpha : x \to y$ in $\vec{\Delta}$ yields a non-zero non-isomorphism $P(x) \to P(y)$ where $P(x)$ is the indecomposable projective module associated with the vertex x, the quiver of a representation-directed algebra A has to be directed (i.e. does not admit oriented cycles). The ideal I is not an invariant of A but the number $b(x, y)$ of minimal generators of I starting in x and ending in y does not depend on the particular ideal I. If we denote by $a(x, y)$ the number of arrows form x to y in $\vec{\Delta}$ and label the vertices of $\vec{\Delta}$ by $1, \dots, r$, then it is shown in [Bo2] that the quadratic form $q : \mathbb{Z}^r \to \mathbb{Z}$ called *Tits form* given by $q(x) = \sum_{i=1}^r x_i^2 - \sum_{i,j=1}^r a(i,j)x_ix_j + \sum_{i,j=1}^r b(i,j)x_ix_j$ for $x = (x_1, \dots, x_r) \in \mathbb{Z}^r$ is weakly positive (i.e. $q(x) > 0$ for all $0 \neq x \in \mathbb{Z}^r$ with non-negative coefficients). Consequently, q has only finitely many *positive* 1*-roots* which are the vectors $x \in \mathbb{Z}^r$ with non-negative coefficients satisfying $q_A(x) = 1$.

The positive 1-roots are closely related to the indecomposable A-modules. We remember that we can identify the A-modules with the contravariant representations X of $\vec{\Delta}$ such that $X(\varrho) = 0$ for all elements ϱ of I. Using this identification the *dimension vector* $\mathbf{dim}\, X \in \mathbb{Z}^r$ is given by $(\mathbf{dim}\, X)_i = \dim_k X(i)$ for all vertices $i = 1, \dots, r$. By [Bo2] the map $\mathbf{dim}$ yields a bijection from the set of isomorphism classes of indecomposable A-modules to the set of positive 1-roots of q. A vector x in $\mathbb{Z}^r$ is called *sincere* if $x_i \neq 0$ for all $i = 1, \dots, r$. Analogously, an A-module X is called *sincere* provided $X(i) \neq 0$ for all $i = 1, \dots, r$. Thus the map $\mathbf{dim}$ yields a bijection between the set of isomorphism classes of sincere indecomposable A-modules and the set of sincere positive 1-roots of q_A. It is well-known (see e.g. [Ri1]) that an indecomposable module over a representation-directed algebra is faithful if and only if it is sincere. Moreover, it is shown in [Ri1] that a representation-directed algebra which has an indecomposable sincere module is *simply connected* (see [BG]). Hence A is *completely separating* in the notation of [Dr2] and therefore can be written as BS/J where S is a finite partially ordered set and J is an ideal of the incidence algebra kS generated by elements (y, x) such that there is z in S satisfying $y < z < x$. Recall, that the incidence algebra kS is the vector space with the basis given by all pairs (y, x) such that $y \leq x$ in S. The product $(z, y)(y', x)$ in kS is (z, x) for $y = y'$

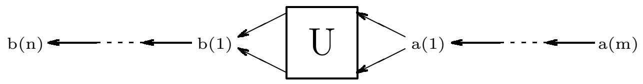
and 0 otherwise. For $A = kS/J$ the quiver $\vec{\Delta}$ of A is the Hasse diagram of S and we can also write A as $A = k\vec{\Delta}/I$ where I is the ideal of $k\vec{\Delta}$ generated by all differences $u - v$ of paths in $\vec{\Delta}$ with the same origin and terminus together with all paths w starting in x and ending in y such that there is a generator (y, x) of J.

Let A be an algebra of the shape $A = kS/J$ for a finite partially ordered set S. With any subset T of S which is convex and *relation-free* (i.e. for each generator (y, x) of J not both x and y may lie in J) there is associated an *indicator module* δ_T which is defined by $\delta_T(x) = k$ for all $x \in T$ and $\delta_T(x) = 0$ otherwise. Moreover, the arrows $\alpha : x \to y$ of $\vec{\Delta}$ are sent to the identity of k for x, y in T and 0 otherwise. Observe, that the module δ_T is indecomposable iff the set T is a connected subset of S.

Proposition. *If $A = kS/J$ is a representation-directed algebra with a sincere diamond X, then $J = 0$, S has a unique minimal and a unique maximal element, X is isomorphic to δ_S and X is up to isomorphism the only sincere indecomposable A-module.*

Proof. Since X is a diamond, there is an epimorphism $\phi : P(x) \to X$ for some element x of S. It is easy to see that $P(x) \cong \delta_T$ where T is the subset of all y in S such that $y \geq x$. The sincerity of X shows $S = \operatorname{supp} X \subseteq \operatorname{supp} P(x) = T$. (For an A-module X we denote by $\operatorname{supp} X$ the set of all elements y of S with $X(y) \neq 0$.) Hence x is the unique minimal element of S and moreover $J = 0$ because $T = S$ is relation-free. Dually, S has a unique maximal element. If ϕ would not be an isomorphism, then its kernel would be non-zero and X would be non-sincere. Let finally N be another sincere indecomposable A-module. Since X is projective and dually also injective, there exist non-zero homomorphisms $X \to N$ and $N \to X$. Consequently, X and N have to be isomorphic because A is representation-directed. $\square$

The above lemma shows that, if $A = kS$ is a representation-directed algebra with a sincere diamond, then there are two possible cases. Either S is a finite chain or the Hasse diagram $\vec{\Delta}$ of S has the following shape where $a(1)$ has at least 2 lower neighbors, $b(1)$ has at least 2 upper neighbors, and all elements of u of U satisfy $a(1) \geq u \geq b(1)$.



If S is a chain, then the graph Δ underlying the Hasse diagram $\vec{\Delta}$ of S is of type $\mathbb{A}_q$. All algebras $A' = kS'$ where the graph Δ' underlying the Hasse diagram $\vec{\Delta}'$ of S' coincides with Δ are representation-directed and the Tits from $q_{A'}$ coincides with q_A. Analogously, in the second case any algebra kS' where the Hasse diagram $\vec{\Delta}'$ of S' is obtained by reorientation of the linear quivers $a(1) \leftarrow a(2) \leftarrow \cdots \leftarrow a(m)$ and $b(n) \leftarrow b(n-1) \leftarrow \cdots \leftarrow b(1)$ in an arbitrary way is representation-directed and has the same Tits form as A.

The algebras kS' obtained in both cases are said to be obtained by *reorientation of arms* from kS. Thus we have shown:

Corollary. *If $A = kS$ is a representation-directed algebra with a sincere diamond and $A' = kS'$ is obtained from A by reorientation of arms, then A' is a representation-directed algebra such that $\varepsilon = (1, \ldots, 1)$ is the only sincere positive 1-root of the Tits form $q_{A'}$.*

3. The Combinatorial Part of the Proof

Lemma. *If $A = kS/J$ is a representation-directed algebra and $\varepsilon = (1, \dots, 1)$ is the only sincere positive 1-root of q_A, then $J = 0$.*

Proof. If X is the indecomposable A-module with $\mathbf{dim}\, X = \varepsilon$, then by [Dr2] we know $X \cong \delta_{\operatorname{supp} X} = \delta_S$. Hence S is relation-free and therefore $J = 0$. □

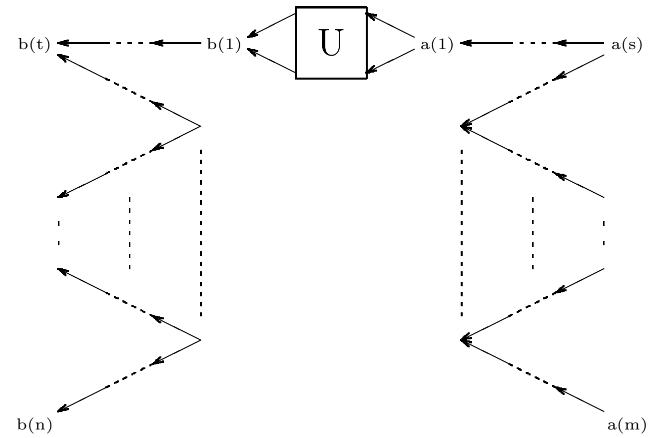
Before continuing, we need another prerequisite. Let $(-,-)_A$ be the symmetric bilinear form associated with the quadratic form q_A and σ_i the reflection with respect to $(-,-)_A$ along the canonical base vector $e(i)$ for $i = 1, \dots, r$. This means that $\sigma_i(x) = x - 2(e(i), x)e(i)$ for all x in $\mathbb{Z}^r$. For x a 1-root the vector $\sigma_i(x)$ is also a 1-root of q_A. In particular, if ε is the only sincere positive 1-root of q_A, then $2(e(i), \varepsilon)_A \geq 0$ for all $i = 1, \dots, r$ because otherwise $\sigma_i(\varepsilon)$ would be another sincere positive 1-root.

Proposition. *If $A = kS$ is a representation-directed algebra such that $\varepsilon = (1, \dots, 1)$ is the only positive 1-root of the Tits form q_A, then A is obtained from a representation-directed algebra with a sincere diamond by reorientation of arms.*

Proof. We proceed by induction on the cardinality r of S and observe that for $r = 1$ nothing is to prove. For $r > 1$ we first consider the case that any element of S is either maximal or minimal. Thus $kS = k\vec{\Delta}$ is a hereditary algebra of finite representation type. By Gabriel's theorem (see [Ga1]) Δ has to be one of the Dynkin diagrams $\mathbb{A}_q$, $\mathbb{D}_q$ or $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. But for all of these diagrams but $\mathbb{A}_q$ the corresponding Tits forms have more than one sincere positive 1-root.

Now we have to deal with the case that there exists an element x of S which is neither minimal nor maximal. We denote by S' the full subposet of S associated with $S \setminus \{x\}$. The partially ordered set S' is connected as well, the algebra kS' is representation-directed, and $X' = \delta_{S'}$ is a sincere indecomposable kS'-module. We assume that there is another sincere indecomposable kS'-module Y' different from X'. By [Dr2] there has to be an element y of S' satisfying $\dim_k Y'(y) \geq 2$. Let L be the left adjoint of the restriction functor from the category of kS-modules to the category of kS'-modules. Hence LY' is an indecomposable kS-module such that $LY'(z) = Y'(z)$ for all elements z of S different from x. By [Bo2] the support of LY' is convex and therefore LY' is a sincere module not isomorphic to δ_S, a contradiction.

Thus we can apply induction to kS' and consider the Hasse diagram $\vec{\Delta}'$ of S'. The case that Δ' is a graph of type $\mathbb{A}_q$ is clear. Otherwise $\vec{\Delta}'$ has the following shape:



We denote by S^u the set of upper neighbors and by S^l the set of lower neighbors of x in S. The sets S^u and S^l are disjoint non-empty antichains in S' such that $z \leq y$ for each z in S^l and y in S^u. That kS is representation-directed implies immediately that $|S^l| + |S^u| \leq 3$. If $S^l \cup S^u$ is contained in $U' := H \cup \{a(1), \ldots, a(s), b(1), \ldots, b(t)\}$, then the claim is clear. So we assume otherwise and distinguish several cases.

Case 1: $|S^l| + |S^u| = 2$. Since both S^l and S^u are non-empty, we have $S^l = \{z\}$ and $S^u = \{y\}$ with $z < y$ in S'. Up to duality we may assume $y \notin U'$.
Case 1.1: $y \in \{b(t+1), \ldots, b(n)\}$, hence $z \in \{b(t), \ldots, b(n)\}$. If there is an arrow $y \to z$ in $\vec{\Delta}'$, then in $\vec{\Delta}$ it is replaced by two arrows $y \to x$ and $x \to z$. Thus $\vec{\Delta}$ has the correct shape. Otherwise y has two lower neighbors and z has two upper neighbors in $\vec{\Delta}$. Consequently Δ contains a subgraph of type $\widetilde{\mathbb{D}}_q$ which is not bound by relations. We arrive at a contradiction to A being of finite representation type.
Case 1.2: $y \in \{a(s+1), \ldots, a(n)\}$, hence $z \in \{a(s+1), \ldots, a(m)\}$. The same arguments as in case 1.1 can be applied.

Case 2: $|S^l| + |S^u| = 3$. Up to duality we now may assume $S^l = \{z_1, z_2\}$ and $S^u = \{y\}$. Thus either $y \notin U'$ or without loss of generality $z_2 \notin U'$. In both situations we observe that there does not exist any element w of S' satisfying $z_1 \geq w \leq z_2$. Therefore $\vec{\Delta}$ contains a full subquiver of the following shape where no other arrows and no relations start or stop at x.

$$\begin{array}{ccccc} & & y & & \\ & & \downarrow & & \\ z_1 & \leftarrow & x & \rightarrow & z_2 \end{array}$$

We obtain a contradiction by the calculation $2(e(x), \varepsilon)_A = 2 - 3 = -1$. $\square$

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