INFINITE QUIVERS AND COHOMOLOGY GROUPS

PU ZHANG

By a theorem of Gabriel (cf. [G]), any finite-dimensional basic algebra over an algebraically closed field k is of the form kQ/I, where Q is a finite quiver, I is an ideal of the path algebra kQ satisfying $J^N \subseteq I \subseteq J^2$ for some $N \geq 2$, and J is the ideal of kQ generated by arrows of Q. In dealing with infinite-dimensional algebras, it is also natural to consider the algebras given by infinte quivers and relations. On the hand, in the recent years, there are a lot of papers to study the Hochschild cohomology groups of finite-dimensional algebras by using some methods in the representation theory of finite-dimensional algebras. This inspires us to consider the Hochschild cohomology groups of some infinite-dimensional algebras given by infinite quivers and relations.

The aim of this paper is to report some results on cohomology groups of algebras given by arbitrary quivers (not necessarily finite quivers) and relations. Different from the finite-dimensional case, the algebras A given by infinte quivers and relations may have no units, and hence the Hochschild cohomology groups $H^n(A)$ are no longer isomorphic to the cohomology groups $\mathrm{E}xt^n_{A^e}(A, A)$ (see e.g. 1.5 below), thus, we should distingush the two cohomology groups for algebras given by infinite quivers and relations. The main results reported here are Theorems 2.2 and 3.1, and some other vanishing conditions on the first Hochschild cohomology group, see Theorems 4.1-4.3.

1. PRELIMINARIES

1.1. A quiver is a datum $Q = (Q_0, Q_1, h, t)$, where Q_0, Q_1 are two sets, which are respectively called the set of vertices and the set of arrows of Q, and $h, t : Q_1 \longrightarrow Q_0$ are two maps, for which $h(\alpha)$ and $t(\alpha)$ are respectively called the head and the tail of arrow α. Thus, for $\alpha \in Q_1$, we write $\alpha : h(\alpha) \longrightarrow t(\alpha)$. We emphasize that the quiver Q considered here may be an infinite quiver, i.e. at least one of Q_0 and Q_1 is an infinite set.

A path p in Q of length l means a sequence of arrows $p = \alpha_1 \cdots \alpha_l$ with $t(\alpha_i) = h(\alpha_{i+1})$ for $1 \leq i \leq l-1$. Call $h(p) = h(\alpha_1)$, $t(p) = t(\alpha_l)$, and $l(p) = l$ respectively the head, the tail, and the length of p. If we regard a vertex $i \in Q_0$ as a path of length 0, it will be denoted by e_i. A path p with $l(p) \geq 1$ is called an oriented cycle provided $h(p) = t(p)$. Denote by Q_p the set of all paths in Q. We emphasize that the quiver Q considered here may contain an oriented cycle.

A quiver is called locally finite provided that for any $i \in Q_0$ there are only finitely many $\alpha \in Q_1$ with $h(\alpha) = i$, and there are only finitely many $\beta \in Q_1$ with $t(\alpha) = i$.

For an arrow $\alpha \in Q_1$, consider the formal inverse α^{-1}. Define $h(\alpha^{-1}) = t(\alpha)$ and $t(\alpha^{-1}) = h(\alpha)$. A walk w is an "unoriented path", i.e. a sequence $w = \beta_1 \cdots \beta_l$, where β_i is an arrow α_i or a formal inverse α_i^{-1} of an arrow α_i such that $t(\beta_i) = h(\beta_{i+1})$, $1 \leq i \leq l-1$,

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and that w contains no subsequences of the form $\alpha\alpha^{-1}$ and the form $\alpha^{-1}\alpha$, where α is an arrow. Define the head $h(w)$ of w to be $h(\beta_1)$, and the tail $t(w)$ of w to be $t(\beta_l)$. A walk of length l is denoted by $i_1 - i_2 - \cdots - i_{l+1}$, where $i_j - i_{j+1}$ means the arrow can go in either direction. In particular, we regard a vertex as a walk of length 0. Two walks w_1 and w_2 are said to be parallel provided that they have the same head and the same tail. A walk $w : i_1 - i_2 - \cdots - i_{l+1}$ is called an unoriented cycle provided that $l > 0, h(w) = i_1 = i_{l+1} = t(w)$, and that the vertices $i_1, \cdots, i_l$ are pairwise distinct. Thus, by definition, for any arrow α, at most one of α and α^{-1} occurs in an unoriented cycle. A quiver Q is called a tree provided that Q contains no unoriented cycles. A quiver is called connected provided that for any two vertices i and j, there is a walk connecting i and j.

For a field k and a quiver Q, let $A = kQ$ be the $k-$space with basis of all paths in Q. For $p = \alpha_1 \cdots \alpha_m, q = \beta_1 \cdots \beta_n \in Q_p$, define the multiplication

$$pq = \begin{cases} \alpha_1 \cdots \alpha_m \beta_1 \cdots \beta_n & t(p) = h(q) \\ 0, & t(p) \neq h(q). \end{cases}$$

Then $A = kQ$ becomes a k-algebra, which is called the path algebra of Q. Note that A may have no unit, however $A^2 = A$; and that A has the unit if and only if Q_0 is a finite set, and in this case $1 = \sum\limits_{i \in Q_0} e_i$; also note that A is finite-dimensional if and only if Q is a finite quiver (i.e. both Q_0 and Q_1 are finite sets) and Q contains no oriented cycles.

We are interested in considering the monomial algebras, which is by definition of the form $A = kQ/I$, where Q is an arbitrary quiver, and I is an ideal of kQ generated by a set of paths of length bigger than 1. In particular, path algebras are monomial algebras. Note that if $A = kQ/I$ is a monomial algebra, then $I \subseteq J^2$, where J is the ideal of kQ generated by all arrows of Q. We emphasize that monomial algebras considered here may be infinite-dimensional.

1.2. Let R be a ring and $e^2 = e \in R$. Then Re and eR are respectively left and right projective R-module. We do not assume that R has a unit, but assume that R has a set of orthogonal idempotents $\{e_i | i \in I\}$ such that $R = \underset{i \in I}{\oplus} Re_i = \underset{i \in I}{\oplus} e_i R$. Consider the category $R - Mod$ of all left R-modules X with $RX = X$, or equivalently, $X = \underset{i \in I}{\oplus} e_i X$ (the morphisms in this category are just the R-module homomorphisms). Clearly, Re_i and R are objects of $R - Mod$; and $R - Mod$ is an extension closed abelian category. Such a ring R is called (left) hereditary provided that every submodule $X \in R - Mod$ of a projective module $P \in R - Mod$ is also projective.

Now, let $Q = (Q_0, Q_1, h, t)$ be a quiver. Then $A = kQ = \underset{i \in Q_0}{\oplus} Ae_i = \underset{i \in Q_0}{\oplus} e_i A$. Note that Ae_i is the $k-$space with basis the set of all paths in Q with tail i. Let $X \in A - Mod$. The following construction of a projective resolution of X is the explicit form of Happel's resolution in [Ha, 1.6], and its proof is due to Crawley-Boevey [CB], both are stated for A being finite-dimensional. Fortunately, it is also valid for infinite quivers. For the convenience of the reader we include the proof here. The tensor product $\otimes$ will mean $\otimes_k$.

Lemma. *(Happel and Crawley-Boevy) We have the following projective resolution of $X \in A - Mod$*

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} Ae_{h(\alpha)} \otimes e_{t(\alpha)} X \xrightarrow{f} \bigoplus_{i \in Q_0} Ae_i \otimes e_i X \xrightarrow{g} X \longrightarrow 0, \qquad (1)$$

where g and f are A-homomorphisms defined by

$$g(a \otimes x) = ax \quad \text{for} \quad a \in Ae_i \quad \text{and} \quad x \in e_i X;$$
$$f(a \otimes x) = a\alpha \otimes x - a \otimes \alpha x \quad \text{for} \quad a \in Ae_{h(\alpha)} \quad \text{and} \quad x \in e_{t(\alpha)} X.$$

Proof. Note that $Ae_i \otimes V$ is always A-projective for any k-space V. Since $X \in A - Mod$, it follows that g is epic. We claim that f is mono. In fact, let $0 \neq \xi \in \bigoplus_{\alpha \in Q_1} Ae_{h(\alpha)} \otimes e_{t(\alpha)} X$. Then ξ can be written as a finite sum

$$\xi = \sum_{\alpha \in Q_1} \sum_{p \in Q_p, t(p)=h(\alpha)} p \otimes x_{\alpha,p} \quad \text{with} \quad x_{\alpha,p} \in e_{t(\alpha)} X.$$

Let p be a path of maximal length such that $x_{\alpha,p} \neq 0$ for some α. Then

$$f(\xi) = \sum_{\alpha \in Q_1} \sum_{p \in Q_p, t(p)=h(\alpha)} (p\alpha \otimes x_{\alpha,p} - p \otimes \alpha x_{\alpha,p}).$$

Since $p\alpha \otimes x_{\alpha,p} \neq 0$ by assumption, it follows that $f(\xi) \neq 0$.

Clearly, $fg = 0$. Let $\eta \in \bigoplus_{i \in Q_0} Ae_i \otimes e_i X$. Then η can be uniquely written as a finite sum

$$\eta = \sum_{i \in Q_0} \sum_{p \in Q_p, t(p)=i} p \otimes x_p \quad \text{with} \quad x_p \in e_i X.$$

Define $deg(\eta)$ to be the maximal length of the paths p with $x_p \neq 0$. Write $p \in Q_p$ with $t(p) = i$ and $l(p) \geq 1$ as $p = p'\alpha$ with $\alpha \in Q_1$. Then

$$f(p' \otimes x_p) = p \otimes x_p - p' \otimes \alpha x_p.$$

Now we can claim that $\eta + Im(f)$ contains an element of degree 0: if $deg(\eta) = d > 0$, then $\eta - f(\sum_{i \in Q_0} \sum_{p \in Q_p, t(p)=i, l(p)=d} p' \otimes x_p)$ is of degree less than d, and the assertion follows from induction.

Let $\eta \in Ker(g)$, and $\eta' \in \eta + Im(f)$ with $deg(\eta') = 0$. Then we can write $\eta' = \sum_{i \in Q_0} e_i \otimes x'_{e_i}$ with $x'_{e_i} \in e_i X$, and then

$$0 = g(\eta) = g(\eta') = g(\sum_{i \in Q_0} e_i \otimes x'_{e_i}) = \sum_{i \in Q_0} x'_{e_i} \in \bigoplus_{i \in Q_0} e_i X,$$

it follows that every component $x'_{e_i} = 0$, and hence $\eta' = 0$ and $\eta \in Im(f)$. $\square$

Corollary. *Let $A = kQ$ with Q a quiver. Then*

(i) For $X \in A - Mod$, the projective dimension $p.d.X \leq 1$.

(ii) A is hereditary.

1.3. Let Λ be an algebra over a field k. We will not insist on Λ to have a unit. Let $\Lambda^e = \Lambda \otimes \Lambda^*$ be the enveloping algebra of Λ, where Λ^* is the opposite algebra of Λ. For $a \in \Lambda$, the corresponding element in Λ^* is denoted by a'. Thus, in Λ^e we have $(a \otimes b')(c \otimes d') = ac \otimes (db)'$. Regard Λ as a left Λ^e-module in a natural way: $(a \otimes b')x = axb$ for $a \otimes b' \in \Lambda^e, x \in \Lambda$.

Now, we consider the path algebra $A = kQ$, where Q is an arbitrary quiver. Then $A^e = \bigoplus_{i,j \in Q_0} A^e(e_i \otimes e'_j) = \bigoplus_{i,j \in Q_0} Ae_i \otimes (e_j A)'$.

Corollary. *Let $A = kQ$ with Q an arbitrary quiver. Then we have the following projective resolution of A over A^e*

$$0 \longrightarrow \bigoplus_{\alpha\in Q_1} Ae_{h(\alpha)} \otimes (e_{t(\alpha)}A)' \xrightarrow{f} \bigoplus_{i\in Q_0} Ae_i \otimes (e_iA)' \xrightarrow{g} A \longrightarrow 0, \tag{2}$$

where g and f are A^e-homomorphisms defined by

$$\begin{aligned} g(a\otimes b') &= ab \quad \text{for} \quad a\in Ae_i \quad \text{and} \quad b\in e_iA;\\ f(a\otimes b') &= a\alpha\otimes b - a\otimes \alpha b \quad \text{for} \quad a\in Ae_{h(\alpha)} \quad \text{and} \quad b\in e_{t(\alpha)}A. \end{aligned}$$

Proof. Take X in (1) in Lemma 1.2 to be ${}_{A^e}A$, then the exact sequence (1) gives a projective resolution of the A^e-module ${}_{A^e}A$, and hence the assertion follows. □

1.4 Let Λ be an algebra over a field k. We will not insist on Λ to have a unit. Let $\Lambda^{\otimes n}$ denote the n-fold tensor product of Λ with itself over k, and X be a Λ^e-module. Regard X as a Λ^e-bimodule by $axb =: (a\otimes b')x$. Recall that the Hochschild complex (C^n, d^n) introduced in [Ho] is defined as follows:

$$\begin{aligned} &C^n = 0 \ \text{ for } \ n<0; \quad C^0 = X; \quad C^n = \mathrm{Hom}_k(\Lambda^{\otimes n}, X) \quad \text{for} \quad n>0;\\ &d^0:\ X \longrightarrow \mathrm{Hom}_k(\Lambda, X) \quad \text{with} \quad d^0x(a) = ax - xa \ \text{ for } \ x\in X,\ a\in\Lambda; \end{aligned}$$

and $\quad d^n : \mathrm{Hom}_k(\Lambda^{\otimes n}, X) \longrightarrow \mathrm{Hom}_k(\Lambda^{\otimes(n+1)}, X) \quad$ with

$$\begin{aligned} d^nf(a_1\otimes\cdots\otimes a_{n+1}) = a_1 f(a_2\otimes\cdots\otimes a_{n+1})&\\ + \sum_{1\le j\le n} (-1)^j f(a_1\otimes\cdots\otimes a_ja_{j+1}\otimes\cdots\otimes a_{n+1})&\\ + (-1)^{n+1} f(a_1\otimes\cdots\otimes a_n)a_{n+1}& \end{aligned}$$

for $f\in C^n$ and $a_1,\cdots,a_{n+1}\in\Lambda$; and the $n-$th Hochschild cohomology of Λ with coefficient in X is by definition the k-space $H^n(\Lambda, X) = Ker(d^n)/Im(d^{n-1})$. In particular, $H^n(\Lambda) = H^n(\Lambda,\Lambda)$ is called the $n-th$ Hochschild cohomology of Λ.

It is clear that $H^0(\Lambda, X) = \{\ x\in X \mid xa = ax, \forall a\in\Lambda\ \}$; in particular, $H^0(\Lambda)$ is the center $Z(\Lambda)$ of Λ. Let

$$Der(\Lambda, X) = \{\ \delta\in \mathrm{Hom}_k(\Lambda, X) \mid \delta(ab) = \delta(a)b + a\delta(b)\ \},$$

and

$$Der^0(\Lambda, X) = \{\ \delta_x \in \mathrm{Hom}_k(\Lambda, X) \mid x\in X\ \} \quad \text{with} \quad \delta_x(a) = xa - ax, \forall a\in\Lambda.$$

Then $H^1(\Lambda, X) = Der(\Lambda, X)/Der^0(\Lambda, X)$; in particular, $H^1(\Lambda) = Der(\Lambda)/Der^0(\Lambda)$, where $Der(\Lambda) = Der(\Lambda,\Lambda)$ and $Der^0(\Lambda) = Der^0(\Lambda,\Lambda)$ are respectively the k-spaces of the derivations and inner derivations of Λ.

1.5. Note that in standard literature (see e.g. [CE, W]) the Hochschild cohomology is defined for algebras with unit, and then we have the isomorphism of k-spaces $H^n(A) \cong \mathrm{Ext}^{\mathrm{n}}_{\mathrm{A^e}}(\mathrm{A},\mathrm{A})$ for $n\ge 0$. But for an algebra without unit, this isomorphism is no longer valid. We have the following

Lemma. *Let $A = kQ$ with Q an arbitrary connected quiver. Then the following are equivalent*

(i) *A has a unit.*

(ii) *Q_0 is a finite set.*

(iii) *$Z(A) \neq 0$.*

(iv) *We have the isomorphism of k-spaces $H^n(A) \cong \mathrm{Ext}^{\mathrm{n}}_{\mathrm{A^e}}(\mathrm{A}, \mathrm{A})$ for $n \geq 0$.*

Proof. It is clear that (i) is equivalent to (ii), and (i) implies (iii). The implication of (i) to (iv) is well known, see e.g. [CE]; and the implication of (iv) to (iii) follows from the fact

$$Z(A) = H^0(A) \cong \mathrm{Hom}_{\mathrm{A^e}}(\mathrm{A}, \mathrm{A}) \neq 0.$$

It remains to prove that if Q_0 is infinite, then $Z(A) = 0$. Otherwise, let $0 \neq a = \sum c_i p_i \in Z(A)$ with $p_i \in Q_p$ and $c_i \in k^*$. Let p be a path amoung those p_i which is of maximal length. Then $h(p) = t(p) = j$. Since Q_0 is infinite and Q is connected, it follows that there exists an arrow α, such that $h(\alpha) = j \neq t(\alpha)$; or $t(\alpha) = j \neq h(\alpha)$. Without loss of generality, we assume that $h(\alpha) = j \neq t(\alpha)$. Then we get the contradiction $a\alpha \neq \alpha a$.

2. Cohomology groups $\mathrm{Ext}^{\mathrm{n}}_{\mathrm{A^e}}(\mathrm{A}, \mathrm{A})$

We are interested in the cohomology groups $\mathrm{Ext}^{\mathrm{n}}_{\mathrm{A^e}}(\mathrm{A}, \mathrm{A})$, where $A = kQ$ with Q an arbitrary quiver.

We need the following observation

Lemma 2.1. *Let $Q = (Q_0, Q_1, h, t)$ be an arbitrary quiver. Then Q is a tree if and only if for any $(d_\alpha)_{\alpha \in Q_1} \in \prod_{\alpha \in Q_1} k_\alpha$, where $k_\alpha = k$ for any $\alpha \in Q_1$, the system of linear equations*

$$x_{t(\alpha)} - x_{h(\alpha)} = d_\alpha, \qquad \forall \alpha \in Q_1 \tag{3}$$

has a solution $x_i = c_i, \ \ \forall i \in Q_0$.

Proof. Assume that the system of linear equations (3) has a solution. If Q is not a tree, then there is an unoriented cycle $c = \beta_1 \cdots \beta_n$ in Q, where β_i is arrow α_i, or a formal inverse of arrow α_i. Choose $(d_\alpha) \in \prod_{\alpha \in Q_1} k_\alpha$ with $d_\alpha = 0$ for all $\alpha \neq \alpha_1$, and $d_{\alpha_1} = 1$. Then from (3) we have the contradiction

$$0 = \sum_{1 \leq i \leq n} (-1)^{\sigma(\beta_i)} d_{\alpha_i} = (-1)^{\sigma(\beta_1)},$$

where $\sigma(\beta_i) = 1$ if $\beta_i = \alpha_i$, and $\sigma(\beta_i) = -1$ if $\beta_i = \alpha_i^{-1}$.

Conversely, without loss of generality, we may assume that Q is a connected tree. Start from an arbitrary vertex i, and take a fixed value of x_i, say $x_i = c_i$. Then for any vertex j we have a walk w starting at i and ending at j, since Q is connected. Then by (3) we obtain the value of $x_j = c_j$. Since Q is a tree, it follows that such a walk w is unique, in this way we get a solution $x_j = c_j, \forall j \in Q_0$, of the system of linear equations (3). $\square$

Theorem 2.2. *Let $A = kQ$, where Q is an arbitrary quiver. Then*

(i) *we have* $\mathrm{Ext}^{\mathrm{n}}_{\mathrm{A^e}}(\mathrm{A},\mathrm{A}) = 0$ *for* $n \geq 2$.

(ii) $\mathrm{Ext}^{1}_{\mathrm{A^e}}(\mathrm{A},\mathrm{A}) = 0$ *if and only if Q is a tree.*

Proof. The assertion (i) follows from Corollary in 1.3.

In order to prove (ii), applying $Hom_{A^e}(-, A)$ to the exact sequence (2) in Corollary 1.3, we see that $\mathrm{Ext}^{1}_{\mathrm{A^e}}(\mathrm{A},\mathrm{A}) = 0$ if and only if the map

$$f^* : \ \mathrm{Hom}_{A^e}(\bigoplus_{i\in Q_0} Ae_i \otimes (e_iA)', A) \longrightarrow \mathrm{Hom}_{A^e}(\bigoplus_{\alpha\in Q_1} Ae_{h(\alpha)} \otimes (e_{t(\alpha)}A)', A)$$

induced by f is epic.

Note that

$$\mathrm{Hom}_{A^e}(\bigoplus_{i\in Q_0} Ae_i \otimes (e_iA)', A) \cong \prod_{i\in Q_0} \mathrm{Hom}_{A^e}(Ae_i \otimes (e_iA)', A);$$

$$\mathrm{Hom}_{A^e}(\bigoplus_{\alpha\in Q_1} Ae_{h(\alpha)} \otimes (e_{t(\alpha)}A)', A) \cong \prod_{\alpha\in Q_1} \mathrm{Hom}_{A^e}(Ae_{h(\alpha)} \otimes (e_{t(\alpha)}A)', A).$$

If Q is a tree, then

$$\mathrm{Hom}_{A^e}(\bigoplus_{i\in Q_0} Ae_i \otimes (e_iA)', A) \cong \prod_{i\in Q_0} kf_i,$$

where $f_i \in \mathrm{Hom}_{A^e}(Ae_i \otimes (e_iA)', A)$ is given by $f_i(e_i \otimes e_i') = e_i$; and

$$\mathrm{Hom}_{A^e}(\bigoplus_{\alpha\in Q_1} Ae_{h(\alpha)} \otimes (e_{t(\alpha)}A)', A) \cong \prod_{\alpha\in Q_1} kf_\alpha,$$

where $f_\alpha \in \mathrm{Hom}_{A^e}(Ae_{h(\alpha)} \otimes (e_{t(\alpha)}A)', A)$ is given by $f_\alpha(e_{h(\alpha)} \otimes e'_{t(\alpha)}) = \alpha$. Note that

$$f^*((c_if_i)_{i\in Q_0}) = ((c_{t(\alpha)} - c_{h(\alpha)})f_\alpha)_{\alpha\in Q_1}. \tag{4}$$

Take an arbitrary element $(d_\alpha f_\alpha)_{\alpha\in Q_1} \in \mathrm{Hom}_{A^e}(\bigoplus_{\alpha\in Q_1} Ae_{h(\alpha)} \otimes (e_{t(\alpha)}A)', A)$, by Lemma 2.1 the system of linear equations

$$x_{t(\alpha)} - x_{h(\alpha)} = d_\alpha, \quad \forall \alpha \in Q_1$$

has a solution $x_i = c_i$, $\forall i \in Q_0$. By (4) this means

$$f^*((c_if_i)_{i\in Q_0}) = (d_\alpha f_\alpha)_{\alpha\in Q_1},$$

that is, f^* is epic.

Conversely, if f^* is epic, then for any $(d_\alpha)_{\alpha\in Q_1} \in \prod_{\alpha\in Q_1} k_\alpha$, where $k_\alpha = k$ for any $\alpha \in Q_1$, there exists $\delta = (\delta_i)_{i\in Q_0} \in \prod_{i\in Q_0} \mathrm{Hom}_{A^e}(Ae_i \otimes (e_iA)', A)$, such that $f^*(\delta) = (d_\alpha f_\alpha)_{\alpha\in Q_1}$. Let $\delta_i(e_i \otimes e_i') = \alpha_i \in e_iAe_i$. Then

$$\begin{aligned} f^*(\delta)(e_{h(\alpha)} \otimes e'_{t(\alpha)}) &= \delta(\alpha \otimes e'_{t(\alpha)} - e_{h(\alpha)} \otimes \alpha') \\ &= (\alpha \otimes e'_{t(\alpha)})\delta_{e_{t(\alpha)}}(e_{t(\alpha)} \otimes e'_{t(\alpha)}) - (e_{h(\alpha)} \otimes \alpha')\delta_{e_{h(\alpha)}}(e_{h(\alpha)} \otimes e'_{h(\alpha)}) \\ &= \alpha\alpha_{t(\alpha)} - \alpha_{h(\alpha)}\alpha. \end{aligned}$$

It follows that

$$\alpha\alpha_{t(\alpha)} - \alpha_{h(\alpha)}\alpha = d_\alpha\alpha \quad \text{for} \quad \alpha \in Q_1. \tag{5}$$

Let $\alpha_i = c_i e_i + x_i$ with $x_i \in e_i A e_i \cap J$, where J is the ideal of A generated by all arrows in Q. Then (5) forces $x_i = 0$ and

$$c_{t(\alpha)} - c_{h(\alpha)} = d_\alpha \quad \forall \alpha \in Q_1,$$

it follows from Lemma 2.1 that Q is a tree. □

3. Monomial algebras

The following result generalizes and unifies several well-known corresponding results in [BM, 2.2], [Ha, 1.6, 2.2, 2.3, 3.2].

Theorem 3.1. *Let $A = kQ/I$ be a monomial algebra with Q connected. Then the following are equivalent*

(i) $H^1(A) = 0$*;*

(ii) Q *is a finite tree;*

(iii) $H^n(A) = 0$ *for* $n \geq 1$*.*

Note that if Q is a tree, then any ideal of kQ is generated by a set of paths in Q. It follows that we have

Corollary 3.2. *Let $A = kQ/I$ with Q a connected tree, and $I \subseteq J^2$, where J is the ideal of kQ generated by all arrows of Q. Then $H^1(A) = 0$ if and only if Q_0 is a finite set.*

In order to prove the theorem, we need the following construction of derivations, which is introduced by Happel [Ha, 3.2] and developed by Bardzell and Marcos [BM, 2.1].

Given any function $w : Q_1 \longrightarrow k$, one can easily see that w can be extended to be a function from Q_p to k, again denoted by w, such that $w(e_i) = 0$ for $i \in Q_0$ and $w(pq) = w(p) + w(q)$ for $p, q \in Q_p$.

Define the k-map $d(w) : kQ \longrightarrow kQ$ by $d(w)(p) = w(p)p$ for $p \in Q_p$. Then it is clear that $d(w) \in Der(kQ)$.

Further, the function w can be extended to the set of all walks in Q by $w(\alpha^{-1}) = -w(\alpha)$ for $\alpha \in Q_1$, again denoted by w.

Lemma 3.3. *Let $A = kQ/I$ with Q an arbitrary quiver and I an ideal of kQ with $I \subseteq J^2$. Let $w : Q_1 \longrightarrow k$ be a function such that $d(w)(I) \subseteq I$. If $H^1(A) = 0$, and q_1, q_2 are two parallel walks in Q, then $w(q_1) = w(q_2)$.*

Proof. The proof is same as the one in [BM, 2.1], in which is stated only for finite-dimensional case. For the convenience of the reader, we include the proof.

For any element $q \in kQ$, denote the canonical image of q in A by $\overline{q}$. Since $d(w)(I) \subseteq I$, it follows that $d(w)$ induces a derivation of A, again denoted by $d(w)$. Thus, $d(w)(\overline{p}) = w(p)\overline{p}$ for $p \in Q_p$. Since $H^1(A) = 0$, it follows that $d(w) = \delta_{\overline{a}}$ for some $a = \sum_{i \in Q_0} c_i e_i + x$ (a finite sum) with $x \in J$. Then for any $\alpha \in Q_1$ we have $w(\alpha)\overline{\alpha} = \delta_{\overline{a}}(\overline{\alpha}) = \overline{a}\overline{\alpha} - \overline{\alpha}\overline{a} =$

$(c_{h(\alpha)} - c_{t(\alpha)})\bar{\alpha} + \bar{y}$ with $y \in J^2$. Since $I \subsetneqq J^2$, it follows that $(w(\alpha) - c_{h(\alpha)} + c_{t(\alpha)})\alpha \in J^2$. This forces

$$c_{h(\alpha)} - c_{t(\alpha)} = w(\alpha), \tag{6}$$

which means that for any walk γ we have $w(\gamma) = w(h(\gamma)) - w(t(\gamma))$. From this the assertion follows. $\square$

Corollary 3.4. *Let A and w be in the same situations as in Lemma 3.3. If $H^1(A) = 0$, then $w(c) = 0$ for any unoriented cycle c.*

For the general case, we have the following similar result as in the finite-dimensional case due to Bardzell and Marcos [BM, 2.2]

Lemma 3.5. *Let $A = kQ/I$ be a monomial algebra. If $H^1(A) = 0$, then Q is a tree.*

Proof. Otherwise, let c be an unoriented cycle containing an arrow α. Choose a function $w : Q_1 \longrightarrow k$ to be $w(\alpha) = 1$ and $w(\beta) = 0$ for otherwise. Since A is monomial, it follows that $d(w)(I) \subseteq I$, and hence $w(c) = 0$. But $w(c) = 1$ by the choice of w. $\square$

Lemma 3.6. *Let $A = kQ/I$ be a monomial algebra. If $H^1(A) = 0$, then Q is locally finite.*

Proof. Otherwise, since Q contains no loops (i.e. an arrow α with $h(\alpha) = t(\alpha)$) by Lemma 3.5 it follows that Q must contain either a subquiver $D = (D_0, D_1)$ with $D_0 = \mathbb{N}_0$ and $D_1 = \{\alpha_i : \ 0 \longrightarrow i \mid i \in \mathbb{N}_1 \}$, or a subquiver $D' = (D'_0, D'_1)$ with $D'_0 = \mathbb{N}_0$ and $D'_1 = \{\alpha_i : \ i \longrightarrow 0 \mid i \in \mathbb{N}_1 \}$, here $\mathbb{N}_i = \{ n \in \mathbb{Z} \mid n \geq i \}$.

By the dual argument, it suffices to assume that Q contains a subquiver D. Consider the function $w : Q_1 \longrightarrow k$ by

$$w(\alpha_{2n}) = 1 \ \text{ for } \ n \geq 1; \quad \text{and} \ \ w(\beta) = 0, \quad \forall \beta \in Q_1 - \{ \alpha_{2n} \mid n \in \mathbb{N}_1 \}.$$

Since A is a monomial algebra, it follows that $d(w)(I) \subseteq I$, and hence we have the derivation $d(w)$ of A given by $d(w)(\bar{p}) = w(p)\bar{p}$ for $p \in Q_p$, where $\bar{p}$ denotes the canonical image in A of p. Let $\delta = \delta_{-\bar{\alpha}_1} + d(w)$. Then $\delta \in Der(A)$. We claim that $\delta \notin Der^0(A)$, and hence we get a desired contradiction with the assumption $H^1(A) = 0$.

In fact, if $\delta = \delta_{\bar{a}}$, where $a \in kQ$ is a finite sum of the form

$$a = \sum_{i \in \mathbb{N}_0} c_i e_i + \sum_{i \in \mathbb{N}_1} d_i \alpha_i + \sum_{\alpha \in \Omega} t_\alpha \alpha,$$

with $\Omega = Q_p - \{e_0, e_i, \alpha_i \ \text{ for } \ i \in \mathbb{N}_1 \}$. Then we have

$$\bar{\alpha}_{2n} = \delta(\bar{\alpha}_{2n}) = \delta_{\bar{a}}(\bar{\alpha}_{2n}) = \bar{a}\bar{\alpha}_{2n} - \bar{\alpha}_{2n}\bar{a} = (c_0 - c_{2n})\bar{\alpha}_{2n} + \bar{x}$$

with $x \in J^2$. Since $I \subseteq J^2$, it follows that

$$c_0 - c_{2n} = 1. \tag{7}$$

Similarly, we have

$$c_0 - c_{2n-1} = 0. \tag{8}$$

Since a is a finite sum, it follows that almost all c_i are zero, which contradicts (7) and (8). $\square$

Lemma 3.7. *Let $A = kQ/I$ be a monomial algebra with Q connected and $H^1(A) = 0$. Then Q is a finite tree.*

Proof. Since Q is connected, by Lemmas 3.5 and 3.6 it is enough to prove Q_1 is a finite set.

Consider the k-map $\delta : A \longrightarrow A$ given by $\delta(\bar{p}) = l(p)\bar{p}$ for $p \in Q_p$, where $l(p)$ is the length of p. Then $\delta \in Der(A)$ since A is monomial, and hence $\delta = \delta_{\bar{a}}$ for some $a = \sum_{i \in Q_0} c_i e_i + x$ (a finite sum) with $x \in J$. Since $I \subseteq J^2$, it follows that

$$c_{h(\alpha)} - c_{t(\alpha)} = 1, \quad \text{for} \quad \alpha \in Q_1. \tag{9}$$

Let Q_0' be the set of vertices i with $c_i \neq 0$. Since a is a finite sum, it follows that Q_0' is a finite set. For every arrow α we see from (4) that at least one of $h(\alpha)$ and $t(\alpha)$ belongs to Q_0'.

If Q_1 is an infinite set, then there exists a vertex $i \in Q_0'$ and infinitely many arrows α_j, such that i is the head or the tail of α_j. This is impossible since Q is locally finite by Lemma 3.6. $\square$

Lemma 3.8. *([Ha, 2.2]) Let $A = kQ/I$ with Q a finite tree and $I \subseteq J^2$. Then $H^n(A) = 0$ for all $n \geq 1$.*

Proof of Theorem 3.1. The implication of (i) to (ii) follows from Lemma 3.7; and the implication of (ii) to (iii) follows from Lemma 3.8. $\square$

Remark. For a non-monomial algebra $A = kQ/I$, even if Q is a finite quiver, the Theorem is no longer valid. For example, let Q be the quiver with $Q_0 = \{ 1, 2, 3, 4 \}$ and $Q_1 = \{ \alpha : 1 \longrightarrow 2;\ \beta : 2 \longrightarrow 4;\ \gamma : 1 \longrightarrow 3;\ \delta : 3 \longrightarrow 4 \}$; and $I = \langle \alpha\beta - \gamma\delta \rangle$. Then $H^1(kQ/I) = 0$, but Q is not a tree.

4. Vanishing of the first Hochschild cohomology

Let $A = kQ/I$ with Q a finite quiver and I an ideal of kQ with $J^N \subseteq I \subseteq J^2$ for some positive integer N. Recently, Buchweitz and Liu ([BL]) have constructed an algebra A with a loop with $H^1(A) = 0$. However, in many cases, $H^1(A) = 0$ implies that Q is directed, that is, Q contains no oriented cycles (cf. [Ha]). We include several results towards this direction.

For an arbitrary quiver Q, recall that a relation ρ in Q is a finite combination $\sum c_i p_i$ of paths p_i of length bigger than 1, such that all p_i have the same head, and have the same tail, see [R, p.43]. Note that any ideal I of kQ with $I \subseteq J^2$ is generated by a set of relations in Q. An ideal $I = \langle \rho_i \rangle$ with all $\rho_i = \sum c_{i,j} p_{i,j}$ being relations is called a homogeneous ideal proveded that $l(p_{i,j})$ does not depends on j.

Denote by $\widetilde{A}_{p,q}$ the quiver with $Q_0 = \{ 1, \cdots, p+q \}$ and $Q_1 = \{ \alpha_i : i \longrightarrow i+1$ for $1 \leq i \leq p-1;\ \alpha_p : p \longrightarrow p+q;\ \beta_1 : 1 \longrightarrow p+1;\ \beta_j : p+j-1 \longrightarrow p+j$ for $2 \leq j \leq q \}$.

Theorem 4.1. *Let $A = kQ/I$ with Q an arbitrary quiver and I be a homogeneous ideal of kQ. If $H^1(A) = 0$, then Q does not contain a subquiver $\widetilde{A}_{p,q}$ with $p \neq q$.*

Proof. Otherwise, denote by c the unoriented cycle given by a subquiver $\widetilde{A_{p,q}}$. Consider the length function l, that is, $l(\alpha) = 1$ for $\alpha \in Q_1$. Since I is homogeneous, it follows that $d(w)(I) \subseteq I$ and $l(c) = 0$ by Corollary 3.4. On the other hand, $l(c) \neq 0$ since $p \neq q$. $\square$

Theorem 4.2. *Let $A = kQ/I$ with Q an arbitrary quiver and I an ideal of kQ with $I \subseteq J^2$. If $H^1(A) = 0$, then either Q is a tree, or for any unoriented cycle c and any arrow α occuring in c, there exists a generating relation $g = \sum\limits_{1\leq i\leq n} c_i p_i$ of I with $n \geq 2$, and i, j such that $m_\alpha(p_i) \neq m_\alpha(p_j)$, where $m_\alpha(p_i) =$ the times of the occurances of α in p_i.*

Proof. Assume that Q is not a tree. Then by Theorem 3.1 we see that A is not monomial. Let $I = \langle \rho_i \mid i \rangle$ with all ρ_i being relations. Consider the set S of all generators of I which are not paths. If there exists an oriented cycle c and an arrow α occuring in c, such that $m_\alpha(p_i) = m_\alpha(p_j)$ for any relation $g = \sum\limits_{1\leq i\leq n} c_i p_i \in S$ and all $1 \leq i, j \leq n$, then define $w : Q_1 \longrightarrow k$ by $w(\alpha) = 1$ and $w(\beta) = 0$ for otherwise. Then $d(w)(I) \subseteq I$ and by Corollary 3.4 we have $w(c) = 0$. But $w(c) = 1$ by the choice of w. $\square$

Theorem 4.3. *Let Q be a cyclic quiver, that is, $Q_0 = \{1, 2, \cdots, l\}$ and $Q_1 = \{\alpha_i : i \longrightarrow i+1, \forall 1 \leq i \leq l-1;\ \alpha_l : l \longrightarrow 1\ \}$; and I be an ideal of kQ with $J^N \subseteq I \subseteq J^2$ for some positive integer N. Then I is generated by some paths, and $H^1(kQ/I) \neq 0$.*

Proof. By Theorem 3.1, it is enough to prove the first assertion. Otherwise, assume that $g = \sum\limits_{1\leq i\leq n} c_i p_i \in I$ is a relation with $\prod\limits_{1\leq i\leq n} c_i \neq 0$, $n \geq 2$, $p_i \in Q_p$, and $p_i \notin I$ for $1 \leq i \leq n$, such that n is minimal amoung all such relations in I. Since Q is a cyclic quiver, we see $l(p_1) > \cdots > l(p_n)$.

Since $J^N \subseteq I \subseteq J^2$ for some positive integer N, it follows that kQ/I is finite-dimensional, and hence there exists a unique path q such that $l(p_1 q)$ is minimal amoung $p_1 q \in I$. Denote $p_1 q$ by p. Then

$$p = p_1 q = c_1^{-1}(gq - \sum_{2\leq i\leq n} c_i p_i q),$$

it follows that $\sum\limits_{2\leq i\leq n} c_i p_i q \in I$. By the minimality of n, we see that $p_i q \in I$ for $2 \leq i \leq n$.

Let c be the oriented cycle with $l(c) = l$ and $h(c) = t$, $t(c) = t$, where $t = t(p_i)$. Then $p_1 = p_n c^m$ for some $m \geq 1$. Write q as $q = c^r q_1$ with $r \geq 0$ and $h(q_1) = t$, $l(q_1) < l$.

If $r \geq m$, then we have

$$p_n q = p_n c^r q_1 = (p_n c^m)(c^{r-m} q_1) = p_1(c^{r-m} q_1) \in I,$$

which contradicts to the assumption of q since $l((c^{r-m} q_1)) < l(q)$.

If $m > r$, then there exists a unique $p' \in Q_p$ such that $c^{m-r} = q_1 p'$, and hence we have

$$p_1 = p_n c^m = (p_n c^r) c^{m-r} = (p_n c^r)(q_1 p') = (p_n q) p' \in I,$$

a contradiction. This completes the proof. $\square$

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Department of Mathematics, University of Science and Technology of China

Hefei 230026, P. R. China e-mail: pzhang@ustc.edu.cn