Properly discontinuous groups of affine transformations

A survey

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March 16, 2000

Abstract

In this survey article we will discuss the structure of properly discontinuous groups of affine transformations and in particular of affine crystallographic groups. One of the main open questions here is Auslander's conjecture claiming that every affine crystallographic group is virtually solvable.

1 Introduction

The motivating questions are the following:

Question *(Milnor) Is every properly discontinuous group of affine transformations virtually solvable?*

Question *(L. Auslander) Is every crystallographic group of affine transformations virtually solvable?*

It turns out that already in dimension 3 the answer to Milnor's question is negative. Concerning the Auslander conjecture, a number of positive results have been obtained. They will be described below. But the question in general is still open.

The contents of the paper are as follows. Roughly speaking, sections 2 through 5 present the questions mentioned above and put them into perspective. Sections 6 through 9 deal with answers and section 10 gives further results. In more detail, after establishing notation in section 2, the questions mentioned above are presented and put into context in section 3. Their geometric significance, namely for flat affine manifolds, is discussed in section 4. The case of crystallographic — and more generally properly discontinuous — groups of affine isometries is covered by Bieberbach's theory. One part of this theory is described in section 5, another part in section 10, subsection 5. At these places it is also discussed how and to which extent this theory generalizes for the more general situation of arbitrary affine groups. Coming to the answers, we see in section 6 that every properly discontinuous affine group in dimension ≤ 2 is virtually solvable, we describe all of them and discuss which of their features survive in higher dimensions. For dimension 3, we present in section 7 a proof of the key case of the Auslander conjecture. In section 8 the displacement function α of Margulis is defined. It is essential for his construction of a free properly discontinuous affine group on affine 3-space. We try to give the reader a geometric intuition of the relevance of this invariant. In section 9 we generalize this to $SO(n+1, n)$, a key case for higher dimensions. In section 10 further results are collected.

I would like to thank my coauthors G. Margulis and G. Soifer for arousing my interest in these questions and A. Lubotzky for his encouragement to write this survey. I also thank Yale University and the Isaac Newton Institute for their hospitality during the time that this survey was written. I thank these institutions and the NSF and the SFB 343 in Bielefeld for financial support.

2 Affine space

In this section we give the basic definitions concerning affine spaces and establish notation.

Throughout this article all vector spaces are over $\mathbb{R}$ and of finite dimension. Affine space is a vector space where you forget the zero. To be precise, an *affine space* $\mathbb{E}$ is a set together with a simply transitive action of a vector space V. The action is usually denoted as addition $V \times \mathbb{E} \to \mathbb{E}$, $(v, x) \longmapsto x + v$. So for any two points x, y in $\mathbb{E}$ there is a unique vector $v \in V$ such that $x + v = y$. This vector is usually written as the *difference* of the two points: $v = y - x$. For every $v \in V$ the map $T_v : \mathbb{E} \to \mathbb{E}$, $T_v(x) = x + v$, is called the *translation* by v. And V is called the *vector space of translations* of $\mathbb{E}$ and denoted $V = T\mathbb{E}$. Given two affine spaces $\mathbb{E}$ and $\mathbb{F}$ a map $f : \mathbb{E} \to \mathbb{F}$ is called an *affine map* if there is a point $x_0 \in \mathbb{E}$ and a linear map $A : T\mathbb{E} \to T\mathbb{F}$ such that $f(x_0 + v) = f(x_0) + Av$. Then $f(x + v) = f(x) + Av$ holds for every $x \in \mathbb{E}$ and every $v \in T\mathbb{E}$ with the same linear map A. The map A is called the *linear part* of f and denoted Lf. One can regard $T\mathbb{E}$ as the tangent space of $\mathbb{E}$ at every point of $\mathbb{E}$ and Lf as the tangent map of f at every point of $\mathbb{E}$. We have the chain rule

$$L(f \circ g) = Lf \circ Lg.$$

An affine map $f : \mathbb{E} \to \mathbb{F}$ of affine spaces is an isomorphism iff $Lf : T\mathbb{E} \to T\mathbb{F}$ is a linear isomorphism. Let $\mathrm{Aff}(\mathbb{E})$ be the group of affine automorphisms of $\mathbb{E}$, also called the group of affine transformations of $\mathbb{E}$. We have an exact sequence of groups

$$1 \to T\mathbb{E} \to \mathrm{Aff}(\mathbb{E}) \to GL(T\mathbb{E}) \to 1.$$

For every $x \in \mathbb{E}$ this exact sequence has a splitting homomorphism

$$\begin{aligned} &\sigma_x : GL(T\mathbb{E}) \to \mathrm{Aff}(\mathbb{E}), \\ \text{given by} \quad &\sigma_x(A)(x + v) = x + Av. \end{aligned}$$

So the affine group is isomorphic to the semidirect product of $V = T\mathbb{E}$ with $GL(V)$, where $GL(V)$ acts on V in the natural way:

$$\mathrm{Aff}(\mathbb{E}) \cong V \rtimes GL(V).$$

Homogenization. It is sometimes useful to think of an affine space as an affine hyperplane in a vector space of one dimension more, as follows. Let V be a vector space and let $\mathbb{F}$ be the affine hyperplane

$$\mathbb{F} = \{(v, 1)\ ;\ v \in V\}$$

in the vector space $V \oplus \mathbb{R}$. Then $\mathbb{F}$ is an affine space with $V = T\mathbb{E}$. Conversely, given an affine space $\mathbb{E}$ with $T\mathbb{E} = V$ then for every $x \in E$ the map $\iota_x : \mathbb{E} \to \mathbb{F}$, $\iota_x(x + v) = (v, 1)$, is an isomorphism of affine spaces. As a consequence we obtain a group isomorphism between $\mathrm{Aff}(\mathbb{E})$ and the subgroup

$$\left\{\begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix}\ ;\ A \in GL(V), t \in V\right\}$$

of $GL(V \oplus \mathbb{R})$ given by sending the above matrix in $GL(V \oplus \mathbb{R})$ to $g(A, t) \in \mathrm{Aff}(\mathbb{E})$ with $g(A, t)(x+v) = x+t+A\,v$. In particular, we may regard $\mathrm{Aff}(\mathbb{E})$ as a linear group, namely as a subgroup of $GL(V \oplus \mathbb{R})$.

3 The questions

3.1 Definition Let Γ be a group acting on a locally compact space X. The action is called *properly discontinuous* if for every compact subset K of X the set of returns $\{\gamma \in \Gamma\ ;\ \gamma K \cap K \neq \emptyset\}$ is finite. The action is called *crystallographic* if it is properly discontinuous and the orbit space $\Gamma \backslash X$ is compact.

Now let $\mathbb{E}$ be an n–dimensional real affine space and let Γ be a subgroup of $\mathrm{Aff}(\mathbb{E})$. Since $\mathrm{Aff}(\mathbb{E})$ acts on $\mathbb{E}$ it makes sense to say that the subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ is properly discontinuous or crystallographic. The question is: What is the structure of Γ? More precisely:

3.2 Question (L. Auslander 1964 [Au 1]) *Is every crystallographic subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ virtually solvable?*

Actually, Auslander stated as a theorem a positive answer to this question. But the proof turned out to be false. The content of the *Auslander conjecture* is that the answer is yes, i.e. that every affine crystallographic group is virtually solvable.

Later, Milnor asked the following more general question.

3.3 Question (Milnor 1977 [Mi]) *Is every properly discontinuous subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ virtually solvable?*

Here and in what follows a group Γ is said to have a certain property P *virtually* if Γ contains a subgroup Δ of finite index which has the property P.

To put these questions into perspective recall the following

3.4 Theorem (Tits alternative [Ti]) *Let Γ be a subgroup of $GL(n, \mathbb{C})$. Then Γ is either virtually solvable or contains a free non–abelian subgroup.*

The Tits alternative applies to $\mathrm{Aff}(\mathbb{E})$ and its subgroups since $\mathrm{Aff}(\mathbb{E})$ is isomorphic to a subgroup of $GL(n+1, \mathbb{R})$, as explained at the end of section 2. So the questions above really ask to which of the two types of groups described by the Tits alternative do our groups Γ belong. It was expected that the answers to these questions were yes. It turned out that the answer to Milnor's question is no. The first counterexample is due to Margulis [Ma 1, Ma 2], and we will explain below the geometry behind this counterexample, see section 8. The answer to Auslander's question is not known, in general. So far only positive answers have been obtained.

Why is one interested in these questions? We will give an algebraic motivation in this paragraph and a geometric motivation in the next section. Much is known about discrete subgroups and in particular lattices in solvable Lie groups, see e.g. Raghunathan's book [Ra], and also for lattices in semisimple Lie groups, see Margulis's book [Ma 4]. The affine group is a Lie group which is of none of these types, neither solvable nor semisimple, it is of what is sometimes called mixed type. And it is one of the simplest and most natural groups of mixed type, namely the semidirect product of $GL(n, \mathbb{R})$ with $\mathbb{R}^n$. And yet we do not know the answer to Auslander's question.

A remark concerning the relation between the notions "discrete" and "properly discontinuous" is in order here. The group $\mathrm{Aff}(\mathbb{E})$ is a Lie group in a natural way. Every properly discontinuous subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ is discrete, as follows immediately from the definition. The converse is not true. For instance, if we regard $GL(n, \mathbb{R})$ as the group of affine transformations fixing a point $O \in \mathbb{R}^n$, then every discrete infinite subgroup of $GL(n, \mathbb{R})$ is not properly discontinuous, since the very definition of proper discontinuity implies that for a properly discontinuous action the isotropy group $\Gamma_x = \{\gamma \in \Gamma \ ; \ \gamma x = x\}$ of every point $x \in \mathbb{E}$ be finite.

On the other hand, properly discontinuous subgroups of the affine group are geometrically more interesting than just discrete subgroups, as explained in the next section. There is an important case, however, where the two notions coincide, see section 5.

3.5 Remark A group Γ is called *polycyclic* if it contains a sequence of subgroups $\Gamma = \Gamma_0 > \Gamma_1 > \cdots > \Gamma_t = \{e\}$ such that Γ_{i+1} is normal in Γ_i and Γ_i/Γ_{i+1} is cyclic for $i = 0, \ldots, t-1$. Clearly, every polycyclic group is solvable. The converse is not true, in general. But every discrete solvable subgroup of $GL(n, \mathbb{R})$ is polycyclic (follows from [Ra, Proposition 3.8]). So our questions are sometimes stated as: Is every properly discontinuous (resp. crystallographic) affine group virtually polycyclic?

4 Flat affine manifolds

There is a geometric interest in properly discontinuous and in particular crystallographic groups since they are the fundamental groups of manifolds with certain geometric structures, namely complete flat affine manifolds.

To understand what a *flat affine manifold* is, recall the following definition of a C^∞–manifold. A C^∞–manifold M is a Hausdorff topological space and has an atlas $\mathfrak{A}$ of local coordinate systems $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$, where U is an open subset of M and φ is a homeomorphism of U onto an open subset $\varphi(U)$ of a real vector space $\mathbb{R}^n$. Any two local coordinate systems of this atlas are C^∞–compatible, i.e. if $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ and $\psi : V \to \psi(V) \subset \mathbb{R}^m$ are in our atlas $\mathfrak{A}$, then the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a C^∞–map. The definition of a flat affine manifold is exactly the same except that one requires that the atlas $\mathfrak{A}$ be *affine*, that is that the transition maps are locally restrictions of affine maps $\mathbb{R}^n \to \mathbb{R}^m$.

It thus makes sense to say what an *affine line segment* in a flat affine manifold M is, namely it is an injective map $\sigma : I \to M$ from some open interval $I \subset \mathbb{R}$ to M which when composed with any local coordinate system of the given affine atlas $\mathfrak{A}$ of M gives a map from I to some $\mathbb{R}^n$ which is locally the restriction of affine maps. Note that an affine line segment is uniquely determined by its image up to an affine reparametrisation, that is if $\sigma : I \to M$ and $\tau : J \to M$ are two line segments with $\sigma(I) = \tau(J)$ then there is an affine automorphism $g \in \mathrm{Aff}(\mathbb{R})$ of $\mathbb{R}$ such that $\tau = \sigma \circ g$. An *affine line* in M is an affine line segment defined on all of $\mathbb{R}$. A flat affine manifold M is called *complete* if every affine line segment in M is the restriction of an affine line in M.

If M is a flat affine manifold its universal covering manifold $\widetilde{M}$ is easily seen to be a flat affine manifold in a natural way. And M is complete iff $\widetilde{M}$ is complete. Now every simply connected complete flat affine manifold $\widetilde{M}$ is isomorphic qua affine manifold to $\mathbb{R}^n$, if $\dim M = n$. It follows that the group of decktransformations $\Gamma \cong \pi_1 M$ is in a natural way a properly discontinuous subgroup of $\mathrm{Aff}(\mathbb{R}^n)$. The action of Γ on $\mathbb{R}^n$ has the property that no element $\gamma \neq e$ has a fixed point. This property is equivalent to Γ being torsion free. The reason is that every finite group of affine transformations has a fixed point, for instance the center of gravity of an orbit. Conversely, if Γ is a properly discontinuous torsion free subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ then $\Gamma\backslash\mathbb{R}^n$ is a complete flat affine manifold M with $\pi_1 M \cong \Gamma$. We thus have

Question of Milnor, geometric version: *Is the fundamental group of every complete flat affine manifold virtually solvable?*

Question of Auslander, geometric version: *Is the fundamental group of every compact complete flat affine manifold virtually solvable?*

Note that a compact flat affine manifold need not be complete, in contrast to the situation in Riemannian geometry. Compare also section 10.5.

4.1 Remark The argument above shows that the geometric version of the questions of Auslander and Milnor are precisely the special case of the corresponding questions of section 3 for the case that Γ is torsion free. It suffices in fact to answer the questions of section 3 for torsion free Γ, for the following reason. It suffices to answer them for finitely generated Γ. For properly discontinuous Γ this follows from the Tits alternative. And crystallographic Γ are finitely generated anyway. Now apply Selberg's lemma, by which every finitely generated subgroup of $GL(n, \mathbb{C})$ contains a torsion free subgroup of finite index, cf. [Ra, Theorem 6.11].

5 The classical case, groups of affine isometries and Bieberbach's theorems

Let $< \, , \, >$ be a positive definite bilinear form on the vector space V. Then on any affine space $\mathbb{E}$ with $T\mathbb{E} = V$ one can define a metric by $d(x, y) = < y - x, y - x >^{1/2}$. Let $G = \mathrm{Isom}(\mathbb{E})$ be the group of isometries of $\mathbb{E}$. Then G is a subgroup of $\mathrm{Aff}(\mathbb{E})$, in fact it is the group of those affine transformations of $\mathbb{E}$ whose linear part is in the orthogonal group $\mathbb{O}$ of the quadratic form $< \, , \, >$. We thus have an exact sequence

$$1 \to T\mathbb{E} \to \mathrm{Isom}(\mathbb{E}) \to \mathbb{O} \to 1.$$

In this case a subgroup Γ of $\mathrm{Isom}(\mathbb{E})$ is properly discontinuous iff Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{E})$. The reason is that the whole group $G = \mathrm{Isom}(\mathbb{E})$ acts properly on $\mathbb{E}$. This means that for every compact subset K of $\mathbb{E}$ the set of returns $\{g \in G \, ; \, gK \cap K \neq \emptyset\}$ is compact. To see this note that the group $T\mathbb{E}$ of translations acts properly on $\mathbb{E}$ and hence the extension $\mathrm{Isom}(\mathbb{E})$ by the compact group $\mathbb{O}$ does, as is easy to see.

For the case of isometric affine actions the two questions about the structure of discrete and crystallographic subgroups Γ of $G = \mathrm{Isom}(\mathbb{E})$ have long been answered, in response to Hilbert's 18th problem formulated by Hilbert at the International Congress of Mathematicians in 1900. The crystallographic subgroups of G are the groups which are associated with crystals occurring in nature, whence their name.

5.1 Theorem (Bieberbach) *Every discrete subgroup Γ of G is virtually abelian. Every crystallographic subgroup Γ of G is virtually a translation group.*

So for a crystallographic subgroup Γ of G the subgroup $\Delta = \Gamma \cap T\mathbb{E}$ of translations is of finite index in Γ and hence the image $L(\Gamma)$ of Γ under the linear part map L is finite, since $L(\Gamma) \cong \Gamma/\Delta$. Furthermore Δ is crystallographic, too, since its finite index overgroup Γ is. So Δ is a lattice in $T\mathbb{E}$. Choosing a basis for Δ we see that $L(\Gamma)$ is represented by integral matrices with respect to this basis. These facts are the starting point for the classification of crystallographic groups, cf. [Wo, BBNWZ].

Bieberbach's theorem above gives an algebraic result. The papers of Bieberbach [B 1, B 2, B 3, F] give in fact the following description of discrete subgroups Γ of G which contains very precise geometric information about how Γ acts on the affine space.

5.2 Theorem (Bieberbach) *If Γ is a discrete subgroup of G there is a Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$ such that the restriction homomorphism $r : \Gamma \to \Gamma|\mathbb{F}$ has finite kernel and a crystallographic subgroup of $\mathrm{Isom}(\mathbb{F})$ as image.*

One gets an amazingly good intuition of the behaviour of Γ by looking at the very special case of a screw motion in $\mathbb{R}^3$ of a cyclic group around a line $\mathbb{F}$ as axis. One can use the geometric insight of the two Bieberbach theorems to develop an algorithm for deciding the following question, see [Ab 1, Ab 2] . Given a finite subset S of G. Let Γ be the subgroup of G generated by S. When is Γ discrete?

Coming back to $\mathrm{Aff}(\mathbb{E})$, one may ask if one cannot improve on Auslander's question, namely that every crystallographic subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ is virtually abelian or virtually nilpotent. The answer is no, in general, if $\dim \mathbb{E} \geq 3$. For an example, let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^n$ and let Γ be the subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ generated by the $n-1$ translations by $e_1, \dots, e_{n-1}$ and the map $x \longmapsto Ax + e_n$, where $A = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$ and $B \in GL(n-1, \mathbb{Z})$. Then Γ is a crystallographic subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ and Γ is nilpotent iff B is unipotent.

There are generalizations of Bieberbach's theorem for subgroups of $\mathrm{Aff}(\mathbb{E})$. A very general and useful fact is the following

5.3 Theorem *Let G be a Lie group and R a closed connected solvable normal subgroup of G. Let $\pi : G \to G/R$ be the natural map. Let H be a closed subgroup of G such that H^0, the identity component of H is solvable. Let $U = \overline{\pi(H)}$ be the closure of $\pi(H)$. Then the identity component U^0 of U is solvable.*

For a proof see [Ra, Theorem 8.24]. Bieberbach's theorem 5.1 is an immediate corollary. The following corollary is proved as in [Ra, Corollary 8.27]. It is often useful as a reduction step when dealing with discrete groups. For a typical application see the proof of 7.1. The reader should be warned that [Ra, Corollary 8.25 and 8.28] are false. I thank D. Witte for pointing this out to me.

5.4 Corollary *Let G be a connected Lie group and R its radical. Let Γ be a discrete subgroup of G and let $\pi : G \to G/R$ be the natural map. Suppose $\pi(\Gamma)$ is Zariski dense in G/R. Then $\pi(\Gamma)$ is discrete.*

For the case of $\mathrm{Aff}(\mathbb{E})$ there is an even closer generalization of Bieberbach's theorem proved by Carrière and Dal'bo. Let Γ be a subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ and put $\Gamma_{\mathrm{nd}} = \Gamma \cap L^{-1}(\overline{L(\Gamma)}^0)$. Think of Γ_{nd} or rather $L(\Gamma)_{\mathrm{nd}} = L(\Gamma) \cap \overline{L(\Gamma)}^0$ as the non discrete part of $L(\Gamma)$.

5.5 Theorem [CD] *If Γ is a discrete subgroup of $\mathrm{Aff}(\mathbb{R}^n)$, then Γ_{nd} is nilpotent and finitely generated. If Γ is crystallographic then Γ_{nd} is unipotent, i.e. $L(\Gamma_{\mathrm{nd}})$ is unipotent.*

In 10.2 f. we come back to comparing Bieberbach's theory for Euclidean crystallographic groups with the more general situation of affine crystallographic groups.

6 A lemma, dimension 2

Let us return to the full group $\mathrm{Aff}(\mathbb{E})$ of affine transformations. Let Γ be a properly discontinuous subgroup of $\mathrm{Aff}(\mathbb{E})$. Passing to a subgroup of finite index, we may assume that Γ is torsion free and hence every element $\gamma \neq e$ of Γ has no fixed point, see remark 4.1. The following lemma is easy but basic.

6.1 Lemma *If $\gamma \in \mathrm{Aff}(\mathbb{E})$ has no fixed point then 1 is an eigenvalue of $L(\gamma)$.*

We can give the following more precise description of γ: After choosing a base point in $\mathbb{E}$ and a basis $e_1, \ldots, e_n$ of $T\mathbb{E}$ suitably, an element $\gamma \in \mathrm{Aff}(\mathbb{E})$ with no fixed point can be written in the form

$$\gamma x = Ax + t \tag{6.2}$$

with

$$A = \begin{pmatrix} B & 0 \\ 0 & J_r \end{pmatrix}, t = e_n, \tag{6.3}$$

where $B \in GL(n-r, \mathbb{R})$ and J_r is an $r \times r$ Jordan matrix

$$J_r = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

with $r \geq 1$. So embedding $\mathrm{Aff}(\mathbb{E})$ into $GL(n+1, \mathbb{R})$ as at the end of section 2 we have $\gamma = \begin{pmatrix} B & 0 \\ 0 & J_{r+1} \end{pmatrix}$.

It is worth giving a proof. After choosing a base point for $\mathbb{E}$ we can write γ in the form $\gamma x = Ax + t$ with $A \in GL(T\mathbb{E})$ and $t \in T\mathbb{E}$. Then γ has a fixed point iff $Ax + t = x$ has a solution iff $t \in \mathrm{Im}(A - I)$. Thus if γ has no fixed point one arrives at the form 6.3 using the following two facts. By choosing an appropriate base point for $\mathbb{E}$ one can change t to an arbitrary element $t' \equiv t \mod \mathrm{Im(A - I)}$. Thus if $t \notin \mathrm{Im}(A - I)$ then 1 is an eigenvalue of A and one can take t as the last vector e_n of a Jordan basis of the primary space corresponding to the eigenvalue 1.

As an application of the lemma we see

6.4 Corollary *Γ is virtually solvable if $\dim \mathbb{E} \leq 2$.*

Proof The case $\dim \mathbb{E} = 1$ is trivial since $\mathrm{Aff}(\mathbb{E})$ is solvable. For $\dim \mathbb{E} = 2$ we use a few basic facts about algebraic groups. We may assume that Γ is torsion free, so 1 is an eigenvalue of $L(\gamma)$ for every $\gamma \in \Gamma$. The same then holds for the algebraic closure G of $L(\Gamma)$, hence G is of codimension at least one in $GL(2,\mathbb{R})$. The only possibility for a non–solvable algebraic subgroup of $GL(2,\mathbb{R})$ of dimension ≤ 3 is $SL(2,\mathbb{R})$, which is impossible by the eigenvalue 1 criterion.

Using the more precise description of formula 6.3 one can classify the torsion free elements of Aff(2). One can go further and describe all the properly discontinuous subgroups of Aff(2), virtually, see [Ku]. Here is the result. Consider the following three subgroups of $\mathrm{Aff}(\mathbb{R}^2)$

$$\begin{aligned} H &= \left\{ \begin{pmatrix} e^t & \\ & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ t \end{pmatrix} ; \quad t \in \mathbb{R} \right\}, \\ T &= \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} ; s,t \in \mathbb{R} \right\}, \\ P &= \left\{ \begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}, \begin{pmatrix} t \\ s \end{pmatrix} ; s,t \in \mathbb{R} \right\}. \end{aligned}$$

H, T and P stand for “hyperbolic”, “translation” and “parabolic”, respectively.

6.5 Proposition *Each of these three subgroups of* $\mathrm{Aff}(\mathbb{R}^2)$ *acts properly on* $\mathbb{R}^2$*, hence every discrete subgroup of any one of them acts properly discontinuously on* $\mathbb{R}^2$*. Conversely, every properly discontinuous subgroup* Γ *of* $\mathrm{Aff}(\mathbb{R}^2)$ *contains a subgroup* Δ *of finite index which is a discrete subgroup of* H*,* T *or* P*, after an appropriate choice of a basepoint and a basis for* $T\mathbb{E}$*.*

Recall from section 5 that an action of a locally compact topological group G on a locally compact topological space X is called *proper* if the set of returns $\{g \in G; gK \cap K \neq \emptyset\}$ of G is (relatively) compact for every compact subset K of X. If $\Gamma \neq \{e\}$ the three cases of proposition 6.5 are disjoint with the following exception: A cyclic group of translations is in $T \cap P$ in suitable coordinates. Incidentally, the groups T and P are conjugate by the polynomial automorphism f of $\mathbb{R}^2$ where $f(x,y) = (x + \frac{y^2}{2}, y)$.

There are numerous consequences of this classification in dimension 2 and it is interesting to ask if the corresponding statements are true in higher dimensions. Let Γ be a properly discontinuous subgroup of $\mathrm{Aff}(\mathbb{E})$. The statements of the corollaries are for $\dim \mathbb{E} = 2$.

6.6 Corollary $(\dim \mathbb{E} = 2)$ Γ *is virtually abelian.*

We had already seen that this is no longer true if $\dim \mathbb{E} \geq 3$, for an example see the remark before 5.3.

6.7 Corollary $(\dim \mathbb{E} = 2)$ Γ *is virtually a discrete cocompact subgroup of a connected Lie subgroup* G *of* $\mathrm{Aff}(\mathbb{E})$ *and* G *acts properly on* $\mathbb{E}$*.*

For virtually solvable properly discontinuous groups this is true for arbitrary dimensions of $\mathbb{E}$. But no uniqueness of G can be achieved, see [FG].

6.8 Corollary $(\dim \mathbb{E} = 2)$ *Suppose* Γ *is properly discontinuous. Then* Γ *is crystallographic iff* Γ *is virtually isomorphic to* $\mathbb{Z}^2$*.*

In higher dimensions, there is of course a purely group theoretical characterization of the crystallographic groups among the properly discontinuous ones, as follows. A properly discontinuous subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ is crystallographic iff the virtual cohomological dimension of Γ equals $\dim \mathbb{E}$. This criterion is essential for one of the proofs of the Auslander conjecture in dimension 3, see the proof of 7.1 below.

6.9 Corollary ($\dim \mathbb{E} = 2$) *A subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ is crystallographic iff it is virtually a cocompact subgroup of a connected Lie group $G \subset \mathrm{Aff}(\mathbb{E})$ acting properly and simply transitively on $\mathbb{E}$. Actually, then G is an algebraic subgroup of $\mathrm{Aff}(\mathbb{E})$.*

In higher dimensions, the first statement is still true for virtually solvable crystallographic subgroups Γ of $\mathrm{Aff}(\mathbb{E})$, see [FG] and 10.2. The second statement is not true in higher dimensions. And no uniqueness of G can be achieved.

6.10 Corollary ($\dim \mathbb{E} = 2$) *If Γ is a torsion free properly discontinuous subgroup of $\mathrm{Aff}(\mathbb{E})$ then $\Gamma \backslash \mathbb{E}$ is diffeomorphic to a cylinder $S^1 \times \mathbb{R}$ or a torus $S^1 \times S^1$.*

The second case occurs of course iff Γ is crystallographic. In dimension 3, every crystallographic subgroup of $\mathrm{Aff}(\mathbb{E})$ contains a subgroup Δ of finite index such that $\Delta \backslash \mathbb{E}$ is a differentiable 2–torus bundle over the circle, see [FG].

7 The Auslander conjecture in dimension 3

We have all the ingredients to give a proof of at least the key case of the Auslander conjecture in dimension 3.

7.1 Theorem [FG] *Every crystallographic subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ is virtually solvable if $\dim \mathbb{E} = 3$.*

Proof of one case Using the eigenvalue 1 criterion of lemma 6.1 one sees that the theorem is true unless the semisimple part of the algebraic hull of Γ is $SL(2,\mathbb{R}) \times \{1\}$ or $SO(2,1)$ in appropriate coordinates. In the first case, one can show that the image of Γ cannot be Zariski dense in $SL(2,\mathbb{R}) \times \{1\}$ if Γ is properly discontinuous. The proof uses dynamical properties of such affine maps and is thus similar in spirit to the considerations of the next section. Now let us only consider the second case. So suppose $L(\Gamma)$ is contained and Zariski dense in $G = SO(2,1)$. Then $L(\Gamma)$ is a discrete subgroup of G, by 5.4. We claim that $L : \Gamma \to G$ is injective. If $\ker(L|\Gamma) \neq 0$, then $\Theta := \ker(L|\Gamma)$ is a lattice in the group T of translations of $\mathbb{E}$, since the representation of G, and hence of $L(\Gamma)$, on T is irreducible and Θ is an $L(\Gamma)$–module and a discrete subgroup of T. But then Θ must be of finite index in Γ, since $\Theta \backslash \mathbb{E} \to \Gamma \backslash \mathbb{E}$ is a covering map of compact spaces — assuming Γ is torsion free. Hence $L(\Gamma) \cong \Gamma/\Theta$ is finite, contradicting that $L(\Gamma)$ is Zariski dense in G. This shows that $L : \Gamma \to G$ is injective. So Γ is isomorphic to the discrete subgroup $L(\Gamma)$ of G. But every discrete subgroup of G acts properly discontinuously on the symmetric space X of G. The complex upper half plane is a model of X, so X is a 2–dimensional contractible manifold. It follows that $L(\Gamma)$ has virtual cohomological dimension at most 2. On the other hand, Γ acts crystallographically on $\mathbb{E} = \mathbb{R}^3$, hence has virtual cohomological dimension 3, a contradiction. Note that for this last step of the argument it is crucial that Γ is crystallographic.

For a geometric proof using the dynamics of affine maps, see [S 2]. The author actually shows the result for semigroups.

7.2 In [FG] Fried and Goldman proceed to classify all the crystallographic subgroups Γ of $\mathrm{Aff}(\mathbb{E})$ for $\dim \mathbb{E} = 3$. As an abstract group Γ is virtually isomorphic to $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{Z}^2$ by the powers of a matrix $A \in SL(2,\mathbb{Z})$ with positive eigenvalues ([FG] end of 5.4). This group Γ sits naturally in the Lie group $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$. Every such G can be embedded into $\mathrm{Aff}(\mathbb{R}^3)$ in such a way that the resulting affine action of G on $\mathbb{R}^3$ is proper and simply transitive. These embeddings are not unique up to conjugation in $\mathrm{Aff}(\mathbb{R}^3)$, in none of the different cases distinguished. The cases are: If A is hyperbolic, i.e. $\mathrm{tr}\, A > 2$, then G is solvable not nilpotent. If A is the identity then G is abelian. If A is neither, then G is isomorphic to the 3–dimensional Heisenberg group. One such embedding for every case is given by sending $((u,v),t)$ in $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ or $\mathbb{R}^2 \rtimes_A \mathbb{R}$ to

$$\begin{pmatrix} 1 & 0 \\ 0 & A^t \end{pmatrix}, \begin{pmatrix} t \\ u \\ v \end{pmatrix}.$$

The proof of the classification relies on Lie theory, see 10.2 f.

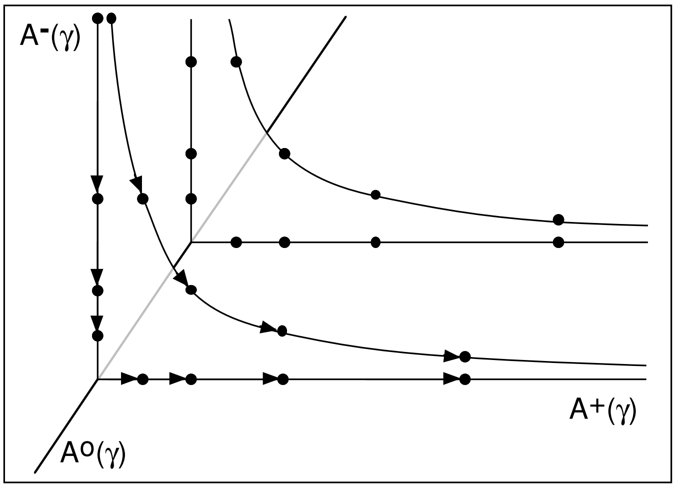
8 A free properly discontinuous affine group in dimension 3

In this section I will describe an invariant for certain affine transformations due to Margulis. This invariant is crucial for the counterexample to Milnor's question in dimension 3. Higher dimensional analogues of this invariant are similarly essential for the Auslander and Milnor questions in those dimensions.

So suppose $\dim \mathbb{E} = 3$. We mentioned already, in the proof of 7.1, that a properly discontinuous subgroup Γ of $\mathrm{Aff}(\mathbb{E})$ is virtually solvable unless $L(\Gamma)$ is virtually contained in $SO(2,1)$.

8.1 So let $\gamma \in \mathrm{Aff}(\mathbb{R}^3)$, $A := L(\gamma) \in SO(2,1)$ and suppose $\mathrm{tr}\, A > 3$. Then A has three real eigenvalues: 1, $\lambda > 1$ and $\lambda^{-1} < 1$. The action of A on $\mathbb{R}^3$ is easy to understand. A fixes the line $A^\circ(\gamma) := \mathrm{Eig}(A, 1)$, it acts by expansion by the factor λ on

$A^+(\gamma) := \mathrm{Eig}(A,\lambda)$ and by contraction by λ^{-1} on $A^-(\gamma) := \mathrm{Eig}(A,\lambda^{-1})$.



The orbit under $<A>$ of any point $\notin (A^+(\gamma)\cup A^-(\gamma))\oplus A^0(\gamma)$ is contained in a hyperbola contained in a plane parallel to $A^+(\gamma)\oplus A^-(\gamma)$. The element A is therefore called *hyperbolic*.

8.2 It is equally easy to understand the action of an *affine transformation* γ with $L(\gamma)$ hyperbolic. There is a unique γ–invariant affine line in $\mathbb{E}$, called the *axis* of γ and denoted $C(\gamma)$. We have $TC(\gamma) = A^0(\gamma)$ and γ induces a translation on $C(\gamma)$, denoted $\tau(\gamma)$. Note that $\tau(\gamma)\in A^0(\gamma)$. So for a point

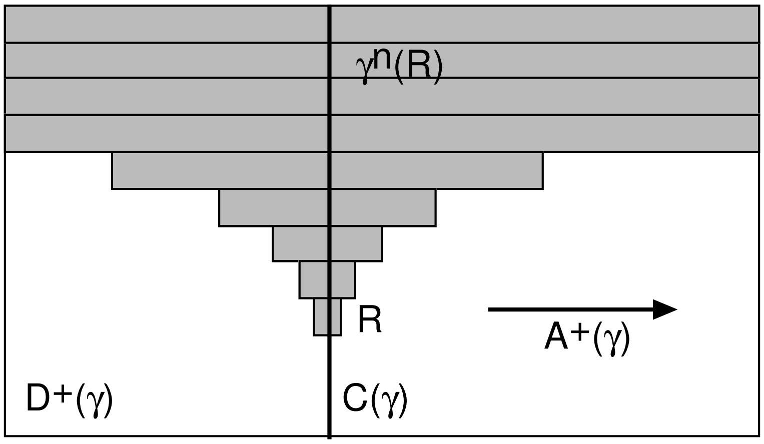
$$x = x^0 + v^+ + v^- \in \mathbb{R}^3$$

with $x^0\in C(\gamma)$, $v^\pm \in A^\pm(\gamma)$ we have

$$\gamma^n x = x^0 + n\tau(\gamma) + \lambda^n v^+ + \lambda^{-n} v^-.$$

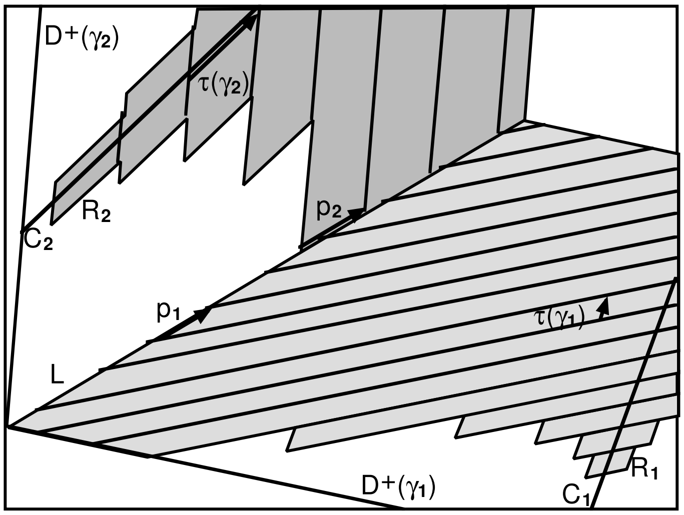
γ has a fixed point iff $\tau(\gamma) = 0$. We thus assume $\tau(\gamma)\neq 0$.

Let us see what happens to a rectangle $R = \{x^0 + s\tau(\gamma) + tv^+;\ s\in[0,1],\ t\in[-\varepsilon,+\varepsilon]\}$ in the plane $D^+(\gamma) := C(\gamma) + A^+(\gamma)$ of length $\tau(\gamma)$ and small width 2ε with symmetry axis $C(\gamma)$. After n times applying γ we obtain the rectangle $\gamma^n R = \{x^0 + s\tau(\gamma) + tv^+;\ s\in[n,n+1],\ t\in[-\lambda^n\varepsilon,\lambda^n\varepsilon]\}$ of the same length but exponentially growing width $2\lambda^n\varepsilon$ for $n\in\mathbb{N}$.



8.3 Let us now look at two affine transformations. So suppose γ_1, γ_2 are in $\mathrm{Aff}(\mathbb{R}^3)$ such that $L(\gamma_i) \in SO^o(2,1)$ are hyperbolic and $\tau(\gamma_i) \neq 0$ for $i = 1, 2$. Let us assume furthermore that they are in general position. To be precise we shall need only that $A^+(\gamma_1) \not\subset A^0(\gamma_2) \oplus A^+(\gamma_2)$ and $A^+(\gamma_2) \not\subset A^0(\gamma_1) \oplus A^+(\gamma_1)$. Then $D^+(\gamma_1) \cap D^+(\gamma_2)$ is a line, say L. Let us see what happens to our rectangles R_1 and R_2 defined as above: $\gamma_i^n R_i$ intersects L for $n \in \mathbb{N}$ large since $TL \not\parallel A^+(\gamma_i)$, and the intersection $\gamma_i^n R_i \cap L$ is a line segment in L of the form $\{y_i + tp_i \ ; \ t \in [n, n+1]\}$ for some point $y_i \in L$ for $n >> 0$, where $p_i \in TL$ is the projection of $\tau(\gamma_i)$ onto TL along $A^+(\gamma_i)$. We have $p_i \neq 0$ since $0 \neq \tau(\gamma_i) \in A^0(\gamma_i)$, hence $\tau(\gamma_i) \notin A^+(\gamma_i)$. So p_1 and p_2 are non zero multiples of each other. Let us consider the case that they are positive multiples of each other, i.e. $p_2 = r \cdot p_1$ with $r > 0$.

In this case, as in the picture below, there are an infinite number of pairs (n_1, n_2), $n_1 \to +\infty$, $n_2 \to +\infty$, such that $\gamma_1^{n_1} R_1 \cap \gamma_2^{n_2} R_2 \neq \emptyset$. But this cannot happen if Γ is properly discontinuous, since it is easy to see that there are infinitely many different elements among the $\gamma_1^{-n_1} \gamma_2^{n_2}$.



Let us state the result.

Two elements γ_1, γ_2 in $\mathrm{Aff}(\mathbb{R}^3)$ as above are said to form a *positive pair* if $p_2 = r \cdot p_1$ for $r > 0$.

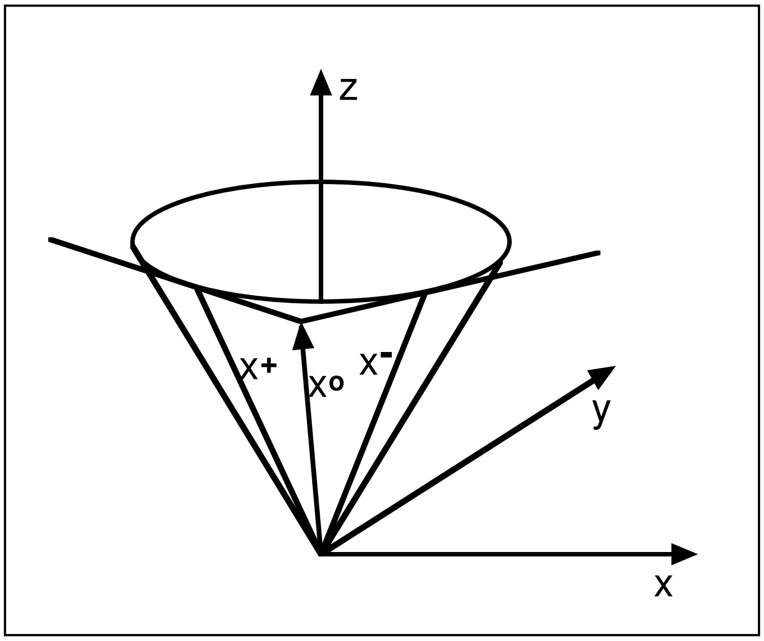
8.4 Lemma *A properly discontinuous subgroup* Γ *of* $\mathrm{Aff}(\mathbb{R}^3)$ *does not contain a positive pair.*

Maybe, if γ_1, γ_2 do not form a positive pair, then γ_1, γ_2^{-1} do? To answer this question and for other purposes it is good to associate an invariant to every single hyperbolic element and to decide if two form a positive pair by comparing their invariants.

To define such an invariant, we need to give $A^0(\gamma)$ an orientation. Let B be the quadratic form

$$B(x, y, z) = x^2 + y^2 - z^2$$

on $\mathbb{R}^3$ with coordinates (x, y, z) corresponding to the group $SO(2,1)$. The set of isotropic vectors $\{(x, y, x) \ ; \ B(x, y, z) = 0\}$ is also called the *light cone*. The set of non zero isotropic vectors has two connected components, one of which we call the positive light cone $\mathcal{N}^+$, say those with $z > 0$. If $L(\gamma)$ is hyperbolic then every eigenvector $x^+ \in A^+(\gamma)$ and every eigenvector $x^- \in A^-(\gamma)$ is isotropic. Note further that every eigenvector $x^0 \in A^0(\gamma)$ is orthogonal to both x^+ and x^- with respect to B and that $B(x^0, x^0) > 0$. Hence there is a unique eigenvector $x^0 \in A^0(\gamma)$ such that $B(x^0, x^0) = 1$ and (x^-, x^+, x^0) form a positively oriented basis of $\mathbb{R}^3$ whenever $x^- \in A^-(\gamma) \cap \mathcal{N}^+$ and $x^+ \in A^+(\gamma) \cap \mathcal{N}^+$.



The invariant is now defined by

(8.5) $$\alpha(\gamma) = B(x^0(\gamma), \tau(\gamma)).$$

This definition is due to Margulis [Ma 1, Ma 2, Ma 3]. We remarked above that $\tau(\gamma) = 0$ iff γ has a fixed point. Thus $\alpha(\gamma) \neq 0$ if γ has no fixed point. Furthermore, if we have $\gamma(x_0 + v) = x_0 + L(\gamma)v + t$ for some $x_0 \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, then

(8.6) $$\alpha(\gamma) = B(x^0(\gamma), \gamma x - x)$$

for every $x \in \mathbb{E}$ since $\gamma x - x \equiv \tau(\gamma) \mod \operatorname{Im}(L(\gamma) - I)$ and $\operatorname{Im}(L(\gamma) - I) = A^+(\gamma) \oplus A^-(\gamma)$ is orthogonal to $x^0(\gamma)$ with respect to B. Note that $A^\pm(\gamma^{-1}) = A^\mp(\gamma)$ and $A^0(\gamma^{-1}) = A^0(\gamma)$, hence $x^0(\gamma^{-1}) = -x^0(\gamma)$ and thus

(8.7) $$\alpha(\gamma^{-1}) = \alpha(\gamma).$$

If we have two elements γ_1, γ_2 as in section 8.3 then it is easy to see that $(x^-(\gamma_1), x^+(\gamma_1), w)$ and $(x^-(\gamma_2),\ x^+(\gamma_2), w)$ have opposite orientations for $0 \neq w \in TL$. Thus γ_1, γ_2 form a positive pair iff $\alpha(\gamma_1)\alpha(\gamma_2) < 0$. We thus have the following consequence of lemma 8.4. One has to mention that the case that γ_1, γ_2 are not in general position is easy to handle by a similar argument using the intersection of $C(\gamma_1) \oplus A^+(\gamma_1)$ with $C(\gamma_2) \oplus A^-(\gamma_2)$.

8.8 Corollary *If Γ is a properly discontinuous subgroup of $\operatorname{Aff}(\mathbb{R}^3)$ then $\alpha(\gamma)$ is positive for every hyperbolic $\gamma \in \Gamma$ or negative for every hyperbolic $\gamma \in \Gamma$.*

Recently Goldman and Margulis [GM] conjectured that the converse of 8.8 holds, as follows.

8.9 Conjecture *Suppose* Γ *is a subgroup of* $\mathrm{Aff}(\mathbb{R}^3)$ *such that* $L(\gamma) \in SO^o(2,1)$ *is hyperbolic for every* $\gamma \neq e$ *in* Γ *and* $\alpha(\gamma) > 0$ *for every* $\gamma \neq e$*. Then* Γ *is properly discontinuous.*

So far this conjecture is open. But the following weaker form was proved by Margulis [Ma 1, Ma 2]. Let γ_1, γ_2 be two elements of $\mathrm{Aff}(\mathbb{R}^3)$ with $L(\gamma_i)$ hyperbolic in $SO^o(2,1)$, $\alpha(\gamma_i) > 0$ and γ_1 and γ_2 in general position. Then the group generated by γ_1 and γ_2 contains a subgroup Γ which is free, acts properly discontinuously and for which $L(\Gamma)$ is Zariski dense in $SO(2,1)$. This proves the following theorem 8.10. The invariant α is used in the proof in two essential ways. First of all it is proved that under the above hypotheses $\alpha(\gamma_1\gamma_2)$ equals $\alpha(\gamma_1) + \alpha(\gamma_2)$ up to a controllable error term. Then this estimate is used to prove the existence of the subgroup Γ. Here one looks at $\alpha(g\gamma)$, $\gamma \in \Gamma$, for a suitably chosen element $g \in SO(2,1) \ltimes \mathbb{R}^3$.

8.10 Theorem *There is a free properly discontinuous subgroup* Γ *of* $\mathrm{Aff}(\mathbb{R}^3)$ *with* $L(\Gamma)$ *Zariski dense in* $SO(2,1)$*.*

The invariant α has a number of further interesting features, as follows.

8.11 Remark Let Γ be a subgroup of $\mathrm{Aff}(\mathbb{R}^3)$ such that $L : \Gamma \to SO^o(2,1)$ is injective and $L(\gamma)$ is hyperbolic for every $\gamma \neq e$. For a given point $x \in \mathbb{R}^3$ the map $u : L(\Gamma) \to \mathbb{R}^3$, $u(L(\gamma)) = \gamma x - x$ defines a cocycle whose cohomology class $[u] \in H^1(L(\Gamma), \mathbb{R}^3)$ is independent of the choice of x. Mapping u to the invariant α gives a map $H^1(L(\Gamma), \mathbb{R}^3) \to \mathbb{R}^{L(\Gamma)}$ from this cohomology group to real valued functions on $L(\Gamma)$. Drumm and Goldman recently showed that this mapping is injective, i.e. α is a complete invariant of the cohomology class given by the translational part.

8.12 Remark One can also interpret α as the derivative of the trace of γ and also as the derivative of the displacement length $\ell(\gamma)$, that is the minimum distance that γ moves a point of the symmetric space of $SO^o(2,1)$. Using this and Teichmüller theory Goldman and Margulis [GM] gave a new proof of the following theorem of Mess [Me].

8.13 Theorem *Let* Γ *be a properly discontinuous subgroup of* $\mathrm{Aff}(\mathbb{R}^3)$ *with* $L(\Gamma) \subset SO^o(2,1)$*. Then* $L(\Gamma)$ *is not cocompact in* $SO^o(2,1)$*.*

8.14 Remark Drumm and Goldman have been pursuing a program of detailed study of the geometry of the manifolds $\Gamma\backslash\mathbb{R}^3$, where Γ is a free properly discontinuous subgroup of $SO(2,1) \ltimes \mathbb{R}^3$, see [Dr, DrG 1, DrG 2, DrG 3].

9 Higher dimensions

The Auslander conjecture has recently been proved for $\dim \mathbb{E} \leq 6$, announced in [AMS 2]. The proof involves a discussion of several cases and the details still have to be published. In this section we discuss the case of $SO(n+1,n)$ which is a key case for dimension 5 with $n = 2$, and is for dimension 7, i.e. $n = 3$, a major unsolved case.

9.1 Much of the discussion in this section is a generalization of section 8. A difference occurs in formula 9.3, namely that $\alpha(g^{-1}) = \pm\alpha(g)$ where the sign depends on the parity of n. This has decisive consequences for our question.

Suppose $\gamma \in \mathrm{Aff}(\mathbb{R}^{2n+1})$ and $L(\gamma) \in SO^o(n+1,n)$. Consider the decomposition of $\mathbb{R}^{2n+1}$ into the direct sum of the subspaces

$$\mathbb{R}^{2n+1} = A^-(\gamma) \oplus A^0(\gamma) \oplus A^+(\gamma)$$

where $A^-(\gamma)$, $A^0(\gamma)$, $A^+(\gamma)$ are determined by the condition that they are the maximal $L(\gamma)$–invariant subspaces such that the eigenvalues of $L(\gamma)|A^-(\gamma)$ (resp. $L(\gamma) \mid A^0(\gamma)$, resp. $L(\gamma) \mid A^+(\gamma)$) are of modulus < 1 (resp.$= 1$, resp. > 1). An element γ is called *pseudohyperbolic* if $\dim A^0(\gamma) = 1$ and the eigenvalue of $L(\gamma) \mid A^0(\gamma)$ is $+1$. Let Ω be the set of pseudohyperbolic elements of $SO^o(n+1,n) \ltimes \mathbb{R}^{2n+1}$.

For every $\gamma \in \Omega$ there is exactly one invariant affine line $C(\gamma)$, called the *axis* of γ, and $C(\gamma)$ is parallel to $A^0(\gamma)$. The restriction of γ to $C(\gamma)$ is a translation by a vector $\tau(\gamma) \in A^0(\gamma)$, called the *translational part* of γ. The affine transformation $\gamma \in \Omega$ has a fixed point iff $\tau(\gamma) = 0$. Let $\Omega_0 = \{\gamma \in \Omega \mid \tau(\gamma) \neq 0\}$. The dynamical properties of $\gamma \in \Omega_0$ are completely analogous to those discussed in 8.2.

9.2 We proceed to define the invariant α for $\gamma \in \Omega_0$. To do this we have to introduce an orientation on $A^0(\gamma)$. Let B be the quadratic form on $\mathbb{R}^{2n+1}$ given by

$$B(x_1, \dots, x_{2n+1}) = x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 - \cdots - x_{2n+1}^2.$$

Let Ψ be the set of all maximal isotropic subspaces of $\mathbb{R}^{2n+1}$ with respect to B. The projection $q : \mathbb{R}^{2n+1} \to \mathbb{R}^n$, $q(x_1, \dots, x_{2n+1}) = (x_{n+2}, \dots, x_{2n+1})$ induces an isomorphism $q : V \to \mathbb{R}^n$ for every $V \in \Psi$. Thus if we give $\mathbb{R}^n$ an orientation, this endows every $V \in \Psi$ with an orientation. Similarly, for $V \in \Psi$ the B–orthogonal subspace $V^\perp := \{w \in \mathbb{R}^{2n+1}$, $B(v,w) = 0$ for every $w \in V\}$ is mapped isomorphically onto $\mathbb{R}^{n+1}$ by the projection $p : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$, $p(x_1, \dots, x_{2n+1}) = (x_1, \dots, x_{n+1})$. Thus giving $\mathbb{R}^{n+1}$ an orientation we can endow every $V^\perp$, $V \in \Psi$, with an orientation. The point is that we then can give every line $\mathbb{R}w \subset V^\perp$, $\mathbb{R}w \not\subset V$, an orientation, depending on V, namely we define w to be positively oriented with respect to V, and write $w >_V 0$, if for every positively oriented basis $(v_1, \dots, v_n)$ of V the basis $(v_1, \dots, v_n, w)$ of $V^\perp$ is positively oriented.

9.3 We of course want to apply this for affine transformations. So let $\gamma \in \Omega$. The subspaces $A^\pm(\gamma)$ are maximal isotropic and the eigenspace $A^0(\gamma)$ is contained in $V^\perp$ for $V = A^\pm(\gamma)$. We have $B(x^0, x^0) > 0$ for every $x^0(\gamma) \neq 0$ in $A^0(\gamma)$. Hence there is a unique eigenvector $x^0(\gamma) \in A^0(\gamma)$ such that $B(x^0(\gamma), x^0(\gamma)) = 1$ and $x^0(\gamma)$ is positive with respect to $A^+(\gamma)$. We now define the invariant

$$\alpha(\gamma) := B(x^0(\gamma), \tau(\gamma)). \tag{9.4}$$

We have

$$\alpha(\gamma) = B(x^0(\gamma), \gamma x - x) \tag{9.5}$$

for every $x \in \mathbb{R}^{2n+1}$ since $(\gamma x - x) - \tau(\gamma) \in \mathrm{Im}(L(\gamma) - I) = A^+(\gamma) \oplus A^-(\gamma) = x^0(\gamma)^\perp$. This definition coincides with the previous one for $\mathbb{R}^3$ by the following observation applied for $V_1 = A^-(\gamma)$, $V_2 = A^+(\gamma)$.

9.6 Observation *If V_1, $V_2 \in \Psi$ with $V_1 \cap V_2 = \{0\}$ then the sum orientation of $V_1 \oplus V_2^\perp = \mathbb{R}^{2n+1}$ is independent of the pair V_1, V_2.*

The sum orientation of the sum $V \oplus W$ of two oriented vector spaces V, W in this order is of course the orientation given by any basis of the form $(v_1, \ldots, v_m, w_1, \ldots, w_n)$ where $(v_1, \ldots, v_m)$ is a positively oriented basis of V and $(w_1, \ldots w_n)$ is a positively oriented basis of W.

All of this is completely analogous to what was said in section 8. A new phenomenon occurs if we compute $\alpha(\gamma^{-1})$:

$$\alpha(\gamma^{-1}) = (-1)^{n+1}\alpha(\gamma), \tag{9.7}$$

since $A^\pm(\gamma) = A^\mp(\gamma^{-1})$, $A^0(\gamma) = A^0(\gamma^{-1})$, $\tau(\gamma^{-1}) = -\tau(\gamma)$ and $x^0(\gamma^{-1}) = (-1)^n\, x^0(\gamma)$, as follows from the observation 9.6 for $V_i = A^\pm(\gamma)$.

9.8 Proposition *Suppose γ_1, $\gamma_2 \in \Omega_0$ and the intersection of any two of the four vector spaces $A^\pm(\gamma_i)$, $i = 1, 2$, is zero. If $(-1)^{n+1}\alpha(\gamma_1)\alpha(\gamma_2) < 0$ then the group Γ generated by γ_1 and γ_2 is not properly discontinuous.*

The proof is the same as for 8.4. The condition that γ_1, γ_2 form a positive pair translates into the condition in 9.8, using the observation 9.6.

A proof generalizing that of 8.10 yields

9.9 Theorem *There is a free properly discontinuous subgroup Γ of $\mathrm{Aff}(\mathbb{R}^{2n+1})$ with $L(\Gamma)$ Zariski dense in $SO(n+1, n)$ if n is odd.*

For n even the situation is completely different:

9.10 Theorem *There is no properly discontinuous subgroup Γ of $\mathrm{Aff}(\mathbb{R}^{2n+1})$ with $L(\Gamma)$ Zariski dense in $SO(n+1, n)$ if n is even.*

Once we found two elements γ_1, γ_2 in Ω_0 fulfilling the hypothesis about the spaces $A^\pm(\gamma_i)$, either γ_1, γ_2 or γ_1, γ_2^{-1} fulfill the hypothesis of 9.8, as follows from 9.7 since n is even. One step in the proof is to find one pseudohyperbolic element. Note that to be pseudohyperbolic is not an algebraic condition, e.g. $A \in SO^o(2, 1)$ is hyperbolic iff $\mathrm{tr}\, A > 3$. Nevertheless one finds a pseudohyperbolic element, in fact many, in every Zariski dense subsemigroup of $SO^0(n+1, n)$. The relevant notion is that of *proximal* elements [GoM, AMS 1]. This notion is due to Furstenberg and was used in his work on boundaries. Proximal elements were also used in Tits' proof of the Tits alternative [Ti], cf. theorem 3.4.

Theorem 9.9 tells us that there is a properly discontinuous subgroup Γ of $\mathrm{Aff}(\mathbb{R}^7)$ with $L(\Gamma)$ contained and Zariski dense in $SO^0(4, 3)$. An open problem is if there is crystallographic Γ with these properties. Auslander's conjecture claims there is none.

10 Further results

The Auslander conjecture has been proved for some further special cases. One of the most general results obtained so far is the following theorem.

10.1 Theorem [S 1, To] *Let Γ be a crystallographic subgroup of* $\mathrm{Aff}(\mathbb{E})$. *Suppose $L(\Gamma)$ is contained in a real algebraic group all of whose simple quotient groups have real rank at most one. Then Γ is virtually solvable.*

For further results under assumptions about $L(\Gamma)$ see [KW, K].

The paper [GrM] contains a classification of those crystallographic groups Γ for which $L(\Gamma)$ is contained in $O(n-1,1)$, up to commensurability.

Bieberbach proved that every Euclidean crystallographic group is a finite extension of its translation group, see theorem 5.1, second part. In the following theorem which holds for an arbitrary affine crystallographic group, supposing it is virtually solvable, the group of all translations is replaced by a group H which acts simply transitively and by affine transformations on affine space. Recall that an action of a group H on a set X is called simply transitive if the map $H \to X$, $h \longmapsto hx$, is a bijection for one — equivalently for every — x in X.

10.2 Theorem [FG] *For every virtually solvable crystallographic subgroup Γ of* $\mathrm{Aff}(\mathbb{E})$ *there is a closed connected solvable Lie subgroup H of* $\mathrm{Aff}(\mathbb{E})$ *with the following properties:*

(a) *H acts simply transitively on $\mathbb{E}$.*
(b) *$\Delta := \Gamma \cap H$ is of finite index in Γ and cocompact in H.*
(c) *Δ and H have the same algebraic hull.*

Fried and Goldman actually have a similar theorem for properly discontinuous virtually solvable subgroups of $\mathrm{Aff}(\mathbb{E})$. Note that H is not unique, in general. Theorem 10.2 is an essential tool in the classification results in [FG] and [GrM]. These authors first classify the possible groups H and then the possible groups Γ or Δ. In view of theorem 10.2 the following problem gains additional relevance: Determine all the connected Lie groups H which have a simply transitive continuous action by affine transformations on some affine space $\mathbb{E}$. Then $H \to \mathbb{E}$, $h \longmapsto hx$, is a diffeomorphism for every $x \in \mathbb{E}$. And it was known for a long time that H must be solvable [Au 2, Mi]. But only much later examples of nilpotent simply connected Lie groups were exhibited which do not have a simply transitive action by affine transformations on some affine space [Be]. As a consequence Benoist gives an example of a finitely generated torsion free nilpotent group which is not an affine crystallographic group.

Here are geometric versions of these results, first of Bieberbach's and then of Fried and Goldman's.

10.3 Corollary *Every compact flat Riemannian manifold has a finite cover by a flat torus. Every complete flat Riemannian manifold is finitely covered by a flat Riemannian vector bundle over a torus.*

10.4 Corollary *Every compact complete flat affine manifold with virtually solvable fundamental group is finitely covered by a solvmanifold.*

A *solvmanifold* (*nilmanifold*) is a manifold of the form G/Γ where G is a connected solvable (nilpotent) Lie group and Γ is a cocompact discrete subgroup of G. Benoist's example shows that there are compact nilmanifolds which do not have a complete flat affine structure.

10.5 The following result is interesting to note: Every virtually solvable affine crystallographic subgroup of $\mathrm{Aff}(\mathbb{E})$ virtually preserves the Euclidean volume of $\mathbb{E}$ [GH]. This supports the Markus conjecture which claims that a compact affine manifold is complete iff it has a parallel volume.

10.6 Bieberbach also showed that in each dimension there are only finitely many isomorphism types of Euclidean crystallographic groups and that isomorphic Euclidean crystallographic groups are conjugate in the affine group. None of these results is true for affine crystallographic groups: There are infinitely many isomorphism types of affine crystallographic groups in dimension 3 already, as follows from the examples in 7.2. Also already in dimension 3 isomorphic affine crystallographic groups may not be conjugate, they may fall into an uncountable number of different conjugacy classes in the affine group, see [FG].

Other features of Bieberbach's theory also break down: Given H and Δ as in theorem 10.2, there may be groups Γ which lie in infinitely many conjugacy classes in $\mathrm{Aff}(\mathbb{E})$ and the indices $|\Gamma : \Delta|$ may be unbounded. An example in dimension 6 is given in the paper [GrS]. In this paper the authors make a detailed study of the structure of virtually solvable affine crystallographic groups. An essential tool is a refined version of 10.2.

Also a simply transitive affine group H may contain infinitely many pairwise abstractly noncommensurable Zariski dense cocompact discrete subgroups Δ, see [GrM]. Here two groups A and B are called *abstractly commensurable* if A contains a subgroup of finite index which is isomorphic to a subgroup of finite index in B.

10.7 *Polynomial automorphisms.* Instead of looking at the group $\mathrm{Aff}(\mathbb{E})$ of affine automorphisms of affine space one can consider the group $P(V)$ of all polynomial automorphisms f of the real vector space V. So with respect to some basis of V the components of f as well as its inverse are given by polynomial functions in the coordinates. Thus $\mathrm{Aff}(\mathbb{E})$ consists of those polynomial automorphisms all of whose component functions have total degree 1. Now that it is known that not every polycyclic group is an affine crystallographic group, one can ask the question if every virtually polycyclic group Γ has a homomorphism into $P(V)$ such that the corresponding action of Γ on V is properly discontinuous and $\Gamma\backslash V$ is compact. The answer is yes, even with polynomial diffeomorphisms of bounded degree [DI]. The authors also asked the following more general version of Auslander's question: Suppose Γ is a subgroup of $P(V)$ such that the corresponding action of Γ on V is properly discontinuously and the orbit space $\Gamma\backslash V$ is compact. Is then Γ virtually polycyclic?

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