

# BLOCK REPRESENTATION TYPE OF REDUCED ENVELOPING ALGEBRAS

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ABSTRACT. Let  $K$  be an algebraically closed field of characteristic  $p$ ,  $G$  a connected, reductive  $K$ -group,  $\mathfrak{g} = \mathrm{Lie}(G)$ ,  $\chi \in \mathfrak{g}^*$  and  $U_\chi$  the reduced enveloping algebra of  $\mathfrak{g}$  associated with  $\chi$ . Assume that  $G^{(1)}$  is simply-connected,  $p$  is good for  $G$  and  $\mathfrak{g}$  has a non-degenerate  $G$ -invariant bilinear form. All blocks of  $U_\chi$  having finite and tame representation type are determined.

## 1. INTRODUCTION

Let  $G$  be a connected, reductive algebraic group over an algebraically closed field of characteristic  $p$  and  $\mathfrak{g} = \mathrm{Lie}(G)$ . The Lie algebra  $\mathfrak{g}$  carries a natural restriction map  $x \mapsto x^{[p]}$ . We assume that the derived group  $G^{(1)}$  of  $G$  is simply-connected,  $p$  is a good prime for the root system of  $G$ , and  $\mathfrak{g}$  has a non-degenerate  $G$ -invariant bilinear form. Given a linear function  $\chi \in \mathfrak{g}^*$  we denote by  $U_\chi(\mathfrak{g})$  the reduced enveloping algebra of  $\mathfrak{g}$  associated with  $\chi$ . Let  $\chi = \chi_s + \chi_n$  be the Jordan decomposition of  $\chi$ . We fix a triangular decomposition  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$  and, without loss of generality, assume that  $\chi$  vanishes on  $\mathfrak{n}_+$  and  $\chi_s$  vanishes on  $\mathfrak{n}_\pm$ . Then the blocks  $\mathcal{B}_{\chi, \lambda}$  of  $U_\chi(\mathfrak{g})$  are parametrised by the set  $W\Lambda_{\chi_s}/W$  where  $W = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$  is the Weyl group of  $\mathfrak{g}$  and  $\Lambda_{\chi_s}$  is the set of all  $\lambda \in \mathfrak{h}^*$  satisfying  $\lambda(h)^p - \lambda(h^{[p]}) = \chi_s(h)^p$  for all  $h \in \mathfrak{h}$ , [3].

In this paper, we show that the rank variety of  $\mathcal{B}_{\chi, \lambda}$  coincides with the intersection of the rank variety of  $\mathcal{B}_{\chi_s, \lambda}$  with the  $p$ -nilpotent cone of the coadjoint stabiliser  $\mathfrak{z}_{\mathfrak{g}}(\chi)$ , see Theorem 4.2. This result implies that the rank variety of  $U_\chi(\mathfrak{g})$  is equal to the  $p$ -nilpotent cone of  $\mathfrak{z}_{\mathfrak{g}}(\chi)$ , hence generalises

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in our class of Lie algebras the main theorem of [35]. In proving Theorem 4.2 we use a deformation argument and some tools from [35]. In particular, we employ the Mil'ner map  $\beta : U(\mathfrak{g}) \rightarrow S(\mathfrak{g})$  and its analogue  $\beta_\chi : U_\chi(\mathfrak{g}) \rightarrow S_\chi(\mathfrak{g})$  constructed in [35].

According to [8], [10] the dimension of the rank variety of a finite dimensional  $\mathcal{B}_{\chi,\lambda}$ -module  $M$  equals the rate of growth of a minimal projective resolution of  $M$ . So Theorem 4.2 enables us to compute the maximum value of the rate of growth of minimal projective resolutions in the category of finite dimensional  $\mathcal{B}_{\chi,\lambda}$ -modules, hence provides valuable information on the homological complexity of the blocks of  $U_\chi(\mathfrak{g})$ , see our analysis in Section 8. Combined with the results of Section 3 on block degeneration and our determination of partial coinvariant algebras of tame representation type in Section 7 this information turns out to be *almost* sufficient for detecting all blocks of  $U_\chi(\mathfrak{g})$  having finite and tame representation type. For  $G$  almost simple and  $\chi$  nilpotent the only case where we have to look closely at the basic algebra of  $\mathcal{B}_{\chi,\lambda}$ , apart from the relatively easy regular case, is the case where  $\chi$  is subregular,  $G$  has type  $A_n$  and the stabiliser of  $\lambda$  in  $W$  has type  $A_{n-1}$  as a Coxeter group. This case, which came as a surprise, might have been anticipated due to the fact that 0 is the subregular nilpotent element in  $\mathfrak{sl}_2$  and the restricted enveloping algebra of  $\mathfrak{sl}_2$  is tame.

The subregular blocks  $\mathcal{B}$  of type  $(A_n, A_{n-1})$  are studied in detail in Section 9. It is shown there that any such  $\mathcal{B}$  is Morita equivalent to a quiver with relations of “special biserial type” hence tame. Notably the quiver of any subregular tame block has the same shape as in the  $\mathfrak{sl}_2$ -case and  $n$ , the rank of  $G^{(1)}$ , appears in the relations only. It is known that any indecomposable non-projective  $U^{[p]}(\mathfrak{sl}_2)$ -module is up to isomorphism either a Weyl module or a dual Weyl module or a maximal submodule of a Weyl module, see [31]. It would be interesting to obtain a purely Lie theoretic description of all indecomposable representations of the subregular tame blocks.

The classification of all blocks  $\mathcal{B}_{\chi,\lambda}$  of finite and tame representation type is given in Theorems 5.2 and 5.3. This should be combined with Proposition 2.7.

## 2. GENERALITIES

2.1. Let  $G$  be a connected, reductive algebraic group over  $K$ , an algebraically closed field of characteristic  $p$ , and let  $\mathfrak{g} = \text{Lie}(G)$ . We assume the following hypotheses are satisfied:

- (A) the derived group  $G^{(1)}$  of  $G$  is simply-connected;
- (B)  $p$  is a good prime for  $G$ ;
- (C)  $\mathfrak{g}$  has a non-degenerate  $G$ -invariant bilinear form.

We will denote the bilinear form on  $\mathfrak{g}$  by

$$B(, ) : \mathfrak{g} \times \mathfrak{g} \longrightarrow K.$$

Let  $T$  be a maximal torus of  $G$  and let  $\mathfrak{h} = \text{Lie}(T)$ . Let  $\Phi$  be the root system of  $G$  with respect to  $T$ . For each  $\alpha \in \Phi$  let  $U_\alpha$  denote the corresponding root subgroup of  $G$  and let  $\mathfrak{g}_\alpha = \text{Lie}(U_\alpha)$  be its Lie algebra, a root subspace of  $\mathfrak{g}$ . We will abuse notation by considering  $\alpha \in \mathfrak{h}^*$  rather than its proper designation  $d\alpha$ . Choose a system  $\Phi^+$  of positive roots and set  $\mathfrak{n}^+$  equal to the sum of all  $\mathfrak{g}_\alpha$  with  $\alpha > 0$ . The subalgebra  $\mathfrak{n}^-$  is similarly defined on  $\Phi^-$ , the negative roots. We have the triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Let  $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ , the Lie algebra of a Borel subgroup of  $G$  containing  $T$ . Let  $\Delta = \{\alpha_1, \dots, \alpha_n\}$  denote the simple roots associated with the choice of positive roots  $\Phi^+$ .

Let  $X = X(T)$  be the character group of  $T$ . This contains the root lattice,  $Q = \mathbb{Z}\Phi$ , as a subgroup. For  $\alpha \in \Phi^+$ , there exists  $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$  such that  $\lambda(h_\alpha) \equiv \langle \lambda, \alpha^\vee \rangle(p)$  for all  $\lambda \in X(T)$ . Let  $W$  be the Weyl group of  $G$ . Then  $W$  is generated by the simple reflections  $s_\alpha$  for all  $\alpha \in \Delta$ . There is an action of  $W$  on both  $X(T)$  and  $\mathfrak{h}^*$ , given by  $s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha$ .

2.2. We write  $g.x$  for the adjoint action of an element  $g \in G$  on an element  $x \in \mathfrak{g}$ . Similarly we will write  $g.\chi$  for the coadjoint action of  $G$  on  $\mathfrak{g}^*$ , defined by  $g.\chi(x) = \chi(g^{-1}.x)$ . Let  $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}^*$  send  $x \in \mathfrak{g}$  to the functional  $\theta(x)$  defined by  $\theta(x)(y) = B(x, y)$  for all  $y \in \mathfrak{g}$ . By Hypothesis (C)  $\theta$  is a  $G$ -equivariant isomorphism.

Recall there is a Jordan decomposition in  $\mathfrak{g}$ : each element  $x \in \mathfrak{g}$  can be written uniquely as  $x = x_s + x_n$  with  $x_s$  semisimple,  $x_n$  nilpotent and

$[x_s, x_n] = 0$ . Given  $x \in \mathfrak{g}$  we can always find  $g \in G$  such that  $g.x \in \mathfrak{b}^+$ , [2, Proposition 14.25]. Let  $\mathfrak{z}_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} : [x, y] = 0\}$  and  $Z_G(x) = \{g \in G : g.x = x\}$ . If  $x$  is semisimple then  $Z_G(x)$  is a connected, reductive algebraic group satisfying Hypotheses (A), (B) and (C), and  $\text{Lie}(Z_G(x)) = \mathfrak{z}_{\mathfrak{g}}(x)$ , [18, Theorem 3.10], [40, II.3.19] and [23, 6.5].

Using  $\theta$  we can transfer the Jordan decomposition to  $\mathfrak{g}^*$ . In particular any element of  $\mathfrak{g}^*$  is conjugate to  $\chi \in \mathfrak{g}^*$  such that  $\chi(\mathfrak{n}^+) = 0$ . For  $\chi \in \mathfrak{g}^*$  let  $\mathfrak{z}_{\mathfrak{g}}(\chi) = \{y \in \mathfrak{g} : \chi([\mathfrak{g}, y]) = 0\}$  and  $Z_G(\chi) = \{g \in G : g.\chi = \chi\}$ . Then  $Z_G(x) = Z_G(\theta(x))$  and  $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(\theta(x))$ .

2.3. Being the Lie algebra of  $G$ ,  $\mathfrak{g}$  has a restriction map  $x \mapsto x^{[p]}$ . The  $p$ -centre  $Z_p = K[x^p - x^{[p]} : x \in \mathfrak{g}]$  is a central subalgebra of  $U = U(\mathfrak{g})$ , the enveloping algebra of  $\mathfrak{g}$ . By the PBW theorem  $Z_p$  is a polynomial ring in  $\dim \mathfrak{g}$  variables and  $U(\mathfrak{g})$  is free over  $Z_p$  of rank  $p^{\dim \mathfrak{g}}$ .

Given  $\chi \in \mathfrak{g}^*$  define  $I_{\chi}$  as the ideal of  $U(\mathfrak{g})$  generated by the elements  $x^p - x^{[p]} - \chi(x)^p$  for  $x \in \mathfrak{g}$ . Set  $U_{\chi} = U_{\chi}(\mathfrak{g}) = U(\mathfrak{g})/I_{\chi}$ , a *reduced enveloping algebra*. This is an algebra of dimension  $p^{\dim \mathfrak{g}}$ . As the isomorphism class of  $U_{\chi}$  depends only on the  $G$ -orbit of  $\chi \in \mathfrak{g}^*$ , [23, 2.9], it suffices, by Section 2.2, to look at  $\chi$  satisfying  $\chi(\mathfrak{n}^+) = 0$ . In this case if  $\chi = \chi_s + \chi_n$  is the Jordan decomposition then, possibly after further conjugation, we also have  $\chi_s(\mathfrak{n}^-) = 0$  and  $\chi_n(\mathfrak{h}) = 0$  and so in particular we can consider  $\chi_s \in \mathfrak{h}^*$ .

2.4. Let  $\chi = \chi_s + \chi_n \in \mathfrak{g}^*$  with  $\chi(\mathfrak{n}^+) = 0$ , and set

$$\Lambda_{\chi_s} = \{\lambda \in \mathfrak{h}^* : \lambda(h)^p - \lambda(h^{[p]}) = \chi_s(h)^p \text{ for all } h \in \mathfrak{h}\}.$$

By Hypothesis (A) the elements  $h_{\alpha}$  for  $\alpha$  simple are linearly independent, so, since  $h_{\alpha}^{[p]} = h_{\alpha}$ , we can find  $\rho \in \Lambda_0$  such that  $\rho(h_{\alpha}) = 1$  for all simple  $\alpha$ . Fix once and for all such a  $\rho$ . Note that  $W$  acts on  $\Lambda_0$ . For each  $\lambda \in \Lambda_0$  let  $W(\lambda) = \{w \in W : w(\lambda) = \lambda\}$ , a parabolic subgroup, [28, Lemma 7]. In general we will only be interested in  $\lambda$  up to  $W$ -conjugacy so we may assume without loss of generality that  $W(\lambda)$  is a standard parabolic subgroup, that is it is generated by simple reflections.

2.5. For each  $\lambda \in \Lambda_{\chi_s}$  one defines a *baby Verma module*

$$Z_{\chi}(\lambda) := U_{\chi}(\mathfrak{g}) \otimes_{U_{\chi}(\mathfrak{b}^+)} K_{\lambda - \rho},$$

where  $\mathfrak{b}^+ = \mathfrak{n}^+ \oplus \mathfrak{h}$  and  $K_{\lambda-\rho}$  is the one dimensional  $U_\chi(\mathfrak{b}^+)$ -module defined by  $\lambda - \rho$ . Every irreducible  $U_\chi(\mathfrak{g})$ -module is a factor of a baby Verma module, although in general a baby Verma module can have several irreducible images, and different choices of  $\lambda$  can yield the same module [23, 6.7, 6.9].

2.6. Let  $Z = Z(U(\mathfrak{g}))$ ,  $Z_1 = U(\mathfrak{g})^G \subset Z$ . By [28, Theorem 2] and [3, Theorem 3.5] there is a  $K$ -algebra isomorphism

$$(1) \quad Z \cong Z_1 \otimes_{Z_p \cap Z_1} Z_p.$$

We have a natural map  $Z \otimes_{Z_p} K_{\chi_s} \longrightarrow U_\chi$  sending  $z \otimes 1$  to  $z + I_\chi$ . By [28, Theorem 10] the primary components of  $Z \otimes_{Z_p} K_{\chi_s}$  are labelled by elements of the orbits of  $W$  on the set  $W\Lambda_{\chi_s}$ , denoted by  $W\Lambda_{\chi_s}/W$ . Thanks to (1) we have an isomorphism

$$(2) \quad Z \otimes_{Z_p} K_{\chi_s} \cong Z_1 \otimes_{Z_p \cap Z_1} K_{\chi_s}.$$

Let  $\{e_{\tilde{\lambda}} \in Z_1 : \tilde{\lambda} \in W\Lambda_{\chi_s}/W\}$  be a set of elements such that  $\{e_{\tilde{\lambda}} \otimes 1 : \tilde{\lambda} \in W\Lambda_{\chi_s}/W\}$  correspond to a set of primitive idempotents of  $Z \otimes_{Z_p} K_{\chi_s}$  under (2). By [3, Theorem 3.18] the elements  $e_{\tilde{\lambda}} + I_\chi \in U_\chi$  give a complete list of central primitive idempotents of  $U_\chi$  as  $\tilde{\lambda}$  runs through  $W\Lambda_{\chi_s}/W$ . In other words we have a block decomposition of  $U_\chi(\mathfrak{g})$

$$(3) \quad U_\chi(\mathfrak{g}) = \bigoplus_{\tilde{\lambda} \in W\Lambda_{\chi_s}/W} \mathcal{B}_{\chi, \tilde{\lambda}}(\mathfrak{g})$$

where  $\mathcal{B}_{\chi, \tilde{\lambda}}(\mathfrak{g}) = e_{\tilde{\lambda}} U_\chi(\mathfrak{g})$ . By [23, 10.11] the baby Verma module  $Z_\chi(\lambda)$  belongs to  $\mathcal{B}_{\chi, \tilde{\lambda}}(\mathfrak{g})$  where  $\tilde{\lambda}$  is a representative for the orbit of  $\lambda$ .

*Remarks.* 1. The reference [3, Theorem 3.5] requires the assumption  $p \neq 2$ . If  $p = 2$  then the block decomposition of  $U_\chi(\mathfrak{g})$  remains valid thanks to [15, Theorem 3.6]. Moreover, if  $G = SL_n(K)$  or  $GL_n(K)$  then [3, Theorem 3.5] continues to hold and the results of this section go through. Thanks to Proposition 6.5 this is sufficient for our applications.

2. If no confusion can occur we will write  $\mathcal{B}_{\chi, \lambda}$  when really we mean  $\mathcal{B}_{\chi, \tilde{\lambda}}(\mathfrak{g})$  for some Lie algebra  $\mathfrak{g}$  and some representative  $\tilde{\lambda}$  of the orbit of  $\lambda \in \Lambda_{\chi_s}$ .

2.7. The following is an easy consequence of [42, Theorem 2] and [10, Theorem 3.2].

**Proposition.** *Let  $\chi = \chi_s + \chi_n \in \mathfrak{g}^*$  and let  $W'$  be the Weyl group of  $Z_G(\chi_s)$ . Then there is a bijection*

$$\pi : W\Lambda_{\chi_s}/W \longrightarrow \Lambda_0/W',$$

*such that for any  $\lambda \in W\Lambda_{\chi_s}/W$  we have an algebra isomorphism*

$$\mathcal{B}_{\chi, \lambda}(\mathfrak{g}) \cong \text{Mat}_{p^d}(\mathcal{B}_{\chi_n, \pi(\lambda)}(\mathfrak{z}_{\mathfrak{g}}(\chi_s))),$$

*where  $d = \frac{1}{2}(\dim G \cdot \chi_s)$ .*

*Proof.* We may assume that  $\chi(\mathfrak{n}^+) = 0$ . Let  $\Phi_\chi = \{\alpha \in \Phi : \chi_s(h_\alpha) = 0\}$ . Then  $\mathfrak{z}_{\mathfrak{g}}(\chi_s)$  is the algebra generated by  $\mathfrak{h}$  and the root spaces  $\mathfrak{g}_\alpha$  for  $\alpha \in \Phi_\chi$ , [23, 7.4].

Choose an element  $\nu \in \Lambda_{\chi_s}$  such that  $\nu(h_\alpha) = 0$  for all  $\alpha \in \Phi_\chi$ . We have a  $W'$ -equivariant bijection between  $\Lambda_0$  and  $\Lambda_{\chi_s}$  sending  $\mu$  to  $\mu + \nu$ . Thus  $\Lambda_0/W'$  is isomorphic to  $W'\Lambda_{\chi_s}/W'$ . Moreover, the inclusion  $\Lambda_{\chi_s} \longrightarrow W\Lambda_{\chi_s}$  induces an isomorphism between  $W'\Lambda_{\chi_s}/W'$  and  $W\Lambda_{\chi_s}/W$ . Combining these isomorphisms yields  $\pi^{-1}$ .

By [42, Theorem 2] and [10, Theorem 3.2], there is an isomorphism

$$U_\chi(\mathfrak{g}) \cong \text{Mat}_{p^d}(U_\chi(\mathfrak{z}_{\mathfrak{g}}(\chi_s))).$$

Let  $\sigma_\nu$  be the *winding automorphism* of  $U(\mathfrak{z}_{\mathfrak{g}}(\chi_s))$  which sends  $h$  to  $h - \nu(h)$  for  $h \in \mathfrak{h}$  and is the identity on the root spaces of  $\mathfrak{z}_{\mathfrak{g}}(\chi_s)$ . Then  $\sigma_\nu$  induces an isomorphism between  $U_{\chi_n}(\mathfrak{z}_{\mathfrak{g}}(\chi_s))$  and  $U_\chi(\mathfrak{z}_{\mathfrak{g}}(\chi_s))$  which sends the baby Verma module  $Z_{\chi_n}(\mu)$  to  $Z_\chi(\mu + \nu)$ . The proposition follows.  $\square$

*Remarks.* 1. Let  $\mathcal{N} = \{x \in \mathfrak{g} : x \text{ nilpotent}\}$  be the nilpotent cone of  $\mathfrak{g}$ . The image of  $\mathcal{N}$  under  $\theta$  is the set  $\{\chi \in \mathfrak{g}^* : \chi \text{ nilpotent}\}$ . We will denote this by  $\mathcal{N}$  also. Proposition 2.7 shows that it is sufficient to consider only blocks  $\mathcal{B}_{\chi, \lambda}$  with  $\chi \in \mathcal{N}$  and  $\lambda \in \Lambda_0/W$ .

2. To ease notation we write  $\Lambda$  for the  $\mathbb{F}_p$ -space  $\Lambda_0$ . That is

$$\Lambda = \{\lambda \in \mathfrak{h}^* : \lambda(h)^p - \lambda(h^{[p]}) = 0 \text{ for all } h \in \mathfrak{h}\}.$$

It is clear that  $\Lambda$  is  $W$ -invariant.

## 3. BLOCK DEGENERATION

3.1. We intend to study the behaviour of  $\mathcal{B}_{\chi,\lambda}$  as we vary  $\chi \in \mathcal{N}$ . Let  $\pi_\Lambda : \mathcal{N} \times \Lambda \longrightarrow \Lambda$  be the projection map.

**Lemma.** *The function  $\phi : \mathcal{N} \times \Lambda \longrightarrow \mathbb{N}$  defined by sending  $(\chi, \lambda)$  to  $\dim \mathcal{B}_{\chi,\lambda}$  is constant on the fibres of  $\pi_\Lambda$ .*

*Proof.* Let  $\chi(x) = B(x, e)$  for some  $e \in \mathcal{N}$ . There exists  $e' \in \mathcal{N}$  such that  $e_\tau = e + \tau e' \in \mathcal{N}$  for all  $\tau \in K$  and such that  $e_1 = e + e'$  is regular nilpotent, [41, Section 5]. Let  $\chi_\tau \in \mathfrak{g}^*$  be defined by  $\chi_\tau(x) = B(x, e_\tau)$ . For  $\lambda \in \Lambda/W$  the function

$$A_\lambda : K \longrightarrow \text{Mat}_d(K)$$

sending  $\tau$  to the matrix of left multiplication in  $U_{\chi_\tau}$  by  $e_\lambda$  is, by construction, a morphism of varieties. Thus the set

$$\mathcal{O}_\lambda = \{\tau \in K : \text{rank } A_\lambda(\tau) \geq \text{rank } A_\lambda(0)\}$$

is open and dense in  $K$ . Let  $\mathcal{O} = \bigcap_{\lambda \in \Lambda/W} \mathcal{O}_\lambda$ . By (3) the rank of  $A_\lambda(\tau)$  equals the dimension of  $\mathcal{B}_{\chi_\tau,\lambda}$ . We deduce that

$$\mathcal{O} = \{\tau \in K : \dim \mathcal{B}_{\chi_\tau,\lambda} = \dim \mathcal{B}_{\chi,\lambda} \text{ for all } \lambda \in \Lambda/W\}.$$

Since the set of regular nilpotent elements,  $\mathcal{N}_{\text{reg}}$ , is dense in  $\mathcal{N}$ , we deduce that  $\mathcal{O} \cap \mathcal{N}_{\text{reg}}$  is dense in  $\mathcal{O}$ . The lemma now follows since the restriction of  $\phi$  to  $\mathcal{N}_{\text{reg}} \times \Lambda$  is constant along the fibres of  $\pi_\Lambda$ , see for example [3, Proposition 3.16].  $\square$

3.2. We recall the following definitions from the theory of finite dimensional algebras, [12] and [26, Chapter II]. Let

$$\text{Bil}(n) = \{\text{bilinear maps } m : K^n \times K^n \longrightarrow K^n\} \cong \mathbb{A}^{n^3},$$

and

$$\text{Alg}(n) = \{\text{associative, bilinear } m \text{ which have an identity}\} \subseteq \text{Bil}(n).$$

As discussed in [12]  $\text{Alg}(n)$  is an affine variety, locally closed in  $\text{Bil}(n)$ . The group  $GL(n)$  acts on  $\text{Alg}(n)$ , the orbits being isomorphism classes of  $n$  dimensional algebras. We let  $\mathcal{O}_A$  denote the orbit in  $\text{Alg}(n)$  of algebras isomorphic to  $A$ . We say that  $A'$  is a *degeneration* of  $A$  if  $\mathcal{O}_{A'} \subseteq \overline{\mathcal{O}_A}$ , the closure of  $\mathcal{O}_A$ .

**Theorem.** *Let  $\chi, \chi' \in \mathcal{N}$  be such that  $\chi$  is in the closure of the orbit  $G.\chi'$ . Then for any  $\lambda \in \Lambda/W$  the algebra  $\mathcal{B}_{\chi, \lambda}$  is a degeneration of  $\mathcal{B}_{\chi', \lambda}$ .*

*Proof.* Let  $d = p^{\dim \mathfrak{g}}$  and let  $w_1, \dots, w_d$  be a free basis of the  $Z_p$ -module  $U$ . Then the cosets  $w_i + I_\zeta$  form a  $K$ -basis of  $U_\zeta$  for all  $\zeta \in \mathfrak{g}^*$ .

Fix  $\lambda \in \Lambda/W$ . Let  $u_1, \dots, u_r \in U$  be such that  $\{e_\lambda u_i + I_\chi : 1 \leq i \leq r\}$  is a basis of  $\mathcal{B}_{\chi, \lambda}$ . We have a morphism

$$\overline{G.\chi'} \longrightarrow \text{Mat}_{r \times d}(K),$$

sending  $\zeta$  to  $M_\zeta$ , the matrix expressing the elements  $e_\lambda u_i + I_\zeta$  in terms of the basis elements  $\{w_i + I_\zeta\}$ . The set

$$\mathcal{O}_\lambda = \{\zeta \in \overline{G.\chi'} : \text{rank } M_\zeta = r\}$$

is a dense open subset of  $\overline{G.\chi'}$  and includes  $\chi$ . Now Lemma 2.1 says that for all  $\zeta \in \mathcal{O}_\lambda$  the block  $\mathcal{B}_{\zeta, \lambda}$  has a basis  $\{e_\lambda u_i + I_\zeta : 1 \leq i \leq r\}$  and so, by definition, we have a morphism

$$\mathcal{O}_\lambda \longrightarrow \text{Alg}(r)$$

sending  $\zeta$  to  $\mathcal{B}_{\zeta, \lambda}$ . Since  $G.\chi'$  is open in its closure the theorem follows.  $\square$

3.3. A finite dimensional  $K$ -algebra  $A$  is said to have *wild representation type*, or be *wild* for short, if for any finite dimensional  $K$ -algebra  $B$ , there is a representation embedding of the module category of  $B$  into that of  $A$ , so that  $A\text{-mod}$  contains  $B\text{-mod}$  although not necessarily as a full subcategory. The algebra  $A$  has *finite representation type* or has *finite type* if it has finitely many isomorphism classes of indecomposable modules. Finally,  $A$  has *tame representation type* or is *tame* if it is neither wild nor of finite type.

Thanks to Theorem 3.2 we can compare the representation type of two blocks.

**Corollary.** *Let  $\chi, \chi' \in \mathcal{N}$  be such that  $\chi$  is in the closure of the orbit  $G.\chi'$ . Let  $\lambda \in \Lambda/W$ .*

1. *If  $\mathcal{B}_{\chi, \lambda}$  has finite representation type then  $\mathcal{B}_{\chi', \lambda}$  has finite representation type.*
2. *If  $\mathcal{B}_{\chi, \lambda}$  has tame representation type then  $\mathcal{B}_{\chi', \lambda}$  has either finite or tame representation type.*



*Proof.* By Theorem 3.2  $\mathcal{B}_{\chi,\lambda}$  is a degeneration of  $\mathcal{B}_{\chi',\lambda}$ . The first assertion follows since the algebras in  $\text{Alg}(d)$  having finite representation type form an open set, [12, Theorem 4.2]. The second assertion is a consequence of the main result in [13] which states that a degeneration of a wild algebra is wild.  $\square$

#### 4. RANK VARIETIES

4.1. Let  $\mathcal{L}$  be a finite dimensional restricted Lie algebra over  $K$  and define

$$\mathcal{N}_p(\mathcal{L}) = \{x \in \mathcal{L} : x^{[p]} = 0\},$$

a Zariski closed, conical subset of  $\mathcal{L}$ , called the *p-nilpotent cone* of  $\mathcal{L}$ . Let  $M$  be a finite dimensional  $U_{\zeta}(\mathcal{L})$ -module. We define the *rank variety* of  $M$  to be

$$\mathcal{V}_{\mathcal{L}}(M) = \{x \in \mathcal{N}_p(\mathcal{L}) : M|_{\langle x \rangle} \text{ is not free}\}$$

where  $\langle x \rangle$  denotes the subalgebra of  $U_{\zeta}(\mathcal{L})$  generated by  $x$ .

Let  $\chi \in \mathfrak{g}^*$  and  $\lambda \in W\Lambda_{\chi_s}/W$ . Let  $S_1, \dots, S_r$  be a list of the simple  $\mathcal{B}_{\chi,\lambda}$ -modules (up to isomorphism). We define the rank variety of the block  $\mathcal{B}_{\chi,\lambda}$  to be

$$\mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = \bigcup_{i=1}^r \mathcal{V}_{\mathfrak{g}}(S_i).$$

By [10, Section 7] the rank variety of any  $\mathcal{B}_{\chi,\lambda}$ -module is contained in  $\mathcal{V}_{\mathfrak{g}}(\chi, \lambda)$ .

*Remark.* For  $G^{(1)}$  of type  $A_n$  the variety  $\mathcal{V}_{\mathfrak{g}}(0, \lambda)$  is described in [22, Proposition 2.6]. This is recently generalised to all types in [30, Theorem 6.1] under the assumption that  $p \geq h$ , the Coxeter number.

4.2. The rest of this section is devoted to a proof of the following result.

**Theorem.** *Let  $\chi = \chi_s + \chi_n \in \mathfrak{g}^*$  and  $\lambda \in W\Lambda_{\chi_s}/W$ . Then*

$$\mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = \mathcal{V}_{\mathfrak{g}}(\chi_s, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(\chi).$$

*Remark.* If  $S_1, \dots, S_t$  is a complete list of the simple  $U_{\chi}$ -modules up to isomorphism, we define

$$\mathcal{V}_{\mathfrak{g}}(\chi) = \bigcup_{i=1}^t \mathcal{V}_{\mathfrak{g}}(S_i).$$

Suppose  $\chi \in \mathcal{N}$ . By [20, Satz 2.14] we have  $V_{\mathfrak{g}}(0) = \mathcal{N}_p(\mathfrak{g})$  so we deduce the principal result of [35], namely

$$\mathcal{V}_{\mathfrak{g}}(\chi) = \mathcal{N}_p(\mathfrak{z}_{\mathfrak{g}}(\chi)).$$

Our class of Lie algebras, however, is smaller than that considered in [35] (the latter includes for example  $\mathfrak{sl}_{np}(K)$ ).

4.3. We recall the Mil'ner map, introduced in [27]. The algebra  $U$  has a natural increasing filtration  $\{U^k\}$  where  $U^k$  denotes the span of all products of at most  $k$  elements of  $\mathfrak{g}$ .

Given a vector space  $V$ , we let  $S(V)$  be the symmetric algebra of  $V$ . The algebra  $S(V)$  also has a natural increasing filtration  $\{S^{\leq k}(V)\}$  where  $S^{\leq k}(V)$  denotes the sum of the homogeneous components  $S^i(V)$  with  $i \leq k$ .

Let  $\underline{x} = (x_1, \dots, x_r) \in \mathfrak{g}$ . For  $I = \{i_1 < \dots < i_k\} \subseteq \{1, \dots, r\}$  we set  $x_I = x_{i_1} \dots x_{i_k} \in U$ . Let

$$\phi_r(\underline{x}) = \sum x_{I_1} \dots x_{I_k} \in S(U),$$

where the summation runs through all decompositions

$$I_1 \cup \dots \cup I_k = \{1, \dots, r\}$$

of  $\{1, \dots, r\}$  into non-empty disjoint subsets. By [9, Section 1.1] there exists a unique linear map  $\phi : U \rightarrow S(U)$  such that  $\phi(1) = 1$  and  $\phi(x_1 \dots x_r) = \phi_r((x_1, \dots, x_r))$  for all  $x_1, \dots, x_r \in \mathfrak{g}$ . Let  $\pi : U \rightarrow \mathfrak{g}$  be an  $\text{ad } \mathfrak{g}$ -equivariant projection, which exists by [35, Section 3.3]. This induces a map  $S(\pi) : S(U) \rightarrow S(\mathfrak{g})$ . We let  $\beta = S(\pi) \circ \phi : U \rightarrow S(\mathfrak{g})$ .

Given  $\chi = \chi_s + \chi_n \in \mathfrak{g}^*$  let  $J_\chi$  be the two-sided ideal of  $S(\mathfrak{g})$  generated by the elements  $x^p - \chi(x)^p$  for all  $x \in \mathfrak{g}$ . We let  $S_\chi = S(\mathfrak{g})/J_\chi$ . There is a natural action of  $\mathfrak{g}$  on  $S_\chi$  induced by the adjoint action on  $\mathfrak{g} \subset S_\chi$ . The filtration  $\{U^k\}$  (respectively  $\{S^{\leq k}(\mathfrak{g})\}$ ) induces a natural increasing filtration on  $U_\chi$  (respectively  $S_\chi$ ).

**Lemma.** [35, Lemma 3.2] *The map  $\beta : U \rightarrow S(\mathfrak{g})$  induces a  $\mathfrak{g}$ -equivariant filtration preserving isomorphism  $\beta_\chi : U_\chi \rightarrow S_\chi$ .*

4.4. We define the map

$$\gamma_\chi : S_\chi \rightarrow S_{\chi_s}$$

by sending  $x \in \mathfrak{g} \subset S_\chi$  to  $x + \chi_n(x) \in S_{\chi_s}$ , and extending algebraically. Since  $\mathfrak{z}_\mathfrak{g}(\chi) \subseteq \mathfrak{z}_\mathfrak{g}(\chi_s)$  the map  $\gamma_\chi$  is a  $\mathfrak{z}_\mathfrak{g}(\chi)$ -equivariant isomorphism. This allows us to construct a  $\mathfrak{z}_\mathfrak{g}(\chi)$ -equivariant isomorphism

$$\phi_\chi = \beta_{\chi_s}^{-1} \circ \gamma_\chi \circ \beta_\chi : U_\chi \longrightarrow U_{\chi_s}.$$

4.5. It will be important for us to vary  $\chi$ . To this end we let  $t$  be a indeterminate. We define  $\mathfrak{g}_t = \mathfrak{g} \otimes_K K[t]$ . We can consider the algebras  $U_{\chi_s+t\chi_n}(\mathfrak{g}_t)$ ,  $U_{\chi_s}(\mathfrak{g}_t)$ ,  $S_{\chi_s+t\chi_n}(\mathfrak{g}_t)$  and  $S_{\chi_s}(\mathfrak{g}_t)$ , all of which are free  $K[t]$ -modules of rank  $p^{\dim \mathfrak{g}}$ . We define a map

$$\beta_{\chi_s+t\chi_n} : U_{\chi_s+t\chi_n}(\mathfrak{g}_t) \longrightarrow S_{\chi_s+t\chi_n}(\mathfrak{g}_t),$$

which, for  $x_1, \dots, x_r \in \mathfrak{g} \subset \mathfrak{g}_t$ , sends  $x_1 \dots x_r$  to  $\sum \pi(x_{I_1}) \dots \pi(x_{I_k})$ , where the summation runs through all decompositions  $I_1 \cup \dots \cup I_k$  of  $\{1, \dots, r\}$  into non-empty disjoint subsets, and is extended to  $U_{\chi_s+t\chi_n}(\mathfrak{g}_t)$  by  $K[t]$ -linearity. As before, it can be shown that this is a  $\mathfrak{g}_t$ -equivariant isomorphism of  $K[t]$ -modules. Moreover,  $\beta_{\chi_s+t\chi_n}$  preserves the natural filtrations on  $U_{\chi_s+t\chi_n}(\mathfrak{g}_t)$  and  $S_{\chi_s+t\chi_n}(\mathfrak{g}_t)$ .

4.6. Similarly we have a  $\mathfrak{z}_{\mathfrak{g}_t}(\chi)$ -equivariant isomorphism of  $K[t]$ -algebras

$$\gamma_{\chi_s+t\chi_n} : S_{\chi_s+t\chi_n}(\mathfrak{g}_t) \longrightarrow S_{\chi_s}(\mathfrak{g}_t),$$

sending  $x \in \mathfrak{g} \subset \mathfrak{g}_t$  to  $x + t\chi_n(x) \in S_{\chi_s}(\mathfrak{g}_t)$ . This allows us to construct a  $\mathfrak{z}_{\mathfrak{g}_t}(\chi)$ -equivariant isomorphism of  $K[t]$ -modules

$$\phi_{\chi_s+t\chi_n} = (\beta_{\chi_s})_t^{-1} \circ \gamma_{\chi_s+t\chi_n} \circ \beta_{\chi_s+t\chi_n} : U_{\chi_s+t\chi_n}(\mathfrak{g}_t) \longrightarrow U_{\chi_s}(\mathfrak{g}_t).$$

4.7. By definition we have that, as elements of  $S_{\chi_s}(\mathfrak{g}_t)$ ,

$$\begin{aligned} & (\gamma_{\chi_s+t\chi_n} \circ \beta_{\chi_s+t\chi_n})(x_1 \dots x_r) \\ &= \sum (\pi(x_{I_1}) + t\chi_n(\pi(x_{I_1}))) \dots (\pi(x_{I_k}) + t\chi_n(\pi(x_{I_k}))), \end{aligned}$$

for  $x_1, \dots, x_r \in \mathfrak{g} \subset \mathfrak{g}_t$ . This expression is equal to

$$\beta_{\chi_s}(x_1 \dots x_r) + t \cdot (\text{linear combination of } y_1 \dots y_d \text{ with } y_i \in \mathfrak{g}_t \text{ and } d < r).$$

Since the Mil'ner map  $(\beta_{\chi_s})_t$  is a filtration preserving isomorphism of  $K[t]$ -modules we see that all  $y_1 \dots y_d$ 's are contained in  $(\beta_{\chi_s})_t(U_{\chi_s}(\mathfrak{g}_t)^d)$ . We deduce that the map  $\phi_{\chi_s+t\chi_n} : U_{\chi_s+t\chi_n}(\mathfrak{g}_t) \longrightarrow U_{\chi_s}(\mathfrak{g}_t)$  is such that

$$\phi_{\chi_s+t\chi_n}(x_1 \dots x_r) \equiv x_1 \dots x_r \pmod{t(U_{\chi_s}(\mathfrak{g}_t)^{r-1})}.$$

4.8. It follows from [32, Theorem 2.5] that one can find a one-dimensional torus  $h(\tau)$  in  $Z_G(\chi_s)$  such that  $h(\tau) \cdot \chi_n = \tau^2 \chi_n$  for all  $\tau \in K^*$ . Since the elements  $e_\lambda$  are  $G$ -invariant the isomorphism between  $U_\chi$  and  $U_{g \cdot \chi}$  for  $g \in G$  restricts to an isomorphism between  $\mathcal{B}_{\chi, \lambda}$  and  $\mathcal{B}_{g \cdot \chi, \lambda}$ . In particular we can find weight vectors for  $h(\tau)$ , say  $u_1, \dots, u_r \in U$ , such that  $\{e_\lambda u_i + I_{\chi_s} : 1 \leq i \leq r\}$  is a basis for  $\mathcal{B}_{\chi_s, \lambda}$ . Arguing as in the proof of Theorem 3.2 we find a dense open subset  $\mathcal{O}$  of  $K$  such that  $\{e_\lambda u_i + I_{\chi_s + v\chi_n} : 1 \leq i \leq r\}$  is a basis of  $\mathcal{B}_{\chi_s + v\chi_n, \lambda}$  for all  $v \in \mathcal{O}$ . Conjugating by  $h(\tau)$  shows that the set  $\{e_\lambda u_i + I_\chi : 1 \leq i \leq r\}$  is a basis of  $\mathcal{B}_{\chi, \lambda}$ .

4.9. For  $1 \leq i \leq r$  we have

$$\phi_{\chi_s + t\chi_n}(e_\lambda u_i) = e_\lambda u_i + tm_{1,i} + t^2 m_{2,i} + \dots$$

where  $m_{j,i} \in U_{\chi_s}(\mathfrak{g}) \subset U_{\chi_s}(\mathfrak{g}_t)$ . Let  $\pi_\lambda : U_{\chi_s}(\mathfrak{g}) \twoheadrightarrow \mathcal{B}_{\chi_s, \lambda}$  be projection, a  $\mathfrak{g}$ -invariant homomorphism. We find that

$$\pi_\lambda \circ \phi_{\chi_s + t\chi_n}(e_\lambda u_i) = e_\lambda u_i + tn_{1,i} + t^2 n_{2,i} + \dots$$

where  $n_{j,i} = \pi_\lambda(m_{j,i})$ . It follows that the matrix expressing  $\pi_\lambda \circ \phi_{\chi_s + t\chi_n}$  in terms of the basis  $\{e_\lambda u_i : 1 \leq i \leq r\}$  has the form

$$C_t = \begin{pmatrix} 1 + ta_{11} & ta_{12} & \dots & ta_{1r} \\ ta_{21} & 1 + ta_{22} & \dots & ta_{2r} \\ \vdots & \vdots & & \vdots \\ ta_{r1} & ta_{r2} & \dots & 1 + ta_{rr} \end{pmatrix}$$

for some  $a_{ij} \in K[t]$ . Thus, for almost all  $v \in K$ , we have that  $\det(C_v)$  is non-zero and, from 4.8, that  $\{e_\lambda u_i + I_{\chi_s + v\chi_n} : 1 \leq i \leq r\}$  is a basis of  $\mathcal{B}_{\chi_s + v\chi_n, \lambda}$ . Thus, for almost all  $v \in K$ , restriction of  $\phi_{\chi_s + v\chi_n}$  yields a  $\mathfrak{z}_{\mathfrak{g}}(\chi_s + v\chi_n)$ -equivariant isomorphism

$$\mathcal{B}_{\chi_s + v\chi_n, \lambda} \longrightarrow \mathcal{B}_{\chi_s, \lambda}.$$

In particular, for almost all  $v \in K^*$ , we have an equality

$$(4) \quad \mathcal{V}_{\mathfrak{z}_{\mathfrak{g}}(\chi_s + v\chi_n)}(\mathcal{B}_{\chi_s, \lambda}) = \mathcal{V}_{\mathfrak{z}_{\mathfrak{g}}(\chi_s + v\chi_n)}(\mathcal{B}_{\chi_s + v\chi_n, \lambda}),$$

where  $\mathcal{B}_{\chi_s, \lambda}$  and  $\mathcal{B}_{\chi_s + v\chi_n, \lambda}$  are modules for the adjoint action of the stabiliser  $\mathfrak{z}_{\mathfrak{g}}(\chi_s + v\chi_n) = \mathfrak{z}_{\mathfrak{g}}(\chi_s) \cap \mathfrak{z}_{\mathfrak{g}}(v\chi_n) = \mathfrak{z}_{\mathfrak{g}}(\chi)$ .

4.10. Let  $L = Z_G(\chi_s)$ , a Levi subgroup of  $G$ , and  $N_L(\mathfrak{z}_{\mathfrak{g}}(\chi)) = \{g \in L : g.z \in \mathfrak{z}_{\mathfrak{g}}(\chi) \text{ for all } z \in \mathfrak{z}_{\mathfrak{g}}(\chi)\}$ . Let  $N$  be the identity component of  $N_L(\mathfrak{z}_{\mathfrak{g}}(\chi))$ . Recall that  $L$ , and hence  $N$ , acts on  $U_{\chi_s}$ . Moreover, if we decompose  $U_{\chi_s}$  as a direct sum of  $\text{ad } \mathfrak{z}_{\mathfrak{g}}(\chi)$ -modules, say  $\oplus M_i$ , then by definition  $N$  permutes the modules  $M_i$ . Since, however,  $N$  is the minimal normal subgroup of finite index in  $N_L(\mathfrak{z}_{\mathfrak{g}}(\chi))$ , we deduce by the Krull–Remak–Schmidt theorem that  $N$  fixes the isomorphism class of the modules  $M_i$ . This implies that the rank variety of  $M_i$  is  $N$ -invariant for any  $i$ .

It can be deduced from [32, Theorem 2.5], that given  $v \in K^*$  there exists  $g \in N$  such that  $g.\chi = \chi_s + v\chi_n$ . As a result we deduce from the above that

$$(5) \quad \mathcal{V}_{\mathfrak{z}_{\mathfrak{g}}(\chi)}(\mathcal{B}_{\chi,\lambda}) = \text{Ad}(g)\left(\mathcal{V}_{\mathfrak{z}_{\mathfrak{g}}(\chi)}(\mathcal{B}_{\chi,\lambda})\right) = \mathcal{V}_{\mathfrak{z}_{\mathfrak{g}}(\chi_s+v\chi_n)}(\mathcal{B}_{\chi_s+v\chi_n,\lambda}).$$

4.11. By [34, Theorem 1.1] and [35, Theorem 2.4] we have that for any  $U_{\chi}$ -module  $M$ ,  $\mathcal{V}_{\mathfrak{g}}(M) \subseteq \mathfrak{z}_{\mathfrak{g}}(\chi)$ . Since  $\mathcal{V}_{\mathfrak{z}_{\mathfrak{g}}(\chi)}(M) = \mathfrak{z}_{\mathfrak{g}}(\chi) \cap \mathcal{V}_{\mathfrak{g}}(M)$  for any  $U_{\chi}$ -module  $M$ , we deduce that

$$(6) \quad \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) \subseteq \mathfrak{z}_{\mathfrak{g}}(\chi).$$

4.12. Arguing as in [35, Proposition 2.2] we have

$$(7) \quad \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = \mathcal{V}_{\mathfrak{g}}(\mathcal{B}_{\chi,\lambda}).$$

The required equality  $\mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = \mathcal{V}_{\mathfrak{g}}(\chi_s, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(\chi)$  now follows by combining (7), (6), (5) and (4).

## 5. REPRESENTATION TYPE

5.1. We come to the description of the representation type of a block,  $\mathcal{B}_{\chi,\lambda}$ , of the reduced enveloping algebra  $U_{\chi}$ . By Proposition 2.7 we can assume without loss of generality that  $\chi$  is nilpotent and  $\lambda \in \Lambda/W$ . Let  $G_1, \dots, G_m$  be the simple (simply-connected) normal subgroups of  $G^{(1)}$  and let  $\mathfrak{g}' = \text{Lie } G^{(1)}$  and  $\mathfrak{g}_j = \text{Lie } G_j$  for  $1 \leq j \leq m$ . Let  $e = \theta^{-1}(\chi)$  be the nilpotent element in  $\mathfrak{g}$  corresponding to  $\chi$ . By [33, Section 2.9]  $e \in \mathfrak{g}'$  so we can decompose  $\chi = \chi_1 + \dots + \chi_m$  where  $\chi_j \in \mathfrak{g}_j^*$  is the restriction of  $\chi$  to  $\mathfrak{g}_j$ . The Weyl group  $W$  can be identified with  $W_1 \times \dots \times W_m$  where  $W_i$  is the Weyl group of  $G_i$ . Under this identification  $W(\lambda)$  decomposes as  $W_1(\lambda) \times \dots \times W_m(\lambda)$  for any  $\lambda \in \Lambda$ . We will retain this notation throughout this section.

5.2. We first classify the blocks of finite representation type, completing work begun in [29].

**Theorem.** *Recall the notation of 5.1. Let  $\chi \in \mathcal{N}$  and  $\lambda \in \Lambda/W$ . Then the block  $\mathcal{B}_{\chi, \lambda}$  has finite representation type if and only if one of the following occurs.*

1.  $W(\lambda) = W$ .
2. There exists an integer  $j$ ,  $1 \leq j \leq m$ , such that  $\chi_j$  is regular,  $W_i(\lambda) = W_i$  for  $i \neq j$  and one of the following conditions holds:
  - (a)  $W_j$  is of type  $A_n$  and  $W_j(\lambda)$  is of type  $A_{n-1}$ ;
  - (b)  $W_j$  is of type  $B_n$  (or  $C_n$ ) and  $W_j(\lambda)$  is of type  $B_{n-1}$  (or  $C_{n-1}$ );
  - (c)  $W_j$  is of type  $G_2$  and  $W_j(\lambda)$  is of type  $A_1$ .

(Here we take  $A_0 = \emptyset$  and  $B_1 = A_1$ .)

5.3. We come to the classification of blocks of tame representation type.

**Theorem.** *Recall the notation of 5.1. Let  $\chi \in \mathcal{N}$  and  $\lambda \in \Lambda/W$ . The block  $\mathcal{B}_{\chi, \lambda}$  has tame representation type if and only if one of the following occurs.*

1. There exists  $j$ ,  $1 \leq j \leq m$ , such that  $\chi_j$  is regular,  $W_i(\lambda) = W_i$  for  $i \neq j$  and one of the following holds:
  - (a)  $W_j$  has rank 2;
  - (b)  $W_j$  is of type  $A_3$  and  $W_j(\lambda)$  is of type  $A_1 \times A_1$ ;
  - (c)  $W_j$  is of type  $B_3$  (or  $C_3$ ) and  $W_j(\lambda)$  has type  $A_2$ ;
  - (d)  $W_j$  is of type  $D_n$  and  $W_j(\lambda)$  is of type  $D_{n-1}$ .
2. There exists  $j$ ,  $1 \leq j \leq m$ , such that  $\chi_j$  is subregular,  $W_i(\lambda) = W_i$  for  $i \neq j$ ,  $W_j$  is of type  $A_n$  and  $W_j(\lambda)$  is of type  $A_{n-1}$ .
3. There exists  $j_1, j_2$ ,  $1 \leq j_1 < j_2 \leq m$ , such that both  $\chi_{j_1}$  and  $\chi_{j_2}$  are regular,  $W_i(\lambda) = W_i$  for  $i \neq j_1, j_2$  and  $W_{j_1} \times W_{j_2}$  is of type  $A_1 \times A_1$  whilst  $W_{j_1}(\lambda) \times W_{j_2}(\lambda)$  is trivial.

(Here we take  $A_0 = \emptyset$  and  $D_3 = A_3$ .)

5.4. Finally, we describe the representation type of a reduced enveloping algebra  $U_\chi$ .

**Theorem.** *Let  $\chi \in \mathcal{N}$ .*

1. *The algebra  $U_\chi$  has finite representation type if  $\chi$  is regular and one of the following holds:*

- (a)  $W$  is trivial;
  - (b)  $W$  has type  $A_1$ ;
  - (c)  $p = 2$  and  $W$  has type  $A_2$ ;
  - (d)  $p = 3$  and  $W$  has type  $B_2$  (or  $C_2$ );
  - (e)  $p = 5$  and  $W$  has type  $G_2$ .
2. The algebra  $U_\chi$  has tame representation type if one of the following holds:
- (a)  $\chi = 0$  and  $W$  has type  $A_1$ ;
  - (b)  $\chi$  is regular and  $W$  has type  $A_1 \times A_1$ ;
  - (c)  $p \neq 2$ ,  $\chi$  is regular and  $W$  has type  $A_2$ ;
  - (d)  $p \neq 3$ ,  $\chi$  is regular and  $W$  has type  $B_2$  (or  $C_2$ );
  - (e)  $p \neq 5$ ,  $\chi$  is regular and  $W$  has type  $G_2$ ;
  - (f)  $p = 2$ ,  $\chi$  is subregular and  $W$  has type  $A_2$ ;
  - (g)  $p = 2$ ,  $\chi$  is regular and  $W$  has type  $A_3$ .
3. In all other cases  $U_\chi$  has wild representation type.

*Remark.* In [35, Proposition 5.2] a list of *possible* tame  $U_\chi$  was given. This list, however, was incomplete and should have also included case 2(f) above. (Indeed the proof of [35, Proposition 5.2] should be adjusted on p.278, line 7 from “If the number of blocks is  $> 1$  ...” to “If the number of blocks of size  $> 1$  is  $> 1$  ...”. Then one must consider  $A_2$  and  $A_3$  with  $\chi$  subregular and  $p = 2$ . A calculation shows that the support variety in the  $A_3$  case is three dimensional, implying wildness.) Of course, Part 2 of the above theorem refines the (corrected) list in [35, Proposition 5.2].

5.5. Theorem 5.2 will be proved in 8.16 and Theorem 5.4 in 8.18. Most of the rest of the paper is concerned with proving Theorem 5.3.

## 6. A REDUCTION

6.1. Let  $G^{(1)}$  be the derived subgroup of  $G$  and let  $\mathfrak{g}' = \text{Lie}(G^{(1)})$ . Then  $\mathfrak{g}_\alpha \subset \mathfrak{g}'$  for all  $\alpha \in \Phi$ . Let  $G_1, \dots, G_m$  be the simple (simply-connected) normal subgroups of  $G^{(1)}$  and let  $\mathfrak{g}_i = \text{Lie}(G_i)$  for  $1 \leq i \leq m$ . Then  $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_s = \mathfrak{g}' \subseteq \mathfrak{g}$  and  $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$  if  $i \neq j$ .

For  $1 \leq i \leq m$ , define  $\tilde{G}_i$  by setting

$$\tilde{G}_i = \begin{cases} GL(V_i), & \text{if } G_i \cong SL(V_i) \text{ and } p \mid \dim V_i, \\ G_i, & \text{otherwise.} \end{cases}$$

Put  $\tilde{G} = \tilde{G}_1 \times \cdots \times \tilde{G}_m$ ,  $\tilde{\mathfrak{g}}_i = \text{Lie}(\tilde{G}_i)$  and  $\tilde{\mathfrak{g}} = \text{Lie}(\tilde{G})$ . We have  $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_1 \oplus \cdots \oplus \tilde{\mathfrak{g}}_m$  and either  $\tilde{\mathfrak{g}}_i = \mathfrak{g}_i$  or  $\mathfrak{g}_i \cong \mathfrak{sl}(V_i)$  and  $\tilde{\mathfrak{g}}_i \cong \mathfrak{gl}(V_i)$  and  $p \mid \dim V_i$ . We identify each  $\mathfrak{g}_i$  with an ideal of codimension at most one in  $\tilde{\mathfrak{g}}_i$ .

6.2. Recall a restricted Lie algebra is called *toral* if it has a basis consisting of toral elements, that is a basis consisting of elements which satisfy  $t^{[p]} = t$ . Thanks to [35, Lemma 4.1] there is a toral Lie algebra  $\mathfrak{t}_0$  and an embedding of restricted Lie algebras

$$\psi : \mathfrak{g} \longrightarrow \tilde{\mathfrak{g}} \oplus \mathfrak{t}_0,$$

such that  $\psi(\mathfrak{g}_i) = \mathfrak{g}_i \subseteq \tilde{\mathfrak{g}}$  and  $\psi(\mathfrak{h}) \subseteq \tilde{\mathfrak{h}} \oplus \mathfrak{t}_0$ , where  $\tilde{\mathfrak{h}}$  is a Cartan subalgebra of  $\tilde{\mathfrak{g}}$  satisfying  $\mathfrak{h} \cap \mathfrak{g}' = \tilde{\mathfrak{h}} \cap \mathfrak{g}'$ . Set  $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathfrak{t}_0$  and  $\hat{\mathfrak{h}} = \tilde{\mathfrak{h}} \oplus \mathfrak{t}_0$ .

**Lemma.** *There exists a toral subalgebra  $\mathfrak{t}_1 \subset \hat{\mathfrak{h}}$  such that  $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{t}_1$  and  $[\mathfrak{t}_1, \hat{\mathfrak{g}}] = 0$ .*

*Proof.* Let  $\mathfrak{c}$  denote the centraliser of  $\mathfrak{g}'$  in  $\hat{\mathfrak{g}}$ . Since  $\mathfrak{g}'$  is invariant under the adjoint action of  $G^{(1)}$  so is  $\mathfrak{c}$ . Now  $\mathfrak{g}_\alpha \subset \mathfrak{g}'$  for any root  $\alpha$  and  $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \ni h_\alpha \neq 0$ . Therefore the maximal torus  $T \cap G^{(1)}$  of  $G^{(1)}$  acts trivially on  $\mathfrak{c}$  implying  $\mathfrak{c} \subseteq \hat{\mathfrak{h}}$ . Since  $\hat{\mathfrak{g}} = \hat{\mathfrak{h}} + \mathfrak{g}'$  and  $\hat{\mathfrak{h}}$  is abelian we deduce that  $\mathfrak{c}$  is a central toral subalgebra of  $\hat{\mathfrak{g}}$ . Let  $\mathfrak{c}^{\text{tor}}$  denote the  $\mathbb{F}_p$ -subspace of toral elements of  $\mathfrak{c}$ .

Identify  $\mathfrak{h}$  with  $\psi(\mathfrak{h}) \subseteq \hat{\mathfrak{h}}$ . Let  $\hat{h} \in \hat{\mathfrak{h}}$  and let  $a_i = \alpha_i(\hat{h})$  where  $1 \leq i \leq n$ . Let  $e_{\pm\alpha_i}$  be root vectors such that  $[e_{\alpha_i}, e_{-\alpha_i}] = h_{\alpha_i}$ . It follows from the  $T$ -invariance of  $B$  that  $B(e_{\alpha_i}, e_{-\alpha_i}) \neq 0$ . Since  $h_{\alpha_1}, \dots, h_{\alpha_n} \in \mathfrak{h}$  are linearly independent and the restriction of  $B$  to  $\mathfrak{h}$  is non-degenerate there is  $h \in \mathfrak{h}$  such that  $B(h_{\alpha_i}, h) = a_i B(e_{\alpha_i}, e_{-\alpha_i})$  for all  $i$ . By the  $\mathfrak{g}$ -invariance of  $B$ ,

$$a_i B(e_{\alpha_i}, e_{-\alpha_i}) = B([e_{\alpha_i}, e_{-\alpha_i}], h) = B(e_{-\alpha_i}, [h, e_{\alpha_i}]) = \alpha_i(h) B(e_{\alpha_i}, e_{-\alpha_i})$$

yielding  $\alpha_i(h) = a_i$  for all  $i$ . As a consequence,  $\hat{h} - h$  centralises the Lie subalgebra generated by all  $e_{\pm\alpha_i}$ . The latter coincides with  $\mathfrak{g}'$ , because  $G^{(1)}$  is simply-connected, see [25, 1.2] for example. Thus for any  $\hat{h} \in \hat{\mathfrak{h}}$  there is  $h \in \mathfrak{h}$  such that  $\hat{h} - h \in \mathfrak{c}$ . In other words  $\hat{\mathfrak{g}} = \mathfrak{c} + \mathfrak{g}$ . Now  $\mathfrak{c} \cap \mathfrak{g}$  is a restricted subalgebra of  $\mathfrak{c}$  hence is spanned by its  $\mathbb{F}_p$ -subspace  $(\mathfrak{c} \cap \mathfrak{g})^{\text{tor}}$  of



toral elements. The latter has a complement in  $\mathfrak{c}^{\text{tor}}$ , say  $\mathfrak{t}_1^{\text{tor}}$ . The  $K$ -span  $\mathfrak{t}_1$  of  $\mathfrak{t}_1^{\text{tor}}$  is a central toral subalgebra of  $\hat{\mathfrak{g}}$  satisfying  $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{t}_1$ .  $\square$

6.3. Let  $T_0$  be an algebraic torus over  $K$  such that  $\mathfrak{t}_0 = \text{Lie}(T_0)$  and let  $\hat{T} = \tilde{T} \times T_0$  be a maximal torus of  $\tilde{G} \times T_0$  such that  $\hat{\mathfrak{h}} = \text{Lie}(\hat{T})$ . There is a decomposition  $\tilde{T} = T_1 \times \cdots \times T_m$  where  $T_j$  is a maximal torus of  $\tilde{G}_j$  for  $1 \leq j \leq m$ . Let  $\mathfrak{h}_j = \text{Lie}(T_j)$  for  $1 \leq j \leq m$ . Define

$$\hat{\Lambda} = \{\lambda \in \hat{\mathfrak{h}}^* : \lambda(h)^p - \lambda(h^{[p]}) = 0 \text{ for all } h \in \hat{\mathfrak{h}}\}.$$

Any element of  $\hat{\Lambda}$  can be uniquely decomposed into  $(\lambda_1, \dots, \lambda_m, \lambda_0) \in \mathfrak{h}_1^* \oplus \cdots \oplus \mathfrak{h}_m^* \oplus \mathfrak{t}_0^*$ . There is an isomorphism between  $W$ , the Weyl group of  $G$  with respect to  $T$  or of  $\tilde{G} \times T_0$  with respect to  $\hat{T}$ , and  $W_1 \times \cdots \times W_m$  where  $W_i$  is the Weyl group of  $\tilde{G}_j$  with respect to  $T_j$  for  $1 \leq j \leq m$ . Under this isomorphism  $W(\lambda)$  is identified with  $W_1(\lambda_1) \times \cdots \times W_m(\lambda_m)$ .

6.4. Since  $\psi : \mathfrak{h} \longrightarrow \hat{\mathfrak{h}}$  is an embedding of restricted Lie algebras the induced map  $\psi^* : \hat{\mathfrak{h}}^* \longrightarrow \mathfrak{h}^*$  is surjective and sends  $\hat{\Lambda}$  onto  $\Lambda$ . Recall that  $\mathfrak{g}'$  contains all root spaces of  $\mathfrak{g}$ . Therefore we can consider any root  $\alpha$  as an element of  $\mathfrak{h}^*$  and as an element of  $\hat{\mathfrak{h}}^*$ . We have

$$\psi^*(s_\alpha \lambda) = \psi^*(\lambda - \lambda(\psi(h_\alpha))\alpha) = \psi^*(\lambda) - \psi^*(\lambda)(h_\alpha)\alpha = s_\alpha(\psi^*\lambda),$$

showing that  $\psi^*$  is  $W$ -equivariant. Hence we have a surjective map

$$\pi : \hat{\Lambda}/W \longrightarrow \Lambda/W.$$

Given  $\xi \in \mathfrak{g}^*$  we denote by  $\hat{\xi}$  the functional on  $\hat{\mathfrak{g}}$  whose restriction to  $\mathfrak{g}$  (respectively to  $\mathfrak{t}_1$ ) equals  $\xi$  (respectively 0). If  $\xi$  vanishes on  $\mathfrak{b}^+$  then  $\hat{\xi}$  vanishes on  $\hat{\mathfrak{h}} + \mathfrak{b}^+$ . In particular,  $\hat{\xi}$  is nilpotent in this case.

**Lemma.** *Keep the above notation and suppose that  $\chi$  vanishes on  $\mathfrak{b}^+$ . Let  $\lambda \in \hat{\Lambda}/W$ . Then there is an isomorphism of algebras*

$$(8) \quad \mathcal{B}_{\hat{\chi}, \lambda}(\hat{\mathfrak{g}}) \cong \mathcal{B}_{\chi, \pi(\lambda)}(\mathfrak{g}).$$

*Proof.* Since  $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathfrak{t}_1$  each  $\mu \in \hat{\Lambda}$  decomposes into  $(\psi^*\mu, \mu_1) \in \mathfrak{h}^* \oplus \mathfrak{t}_1^*$ . Clearly  $\lambda_1(h)^p = \lambda_1(h^{[p]})$  for all  $h \in \mathfrak{t}_1$ . Let  $K_{\lambda_1}$  denote the one-dimensional  $U_0(\mathfrak{t}_1)$ -module corresponding to  $\lambda_1$ . Thanks to Lemma 6.2 and the PBW theorem there exists an algebra isomorphism

$$\phi : U_{\hat{\chi}}(\hat{\mathfrak{g}}) \xrightarrow{\sim} U_{\chi}(\mathfrak{g}) \otimes U_0(\mathfrak{t}_1)$$

such that  $\phi(u) = u \otimes 1$  for any  $u \in U_\chi(\mathfrak{g}) \subset U_{\hat{\chi}}(\hat{\mathfrak{g}})$ . Since  $U_0(\mathfrak{t}_1)$  is a commutative semisimple algebra there is a primitive idempotent  $e \in U_0(\mathfrak{t}_1)$  such that  $e(K_{\lambda_1}) \neq 0$  and

$$\phi(\mathcal{B}_{\hat{\chi}, \lambda}(\hat{\mathfrak{g}})) = \mathcal{B}_{\chi, \mu}(\mathfrak{g}) \otimes e \cong \mathcal{B}_{\chi, \pi(\nu)}(\mathfrak{g})$$

for some  $\nu \in \hat{\Lambda}/W$ . It is straightforward to see that  $\phi$  sends the  $U_\chi(\mathfrak{g}) \otimes U_0(\mathfrak{t}_1)$ -module  $Z_\chi(\psi^*\lambda) \otimes K_{\lambda_1}$  to the baby Verma module  $Z_{\hat{\chi}}(\lambda)$ . It follows that  $\mathcal{B}_{\chi, \pi(\nu)}$  acts non-trivially on  $Z_\chi(\psi^*\lambda)$ . Therefore  $\pi(\nu) = \pi(\lambda)$  as required.  $\square$

6.5. Let  $\chi$  be as above and define  $\chi_i = \hat{\chi}|_{\hat{\mathfrak{g}}_i}$ . Since  $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}}_1 \oplus \dots \oplus \tilde{\mathfrak{g}}_m \oplus \mathfrak{t}_0$  we have

$$\mathcal{B}_{\hat{\chi}, \lambda}(\hat{\mathfrak{g}}) \cong \mathcal{B}_{\chi_1, \lambda_1}(\tilde{\mathfrak{g}}_1) \otimes \dots \otimes \mathcal{B}_{\chi_m, \lambda_m}(\tilde{\mathfrak{g}}_m) \otimes \mathcal{B}_{0, \lambda_0}(\mathfrak{t}_0).$$

Since  $\mathfrak{t}_0$  is toral  $\mathcal{B}_{0, \lambda_0}(\mathfrak{t}_0) \cong K$ . Combining this with 6.3 and Lemma 6.4 we have the following result.

**Proposition.** *Keep the above notation. There is an isomorphism of algebras*

$$\mathcal{B}_{\chi_1, \lambda_1}(\tilde{\mathfrak{g}}_1) \otimes \dots \otimes \mathcal{B}_{\chi_m, \lambda_m}(\tilde{\mathfrak{g}}_m) \cong \mathcal{B}_{\chi, \pi(\lambda)}(\mathfrak{g}),$$

such that  $\chi = \chi_1 + \dots + \chi_m$  (considered as functionals on  $\mathfrak{g}'$ ) and  $W(\lambda) = W_1(\lambda_1) \times \dots \times W_m(\lambda_m)$ .

## 7. REGULAR ALGEBRAS

7.1. We assume throughout this section that  $G = \tilde{G}$  and  $G^{(1)}$  is simple. We determine the representation type of the blocks  $\mathcal{B}_{\chi, \lambda}$  for  $\chi \in \mathfrak{g}^*$  regular nilpotent and  $\lambda \in \Lambda/W$ . Thanks to [28, Theorem 12] and [3, Proposition 3.16] the block  $\mathcal{B}_{\chi, \lambda}$  is Morita equivalent to the partial coinvariant algebra

$$(9) \quad C_\lambda = S(\mathfrak{h})^{W(\lambda)} \otimes_{S(\mathfrak{h})^W} K,$$

where  $W(\lambda)$  is defined as in 2.4. So we only need to calculate the representation type of  $C_\lambda$ . If  $p = 2$  then  $\mathfrak{g} = \mathfrak{sl}(V)$  (respectively  $\mathfrak{g} = \mathfrak{gl}(V)$ ) if  $\dim V$  is odd (respectively even). It is straightforward to check that in these situations the same analysis applies.

7.2. The partial coinvariant algebras of finite representation type are described in [3, Corollary 3.19]. They are as follows.

**Theorem.** *Let  $\lambda \in \Lambda/W$ . The algebra  $C_\lambda$  has finite representation type if and only if one of the following cases occurs:*

1.  $W(\lambda) = W$  ;
2.  $W$  is of type  $A_n$  and  $W(\lambda)$  is of type  $A_{n-1}$ ;
3.  $W$  is of type  $B_n$  (or  $C_n$ ) and  $W(\lambda)$  is of type  $B_{n-1}$  (or  $C_{n-1}$ ) ;
4.  $W$  is of type  $G_2$  and  $W(\lambda)$  is of type  $A_1$ .

In all these cases  $C_\lambda \cong K[X]/(X^r)$  where  $r = [W : W(\lambda)]$ .

7.3. We spend the rest of the section proving the following result.

**Theorem.** *Let  $\lambda \in \Lambda/W$ . The algebra  $C_\lambda$  has tame representation type if and only if one of the following cases occurs:*

1.  $W$  has rank 2 and  $W(\lambda) = 1$  ;
2.  $W$  is of type  $A_3$  and  $W(\lambda)$  is of type  $A_1 \times A_1$ ;
3.  $W$  is of type  $B_3$  (or  $C_3$ ) and  $W(\lambda)$  is of type  $A_2$ ;
4.  $W$  is of type  $D_n$  and  $W(\lambda)$  is of type  $D_{n-1}$ .

We prove this by case-by-case analysis after making several general observations and simplifications.

7.4. Let  $\lambda \in \Lambda/W$ . By [3, 3.8] the partial coinvariant algebra  $C_\lambda$  is a local, symmetric, commutative algebra of dimension  $[W : W(\lambda)]$ . Let  $W_\lambda$  be the subset of  $W$  consisting of minimal length coset representatives for the subgroup  $W(\lambda)$  in  $W$ , [17, 1.10]. It is shown in [3, Lemma 3.19] that  $C_\lambda$  admits a  $\mathbb{N}$ -grading such that its Poincaré series is

$$(10) \quad P(C_\lambda, t) = \sum_{w \in W_\lambda} t^{\ell(w)}.$$

7.5. Tame, local, symmetric, commutative algebras are classified in [5, Theorem III.1] (the proof follows the ideas in [37]). They have the form  $K[X, Y]/I$  where  $I$  is an ideal of the following type:

1.  $I = (X^m - Y^n, XY)$  where  $m \geq n \geq 2$  and  $m + n > 4$ ;
2.  $I = (X^2, Y^2)$ ;
3.  $I = (X^2, Y^2 - XY)$  where  $\text{char } K = 2$ .

In particular any algebra with minimal number of generators greater than two is wild. Note that the first type of algebra has dimension  $m + n > 4$ .

7.6. As a consequence of 7.4 and 7.5 if  $\lambda \in \Lambda/W$  is such that  $C_\lambda$  is tame then the rank of  $W$  and the rank of  $W(\lambda)$  as Coxeter groups must differ by either one or two. Indeed, if the ranks differ by more than two then  $C_\lambda$  has at least three generators by (10), implying wildness.

7.7. The following lemma gives a useful criterion for the wildness of a partial coinvariant algebra.

**Lemma.** *Suppose  $\lambda \in \Lambda/W$  is such that*

$$P(C_\lambda, t) = (1 + t + \dots + t^i) + 2(t^{i+1} + t^{i+2}) + t^{i+1}\mathbb{N}[t],$$

*for some  $i \geq 1$ , or*

$$P(C_\lambda, t) = 1 + t + 3t^2 + t\mathbb{N}[t].$$

*Then  $C_\lambda$  is wild.*

*Proof.* Since  $C_\lambda$  is local it admits a minimal generating set consisting of elements homogeneous with respect to the grading of 7.4.

We consider the first Poincaré series. We have a generator  $X$  in degree one. If  $Y$  is a new generator in degree  $j \leq i$  then  $X^j = 0$ . Therefore the algebra generated by  $X$  and  $Y$  has only one dimensional homogeneous components. Since  $\dim(C_\lambda)_{i+1} \geq 2$  we deduce that  $C_\lambda$  requires a third generator, implying wildness. Similarly if  $X^{i+1} = 0$  or  $X^{i+2} = 0$  then  $C_\lambda$  requires at least three generators, so is wild.

Assume  $X^{i+2} \neq 0$ . Let  $Y$  be any element in  $(C_\lambda)_{i+1}$ , linearly independent from  $X^{i+1}$ . Then, since  $i \geq 1$ , either  $C_\lambda$  requires a third generator in degree  $i + 2$ , implying wildness, or  $X^{i+2}$  and  $XY$  are linearly independent. In this last situation the non-vanishing of  $XY$  implies that  $C_\lambda$  cannot be an algebra of Type 7.5.1. Since  $\dim C_\lambda > 4$  we deduce from 7.5 that  $C_\lambda$  is indeed wild.

For the second Poincaré series we need a generator in degree one and at least two new generators in degree two, implying wildness.  $\square$

7.8. We also have a sufficient criterion for the tameness of a partial coinvariant algebra.

**Lemma.** *Let  $\lambda \in \Lambda/W$  be such that  $C_\lambda$  is generated in degree one with Poincaré series*

$$P(C_\lambda, t) = 1 + 2(t + t^2 + \dots + t^{r-1}) + t^r,$$

*for some  $r \geq 3$ . Assume that the unique quadratic relation in  $C_\lambda$  is square-free. Then  $C_\lambda$  is tame.*

*Proof.* Since the quadratic relation in  $C_\lambda$  is square-free we can find linearly independent elements  $X, Y \in (C_\lambda)_1$  such that  $XY = 0$ . By hypothesis these elements generate  $C_\lambda$ . We deduce from the Poincaré series that the elements  $X^i$  and  $Y^i$  for  $1 \leq i \leq r-1$  are linearly independent. Since  $\dim(C_\lambda)_r = 1$  we see that  $X^r$  and  $Y^r$  are linearly dependent. Therefore either exactly one of  $X^r, Y^r$  is zero or  $0 \neq X^r = cY^r$  for some  $c \in K^*$ . If  $X^r = 0$  (respectively  $Y^r = 0$ ) then both  $X^{r-1}$  (respectively  $Y^{r-1}$ ) and  $Y^r$  (respectively  $X^r$ ) belong to the socle of  $C_\lambda$ , contradicting symmetry. So, possibly after rescaling, we have an algebra isomorphism

$$C_\lambda \cong \frac{K[X, Y]}{(XY, X^r - Y^r)},$$

implying that  $C_\lambda$  has tame representation type, by 7.5.  $\square$

7.9. Let  $\lambda \in \Lambda/W$  be such that the rank of  $W(\lambda)$  is two less than the rank of  $W$ . Consider the following part of a Dynkin diagram

$$\dots \circ \text{---} \bullet \text{---} \bullet \dots$$

Here (and for the rest of this section) the coloured nodes indicate the simple reflections of  $W$  not contained in  $W(\lambda)$ . Using the grading from (10) we see that  $C_\lambda$  has two elements of degree one, namely those corresponding to  $s_{i+1}$  and  $s_{i+2}$ , and at least three elements of degree two, corresponding to  $s_i s_{i+1}$ ,  $s_{i+1} s_{i+2}$  and  $s_{i+2} s_{i+1}$ . It follows from Lemma 7.7 that  $C_\lambda$  is wild in this case.

Now consider the following diagram

$$\dots \bullet \dots \circ \text{---} \bullet \dots$$

By (10) the algebra  $C_\lambda$  has two elements of degree one, corresponding to  $s_i$  and  $s_{j+1}$ , and at least three of degree two, corresponding to  $s_i s_{j+1}$ ,  $s_{i+1} s_i$  and  $s_j s_{j+1}$ . It follows from Lemma 7.7 that  $C_\lambda$  is wild.

7.10. So, in the case  $W(\lambda)$  has rank two less than  $W$ , the only possibility for  $C_\lambda$  to be tame is if  $W$  has rank two, so is one of  $A_2$ ,  $B_2$  or  $G_2$ . In case  $A_2$  (respectively  $B_2$  and  $G_2$ ) a straightforward calculation shows that the hypotheses of Lemma 7.8 are satisfied with  $r = 3$  (respectively  $r = 4$  and  $r = 6$ ) and hence  $C_\lambda$  is tame.

7.11. **Type  $A_n$ .** Assume that the rank of  $W(\lambda)$  is one less than the rank of  $W$ . We know by Theorem 7.2.2 that if  $W(\lambda)$  has type  $A_{n-1}$  then  $C_\lambda$  has finite representation type. Suppose we have the following diagram,

$$(11) \quad \cdots \circ \text{ --- } \circ \text{ --- } \bullet \text{ --- } \circ \cdots \quad .$$

Then  $C_\lambda$  has a unique element of degree one corresponding to  $s_{i+2}$ , whilst there are two elements of degree two corresponding to  $s_{i+1} s_{i+2}$  and  $s_{i+3} s_{i+2}$ . In degree three we have at least the elements corresponding to  $s_i s_{i+1} s_{i+2}$  and  $s_{i+1} s_{i+3} s_{i+2}$ , so  $C_\lambda$  is wild by Lemma 7.7.

7.12. We have only the following case to consider in type  $A_3$ ,

$$\circ \text{ --- } \bullet \text{ --- } \circ \quad .$$

Here  $C_\lambda$  is isomorphic to the algebra with generators  $A = X_1 + X_2$ ,  $B = X_3 + X_4$ ,  $C = X_1 X_2$  and  $D = X_3 X_4$  subject to the relations  $A + B = 0$ ,  $AB + C + D = 0$ ,  $AD + BC = 0$  and  $CD = 0$ . This is a six-dimensional algebra. Calculation shows that this algebra is generated by  $X = A$  and  $Y = A^2 - 2C$  and that  $XY = 0$  and  $X^4 = Y^2$ . We deduce that  $C_\lambda$  is isomorphic to the algebra  $K[X, Y]/(X^4 - Y^2, XY)$  of Type 7.5.1, and hence tame.

7.13. **Types  $B_n$  and  $C_n$ .** We assume that  $W(\lambda)$  has rank one less than  $W$ . If  $W(\lambda)$  has type  $B_{n-1}$  (or  $C_{n-1}$ ) then we know by Theorem 7.2.3 that  $C_\lambda$  has finite representation type. Consider the following diagram,

$$(12) \quad \cdots \circ \text{ --- } \bullet \cdots \cdots \text{ === } \circ \quad .$$

Then  $C_\lambda$  has one element of degree one corresponding to  $s_{i+1}$ , two elements of degree two corresponding to  $s_i s_{i+1}$  and  $s_{i+1} s_{i+2}$  and elements of

degree three corresponding to  $s_{i+2}s_is_{i+1}$  and at least one of  $s_{i+3}s_{i+2}s_{i+1}$  or  $s_{i+1}s_{i+2}s_{i+1}$  (which occurs if  $i = n - 2$ ). Therefore  $C_\lambda$  is wild by Lemma 7.7.

Consider the following diagram

$$\cdots \circ \text{---} \circ \text{---} \circ \text{===} \bullet \cdot$$

Then  $C_\lambda$  has one element of degree one corresponding to  $s_n$ , and one element of degree two corresponding to  $s_{n-1}s_n$ . In degree three we have the elements corresponding to  $s_{n-2}s_{n-1}s_n$  and  $s_ns_{n-1}s_n$ , whilst in degree four we have the elements corresponding to  $s_{n-3}s_{n-2}s_{n-1}s_n$  and  $s_{n-2}s_ns_{n-1}s_n$ . This implies that  $C_\lambda$  is wild by Lemma 7.7.

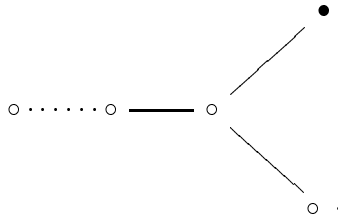
7.14. The only remaining case is the following diagram,

$$\circ \text{---} \circ \text{===} \bullet \cdot$$

In this case it can be checked that  $C_\lambda$  is the algebra generated by elements  $A = X_1 + X_2 + X_3$ ,  $B = X_1X_2 + X_1X_3 + X_2X_3$  and  $C = X_1X_2X_3$  subject to the relations induced by  $X_1^2 + X_2^2 + X_3^2 = 0$ ,  $X_1^2X_2^2 + X_1^2X_3^2 + X_2^2X_3^2 = 0$  and  $X_1^2X_2^2X_3^2 = 0$ . This is an eight dimensional algebra. Calculation shows that  $A^2 = 2B$  and  $B^2 = 2AC$  and  $A^4 = 8AC$  and  $C^2 = 0$ . This implies that  $A^7 = 0$ . Since  $\text{char} K \neq 2$  (this is a bad prime) we deduce that  $C_\lambda$  is a quotient of  $K[A, C]/(A^7, C^2, A^4 - 8AC)$ . Making the substitution  $X = \eta A$  and  $Y = A^3 - 8C$  we find that  $C_\lambda$  is the algebra  $K[X, Y]/(X^6 - Y^2, XY)$  of Type 7.5.1, and so tame.

7.15. **Type  $D_n$ .** We assume that the rank of  $W(\lambda)$  is one less than the rank of  $W$ . Consider the following diagram

(13)



In degree one (respectively two)  $C_\lambda$  has a unique element corresponding to  $s_n$  (respectively  $s_{n-2}s_n$ ). In degree three it has elements corresponding

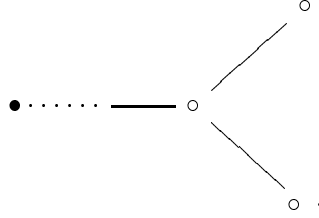
to  $s_{n-1}s_{n-2}s_n$  and  $s_{n-3}s_{n-2}s_n$ , whilst in degree four it has elements corresponding to  $s_{n-3}s_{n-1}s_{n-2}s_n$  and  $s_{n-4}s_{n-3}s_{n-2}s_n$ . This implies that  $C_\lambda$  is wild by Lemma 7.7.

Consider the following diagram,



Then  $C_\lambda$  has an element in degree one corresponding to  $s_{n-2}$  whilst in degree two we have elements corresponding to  $s_n s_{n-2}$ ,  $s_{n-1} s_{n-2}$  and  $s_{n-3} s_{n-2}$ . Therefore  $C_\lambda$  is wild by Lemma 7.7.

7.16. Thanks to (11) the only remaining case is the following diagram,



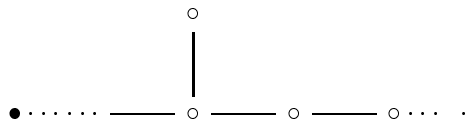
Let  $\sigma_i \in K[X_1, \dots, X_n]$  (respectively  $\sigma'_i \in K[X_2, \dots, X_n]$ ) be the  $i^{\text{th}}$  elementary symmetric polynomial in  $n$  (respectively  $n-1$ ) variables. Then the algebra  $C_\lambda$  is generated by the polynomials  $X_1, \sigma'_i(X_2^2, \dots, X_n^2)$  for  $1 \leq i \leq n-2$  and  $X_2 \dots X_n$  subject to the relations induced by  $\sigma_i(X_1^2, \dots, X_n^2) = 0$  for  $1 \leq i \leq n-1$  and  $X_1 \dots X_n = 0$ . This is a  $2n$  dimensional algebra. Let  $A = X_1$  and  $B = X_2 \dots X_n$ . The formula

$$\sigma_i(X_1^2, \dots, X_n^2) = X_1^2 \sigma'_{i-1}(X_2^2, \dots, X_n^2) + \sigma'_i(X_2^2, \dots, X_n^2),$$

shows by induction that  $A$  and  $B$  generate this algebra. In fact we find that  $\sigma'_i = (-A^2)^i$  for  $1 \leq i \leq n-1$ . It follows that  $B^2 = (-A^2)^{n-1}$  and it is clear that  $AB = 0$ . Thus  $C_\lambda$  is a tame algebra of Type 7.5.1.

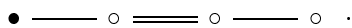


7.17. **Type E.** We assume that the rank of  $W(\lambda)$  is one less than the rank of  $W$ . Consider the following diagram,



Arguing as in the previous subsections we find that  $C_\lambda$  has a Poincaré series of the form described in Lemma 7.7, and is therefore wild. Other possibilities for diagrams are dealt with in (11), (13) and (14).

7.18. **Type  $F_4$ .** Suppose the rank of  $W(\lambda)$  is three. Consider the following diagram



Then in degrees one, two and three respectively  $C_\lambda$  has elements corresponding to  $s_1$ ,  $s_2s_1$  and  $s_3s_2s_1$  respectively, whilst in degree four we have elements corresponding to  $s_2s_3s_2s_1$  and  $s_4s_3s_2s_1$ . In degree five  $C_\lambda$  has elements corresponding to  $s_1s_2s_3s_2s_1$  and  $s_2s_4s_3s_2s_1$ . It follows from Lemma 7.7 that  $C_\lambda$  is wild. The case (12) finishes type  $F_4$ , and therefore the proof of Theorem 7.3.

## 8. COMPLEXITY

8.1. The purpose of this section is to find a necessary condition on  $\chi \in \mathfrak{g}^*$  and  $\lambda \in \Lambda/W$  for  $\mathcal{B}_{\chi,\lambda}$  to be tame, and to prove Theorems 5.2 and 5.4 (we continue assuming that  $\chi$  vanishes on  $\mathfrak{b}^+$ ). Our principal tools will be the results of Sections 3 and 4.

8.2. We assume until 8.14 that  $G = \tilde{G}$  and that  $G^{(1)}$  is simple. In the next lemma we do not assume that the parabolic subgroup  $W(\lambda)$  is standard in  $W$ .

**Lemma.** *Let  $\lambda \in \Lambda/W$ .*

1. Suppose that  $W(\lambda)$  does not contain simple reflections  $s_{i(1)}, \dots, s_{i(d)}$  such that  $i(j)$  and  $i(k)$  are not adjacent on the Dynkin diagram of  $\Delta$  for  $j \neq k$ . Then  $G.(e_{\alpha_{i(1)}} + \dots + e_{\alpha_{i(d)}}) \subseteq \mathcal{V}_{\mathfrak{g}}(0, \lambda)$ .
2. If  $W(\lambda) \neq W$  then  $G.e_{\alpha} \subseteq \mathcal{V}_{\mathfrak{g}}(0, \lambda)$  for some short root  $\alpha \in \Delta$ .

*Proof.* 1. There exists  $\nu \in X(T)$  such that  $0 < \langle \nu, \alpha_i^\vee \rangle \leq p$  for all  $i$  and  $d\nu = \lambda$ . Let  $V$  denote the Weyl module for  $G$  with highest weight  $\nu - \rho$ . Differentiating the action of  $G$  on  $V$  gives  $V$  a natural  $\mathcal{B}_{0,\lambda}$ -module structure. Let  $G_0$  denote the subgroup of  $G$  generated by all  $U_{\pm\alpha_{i(k)}}$ , and  $\mathfrak{g}_0 = \text{Lie}(G_0)$ . Obviously  $e := e_{\alpha_{i(1)}} + \dots + e_{\alpha_{i(d)}} \in \mathfrak{g}_0$ . Let  $V_0$  denote the subspace of  $V$  spanned by all weight spaces  $V_\mu$  with  $\nu - \rho - \mu \in \mathbb{Z}\alpha_{i(1)} \oplus \dots \oplus \mathbb{Z}\alpha_{i(d)}$ . By construction  $V_0$  is a direct summand of the  $\mathfrak{g}_0$ -module  $V$ . It is well-known (and easy to see) that  $\dim V_0 = \prod_k \nu(h_{\alpha_{i(k)}})$ . Since  $s_{i(k)} \notin W(\lambda)$  we have that  $\nu(h_{\alpha_{i(k)}}) \leq p - 1$  for all  $k$ . Therefore  $p \nmid \dim V_0$ . In particular  $V_0$  is not a free  $\langle e \rangle$ -module. Since all  $e_{\alpha_{i(k)}}$  commute we also have that  $e \in \mathcal{N}_p(\mathfrak{g})$ . Using our discussion in 4.1 we now deduce that  $e \in \mathcal{V}_{\mathfrak{g}}(0, \lambda)$ .

2. Since the short roots in  $\Phi$  span the root lattice  $Q$  and  $p$  is a good prime for  $G$  there is a short root  $\beta \in \Phi$  such that  $\langle \lambda, \beta^\vee \rangle \not\equiv 0(p)$ , for otherwise  $W(\lambda) = W$  contradicting our assumption. Since  $G^{(1)}$  is simple all short roots in  $\Phi$  are conjugate under  $W$ . Replacing  $\lambda$  by its  $W$ -conjugate we can assume that there exists a short root  $\alpha \in \Delta$  such that  $\langle \lambda, \alpha^\vee \rangle \not\equiv 0(p)$ . Then  $s_\alpha \notin W(\lambda)$ . Applying the first part of this lemma (with  $d = 1$ ) we now deduce that  $e_\alpha \in \mathcal{V}_{\mathfrak{g}}(0, \lambda)$ . Since this set is  $G$ -invariant the result follows.  $\square$

8.3. In the following subsections we give lower bounds for  $\dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda)$  in a number of important cases. We let  $e = \theta^{-1}(\chi) \in \mathcal{N}$  be the element of  $\mathfrak{g}$  corresponding to  $\chi$ . Recall that by 2.2 and Theorem 4.2  $\dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = \dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e)$ .

Given a subset  $J \subseteq \Delta$  we denote by  $\mathfrak{u}_J$  the Lie algebra of the unipotent radical of the standard parabolic of type  $J$  in  $G$ .

8.4. **Type  $A_n$ .** For  $\alpha = \alpha_i + \dots + \alpha_j \in \Phi$  we choose as a root vector  $e_\alpha$  the matrix  $E_{i,j+1}$  whose  $(i, j)$ <sup>th</sup> entry equals 1 and all other entries are zero and set  $e_{-\alpha} = E_{i,j+1}^T$ . Assume that  $W(\lambda) \neq W$ . First we consider the subregular case.

Assume  $n \geq 2$ . Let  $e = e_{\alpha_1} + \dots + e_{\alpha_{n-1}}$  and  $J = \{2, \dots, n\}$ . The orbit  $G.e_{\alpha_1}$  is minimal in  $\mathcal{N}$  and so is the Richardson class corresponding to  $\mathfrak{u}_J$ , that is it intersects densely with  $\mathfrak{u}_J$ . Thus  $\overline{G.e_{\alpha_1}} = G.\mathfrak{u}_J$ . We have

$$Ke_{\alpha_1+\dots+\alpha_{n-1}} + Ke_{\alpha_1+\dots+\alpha_n} \subseteq G.\mathfrak{u}_J \cap \mathfrak{z}_{\mathfrak{g}}(e),$$

so it follows that  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 2$ .

Now assume  $n = 1$ . Then zero is the subregular orbit and the closure of the minimal orbit is the nilpotent cone. It is immediate that  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) = 2$  in this situation.

8.5. The closure ordering of conjugacy classes in type  $A_n$  is given by the dominance ordering on partitions of  $n$ , [18, 7.19]. It can be checked that for  $n \geq 2$  there is a unique maximal non-regular, non-subregular class. If  $n \geq 3$  this corresponds to the partition  $(n-1, 2)$ . Let  $e = e_{\alpha_1} + \dots + e_{\alpha_{n-2}} + e_{\alpha_n}$  be a representative of this orbit. The element  $e$  is centralised by the set

$$Ke_{\alpha_1+\dots+\alpha_{n-2}} + Ke_{\alpha_1+\dots+\alpha_n} + Ke_{-\alpha_{n-1}} + Ke_{\alpha_n} \subset \mathcal{N}_p.$$

The orbit  $G.e_{\alpha_1}$  is characterised as those matrices of rank one whose square is zero. We deduce that the set

$$\{x_1e_{\alpha_1+\dots+\alpha_{n-2}} + x_2e_{\alpha_1+\dots+\alpha_n} + x_3e_{-\alpha_{n-1}} + x_4e_{\alpha_n} : x_1x_4 - x_2x_3 = 0\}$$

is contained in  $\overline{G.e_{\alpha_1}} \cap \mathfrak{z}_{\mathfrak{g}}(e)$ . Therefore  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$ .

Now assume  $n = 2$ . We have to consider the zero orbit. We have  $\overline{G.e_{\alpha_2}} = G.\mathfrak{u}_{\{2\}}$ . Since any element in  $\mathfrak{u}_{\{2\}}$  has square zero we deduce that  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(0) \geq \dim G.\mathfrak{u}_{\{2\}} = 4$ .

8.6. Suppose  $n = 3$  and  $W(\lambda) = \Sigma_2 \times \Sigma_2$ . Adopt the notation of [22, Proposition 2.6]. It is easy to see that the subsystem  $R_\lambda$  of  $\Phi$  has type  $A_1 \times A_1$  and the partition  ${}^t\pi(\lambda)$  of  $n+1 = 4$  introduced in [22, 2.6] equals  $(2, 2)$ . Combining [22, Proposition 2.6] with our discussion in 4.1 we derive that  $e_{\alpha_1} + e_{\alpha_3} \in \mathcal{V}_{\mathfrak{g}}(0, \lambda)$ . As a consequence,  $\mathcal{V}_{\mathfrak{g}}(0, \lambda)$  contains all matrices in  $\mathfrak{g}$  of rank two whose square is zero. Let  $e = e_{\alpha_1} + e_{\alpha_2}$ , a subregular nilpotent element in  $\mathfrak{g}$ . The element  $e$  is centralised by  $e_{\alpha_1} + e_{\alpha_2}$ ,  $e_{\alpha_1+\alpha_2}$ ,  $e_{\alpha_1+\alpha_2+\alpha_3}$  and  $e_{-\alpha_3}$ . The set

$$\{x_1(e_{\alpha_1} + e_{\alpha_2}) + x_2e_{\alpha_1+\alpha_2} + x_3e_{\alpha_1+\alpha_2+\alpha_3} + x_4e_{-\alpha_3} : x_1^2 + x_3x_4 = 0\}$$

is a three dimensional subvariety of  $\overline{G.(e_{\alpha_1} + e_{\alpha_3})} \cap \mathfrak{z}_{\mathfrak{g}}(e)$ . It follows that  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$ .

8.7. Suppose  $n = 2$  and that  $W(\lambda) = 1$ . Then  $\lambda$  is a regular weight. Hence  $p$  is greater than or equal to 3, the Coxeter number, and  $\mathcal{N}_p = \mathcal{N}$ . Moreover,

$\mathcal{V}_{\mathfrak{g}}(0, \lambda) = \mathcal{N}_p$ , by [22, Proposition 2.6]. Let  $e = e_{\alpha_1}$ , a subregular element. We have

$$Ke_{\alpha_1} + Ke_{\alpha_1+\alpha_2} + Ke_{-\alpha_2} \subseteq \mathcal{N} \cap \mathfrak{z}_{\mathfrak{g}}(e),$$

showing that  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$ .

**8.8. Type  $B_n$ .** We begin by identifying the orbit  $G.e_{\alpha_n}$ . Since  $\alpha_n$  is a short root the closure of  $G.e_{\alpha_n}$  contains the minimal nilpotent orbit strictly, so we deduce that  $\dim G.e_{\alpha_n} \geq 4(n-1) + 2$ . Let  $J = \{2, \dots, n\}$ . We have  $G.\mathfrak{u}_J \supseteq G.e_{\alpha_n}$ . Moreover, by [18, Theorem 5.3],  $\dim G.\mathfrak{u}_J = 2 \dim \mathfrak{u}_J = 2(n^2 - (n-1)^2) = 4n - 2$ . We deduce that  $\overline{G.e_{\alpha_n}} = G.\mathfrak{u}_J$ , the closure of the Richardson class of  $\mathfrak{g}$  corresponding to  $J$ . Since  $e_{\alpha_n}^{[p]} = 0$  we must have  $x^{[p]} = 0$  for all  $x \in \overline{G.e_{\alpha_n}}$ . In particular  $\mathfrak{u}_J^{[p]} = 0$ .

Now let  $e = e_{\alpha_2} + \dots + e_{\alpha_n}$ , a subregular element [25, 3.2]. We mention for a further reference that  $e$  is conjugate under  $N_G(T)$  to  $e_{\alpha_1} + \dots + e_{\alpha_{n-2}} + e_{\alpha_{n-1}+\alpha_n}$ . The centraliser of  $e$  includes the elements  $e_{-\alpha_1}$ ,  $e_{\alpha_1+2\alpha_2+\dots+2\alpha_n}$  and  $e_{\alpha_2+2\alpha_3+\dots+2\alpha_n}$  (if  $n = 2$  take  $e_{-\alpha_1}$ ,  $e_{\alpha_2}$  and  $e_{\alpha_1+2\alpha_2}$ ). Applying a preimage of  $s_1$  in  $N_G(T)$  to these elements yields non-zero multiples of  $e_{\alpha_1}$ ,  $e_{\alpha_1+2\alpha_2+\dots+2\alpha_n}$  and  $e_{\alpha_1+\alpha_2+2\alpha_3+\dots+2\alpha_n}$  (respectively  $e_{\alpha_1}$ ,  $e_{\alpha_1+\alpha_2}$  and  $e_{\alpha_1+2\alpha_2}$ ), all of which lie in  $\mathfrak{u}_J$ . Therefore  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$ .

**8.9. Type  $C_n$ .** We can assume that  $n \geq 3$ . Let us describe  $G.e_{\alpha_{n-1}}$ . To this end let

$$\beta = s_2 s_3 \dots s_{n-1} s_1 s_2 \dots s_{n-2} s_n(\alpha_{n-1}),$$

a short root. We have

$$\langle \alpha_i, \beta^\vee \rangle = \begin{cases} 1 & \text{if } i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

By [40, IV.2.2.8] the dimension of the orbit is independent of  $p$ . Using the grading on  $\mathfrak{g}$  induced by the above we therefore calculate that  $\dim G.e_\beta = 4n - 2$ . Let  $J = \{2, \dots, n\}$ . Then, by [18, Theorem 5.3],  $\dim G.\mathfrak{u}_J = 2 \dim \mathfrak{u}_J = 4n - 2$ . Since  $e_\beta \in \mathfrak{u}_J$  we deduce that  $\overline{G.e_\beta} = G.\mathfrak{u}_J$ , the closure of the Richardson class corresponding to  $J$ .

Let  $e = e_{\alpha_2} + \dots + e_{\alpha_n} + e_{2\alpha_1+\dots+2\alpha_{n-1}+\alpha_n}$ . It is easily seen that  $e$  is conjugate under  $N_G(T)$  to  $e_{\alpha_1} + \dots + e_{\alpha_{n-2}} + e_{\alpha_n} + e_{2\alpha_{n-1}+\alpha_n}$ , a Richardson element in  $\mathfrak{u}_{\{n-1\}}$ . It follows that  $e$  is a subregular nilpotent element of  $\mathfrak{g}$ . The

centraliser of  $e$  includes the elements  $e_{2\alpha_2+\dots+2\alpha_{n-1}+\alpha_n}$ ,  $e_{\alpha_1+2\alpha_2+\dots+2\alpha_{n-1}+\alpha_n}$  and  $e_{2\alpha_1+\dots+2\alpha_{n-1}+\alpha_n}$ . It is straightforward to check that the set

$$U_{-\alpha_1} \cdot (Ke_{\alpha_1+2\alpha_2+\dots+2\alpha_{n-1}+\alpha_n} + Ke_{2\alpha_1+\dots+2\alpha_{n-1}+\alpha_n}) \subseteq G \cdot \mathfrak{u}_J \cap \mathfrak{z}_{\mathfrak{g}}(e),$$

is three dimensional. We deduce that  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$ .

8.10. **Type  $D_n$ .** The present assumption on  $G$  implies that  $G = G^{(1)}$ . Assume that  $W(\lambda)$  is a standard parabolic subgroup of  $W$  of type  $D_{n-1}$ . Then  $\lambda = r\varpi_1$  for some  $r \in \mathbb{F}_p^*$ . Adopt Bourbaki's notation for the root system of type  $D_n$ . Since  $\varpi_1 = \epsilon_1$  we can find  $w \in W$  such that  $w\lambda = r\epsilon_{n-1}$ . Set  $\mu = r\epsilon_{n-1}$ . For  $k = n-1, n$  we have

$$\mu(h_{\alpha_k}) = \frac{2r(\epsilon_{n-1}|\epsilon_{n-1} \pm \epsilon_n)}{(\epsilon_{n-1} \pm \epsilon_n|\epsilon_{n-1} \pm \epsilon_n)} = r \neq 0.$$

Therefore we can assume in what follows that  $\lambda$  is such that  $s_{n-1}, s_n \notin W(\lambda)$ . Then, by Lemma 8.2,  $G \cdot (e_{\alpha_{n-1}} + e_{\alpha_n}) \subset \mathcal{V}_{\mathfrak{g}}(0, \lambda)$ . The present assumption on  $G$  implies that  $G = G^{(1)}$ .

8.11. Let  $\sigma$  denote the outer involution of the algebraic group  $G$  induced by the nontrivial symmetry of  $\Delta$ . It induces an automorphism of  $\mathfrak{g}$ , also denoted by  $\sigma$ , which swaps  $e_{\pm\alpha_{n-1}}$  and  $e_{\pm\alpha_n}$  and fixes  $e_{\pm\alpha_i}$  for  $i \leq n-2$ . Let  $G^0$  denote the connected component of the fixed point group  $G^\sigma$  and  $\mathfrak{g}^\sigma$  the fixed point subalgebra of  $\sigma$ . It is well-known (and not hard to see) that  $\mathfrak{g}^\sigma = \text{Lie}(G^0)$  has type  $B_{n-1}$  and is generated by  $e_{\pm\alpha_1}, \dots, e_{\pm\alpha_{n-2}}$  and  $e_{\pm\alpha_{n-1}} + e_{\pm\alpha_n}$ . Moreover, the elements  $e_{\alpha_i}$  with  $i \leq n-2$  together with  $e_{\alpha_{n-1}} + e_{\alpha_n}$  generate a maximal nilpotent subalgebra of  $\mathfrak{g}^\sigma$  and can be viewed as simple root vectors for  $\mathfrak{g}^\sigma$  with respect to  $\mathfrak{h}^\sigma = \mathfrak{g}^\sigma \cap \mathfrak{h}$ . Let  $\beta_1, \dots, \beta_{n-1}$  denote the corresponding simple roots, so that  $e_{\beta_i} = e_{\alpha_i}$  for  $i \leq n-2$  and  $e_{\beta_{n-1}} = e_{\alpha_{n-1}} + e_{\alpha_n}$ . Note that  $\beta_{n-1}$  is a *short* root of  $\mathfrak{g}^\sigma$ .

8.12. The element  $e = e_{\alpha_1} + \dots + e_{\alpha_{n-3}} + e_{\alpha_{n-3}+\alpha_{n-2}} + e_{\alpha_{n-3}+\alpha_{n-1}} + e_{\alpha_{n-3}+\alpha_n}$ , fixed by  $\sigma$ , is Richardson in  $\mathfrak{u}_{\{n-2\}}$  hence subregular in  $\mathfrak{g}$ . There exist root vectors in  $\mathfrak{g}^\sigma$  such that  $e = e_{\beta_1} + \dots + e_{\beta_{n-3}} + e_{\beta_{n-3}+\beta_{n-2}} + e_{\beta_{n-2}+\beta_{n-1}}$ . This is conjugate under  $U_{\beta_{n-2}} = U_{\alpha_{n-2}}$  to  $e_{\beta_1} + \dots + e_{\beta_{n-3}} + e_{\beta_{n-2}+\beta_{n-1}}$ , a subregular nilpotent element of  $\mathfrak{g}^\sigma$  (see 8.8). Since  $e_{\beta_{n-1}}$  is a short root vector in  $\mathfrak{g}^\sigma$  we have, by 8.8, that

$$\dim \overline{G^0 \cdot e_{\beta_{n-1}}} \cap \mathfrak{z}_{\mathfrak{g}^\sigma}(e) \geq 3.$$

Since  $\mathfrak{z}_{\mathfrak{g}^\sigma}(e) \subseteq \mathfrak{z}_{\mathfrak{g}}(e)$  and  $e_{\beta_{n-1}} = e_{\alpha_{n-1}} + e_{\alpha_n}$  our discussion in 8.10 yields  $\dim V_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$ .

**8.13. Type  $G_2$ .** The element  $e = e_{\alpha_2} + e_{\alpha_1+2\alpha_2}$  is conjugate under  $N_G(T)$  to  $e_{\alpha_1+\alpha_2} + e_{\alpha_1+2\alpha_2}$ , a Richardson element in  $\mathfrak{u}_{\{2\}}$ . Hence  $e$  is subregular in  $\mathfrak{g}$ . Then  $\mathfrak{z}_{\mathfrak{g}}(e)$  has a basis consisting of  $e_{-\alpha_1}$ ,  $e_{\alpha_1+3\alpha_2}$ ,  $e_{2\alpha_1+3\alpha_2}$  and  $e$ . It is easily seen that  $U_{\alpha_1+2\alpha_2}U_{-\alpha_1-\alpha_2}(Ke_{\alpha_2})$  is a dense subset of  $Ke_{\alpha_2} \oplus Ke_{-\alpha_1} \oplus Ke_{\alpha_1+3\alpha_2}$ . From this it is immediate that the Zariski closure,  $Y$ , of  $U_{\alpha_1+\alpha_2}U_{\alpha_1+2\alpha_2}U_{-\alpha_1-\alpha_2}(Ke_{\alpha_2})$  is an irreducible, conical hypersurface in

$$X := Ke_{\alpha_2} \oplus Ke_{-\alpha_1} \oplus Ke_{\alpha_1+2\alpha_2} \oplus Ke_{\alpha_1+3\alpha_2} \oplus Ke_{2\alpha_1+3\alpha_2} \cong \mathbb{A}^5.$$

Since  $\mathfrak{z}_{\mathfrak{g}}(e)$  is a hyperplane in  $X$  all irreducible components of  $Y \cap \mathfrak{z}_{\mathfrak{g}}(e)$  must be at least three dimensional. As a consequence,  $\dim \mathcal{V}_{\mathfrak{g}}(0, \lambda) \cap \mathfrak{z}_{\mathfrak{g}}(e) \geq 3$ .

**8.14.** Let us return to general  $G$  satisfying the hypotheses of 2.1. Recall the *complexity* of module over a finite dimensional algebra is the rate of growth of its minimal projective resolution. If we consider a  $U_{\chi}(\mathfrak{g})$ -module  $M$ , then the complexity of  $M$  equals  $\dim \mathcal{V}_{\mathfrak{g}}(M)$ , see [8, Proposition 3.2] and [10, Section 6]. Thus the following lemma provides the link between rank varieties and representation type.

**Lemma.** *Let  $\chi \in \mathcal{N}$  and  $\lambda \in \Lambda/W$ .*

1. *The algebra  $\mathcal{B}_{\chi, \lambda}$  is simple if and only if  $\mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = 0$ .*
2. [6, Theorem 3.2] *The algebra  $\mathcal{B}_{\chi, \lambda}$  has finite representation type if and only if  $\dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) \leq 1$ .*
3. [36, Theorem 2] *If  $\mathcal{B}_{\chi, \lambda}$  has tame representation type then  $\mathcal{V}_{\mathfrak{g}}(\chi, \lambda)$  is two dimensional.*

**8.15.** Thanks to Proposition 6.5, in order to prove the results of Section 5, we may assume that  $G = \tilde{G} = \tilde{G}_1 \times \cdots \times \tilde{G}_m$ ,  $\mathfrak{g} = \tilde{\mathfrak{g}}_1 \oplus \cdots \oplus \tilde{\mathfrak{g}}_m$ ,  $\chi = \chi_1 + \cdots + \chi_m$  and  $\lambda = (\lambda_1, \dots, \lambda_m)$ . In this situation

$$\mathcal{B}_{\chi, \lambda}(\mathfrak{g}) \cong \mathcal{B}_{\chi_1, \lambda_1}(\tilde{\mathfrak{g}}_1) \otimes \cdots \otimes \mathcal{B}_{\chi_m, \lambda_m}(\tilde{\mathfrak{g}}_m).$$

As in [34, p.242] we have  $\mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = \mathcal{V}_{\tilde{\mathfrak{g}}_1}(\chi_1, \lambda_1) \times \cdots \times \mathcal{V}_{\tilde{\mathfrak{g}}_m}(\chi_m, \lambda_m)$ . In particular

$$(15) \quad \dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = \sum_{j=1}^m \dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j, \lambda_j).$$

Let us remark that thanks to [11, Theorem 4.2] if  $W_i(\lambda_i) = W_i$  then  $\mathcal{B}_{\chi_i, \lambda}(\tilde{\mathfrak{g}}_i)$  is simple (this also follows from Theorem 4.2 combined with the irreducibility and projectivity of the restricted Steinberg module).

**8.16. Proof of Theorem 5.2.** It follows from Theorem 7.2 that the blocks considered in Theorem 5.2 have finite representation type.

Thanks to 8.14.2  $\mathcal{B}_{\chi, \lambda}$  has finite representation type if and only if  $\mathcal{V}_{\mathfrak{g}}(\chi, \lambda)$  has dimension at most 1. So by (15)  $\mathcal{B}_{\chi, \lambda}$  has finite representation type if and only if there exists  $j$ ,  $1 \leq j \leq m$  such that  $\mathcal{V}_{\tilde{\mathfrak{g}}_i}(\chi_i, \lambda_i) = 0$  for  $i \neq j$  and  $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j, \lambda_j) \leq 1$ .

Suppose  $\mathcal{V}_{\tilde{\mathfrak{g}}_i}(0, \lambda)$  is non-zero. Since it is a closed  $\tilde{G}_i$ -invariant variety it contains  $e_{\tilde{\alpha}}$ , where  $\tilde{\alpha}$  is the longest root of  $\tilde{\mathfrak{g}}_i$ . Since  $\chi(\mathfrak{n}^+ \cap \tilde{\mathfrak{g}}_i) = 0$  we see that  $e_{\tilde{\alpha}} \in \mathcal{V}_{\tilde{\mathfrak{g}}_i}(0, \lambda) \cap \mathfrak{z}_{\tilde{\mathfrak{g}}_i}(\chi_i)$ . We deduce that  $\mathcal{V}_{\tilde{\mathfrak{g}}_i}(0, \lambda) = 0$  for all  $i \neq j$ , implying  $W(\lambda_i) = W$  for all such  $i$  by Lemma 8.2.2. By Lemma 8.14.1 we have  $\mathcal{B}_{\chi_i, \lambda_i}(\tilde{\mathfrak{g}}_i)$  is a simple algebra for all such  $i$ .

Suppose  $\mathcal{B}_{\chi, \lambda}$  is of a type not considered in Theorem 5.2. By Theorem 7.2  $\chi_j$  is not regular. If  $\xi_j$  is regular nilpotent then  $\mathcal{B}_{\xi_j, \lambda_j}(\tilde{\mathfrak{g}}_j)$  has finite representation type by Corollary 3.3. Thus the pair  $(W_j, W_j(\lambda_j))$  must be of the type described in Theorem 7.2. Since  $\chi_j$  is not regular, however, the calculations of 8.4, 8.8, 8.9 and 8.13 show that  $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j, \lambda_j) \geq 2$ , a contradiction.

**8.17.** We give a necessary condition for tame representation type.

**Proposition.** *Let  $\chi \in \mathcal{N}$  and  $\lambda \in \Lambda/W$ . If  $\mathcal{B}_{\chi, \lambda}$  has tame representation type, then  $\chi$  and  $\lambda$  satisfy one of the conditions of Theorem 5.3.*

*Proof.* Suppose  $\mathcal{B}_{\chi, \lambda}$  is tame. By Lemma 8.14.3 we must have  $\dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) = 2$ . By (15) there are two cases to consider.

1. There exist  $j_1, j_2$ ,  $1 \leq j_1 < j_2 \leq m$ , such that  $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_i}(\chi_i, \lambda_i) = \delta_{ij_1} + \delta_{ij_2}$ .
2. There exists  $j$ ,  $1 \leq j \leq m$ , such that  $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_i}(\chi_i, \lambda_i) = 2\delta_{ij}$ .

In Case 1, arguing as in 8.16, the blocks  $\mathcal{B}_{\chi_i, \lambda_i}(\tilde{\mathfrak{g}}_i)$  are simple and  $W_i(\lambda_i) = W_i$  for  $i \neq j_1, j_2$ . By Lemma 8.14.2, Theorem 5.2 and Theorem 7.2 the block  $\mathcal{B}_{\chi_{j_k}, \lambda_{j_k}}(\tilde{\mathfrak{g}}_{j_k})$  is isomorphic to  $K[X]/(X^{r_k})$  where  $k = 1, 2$  and  $r_k = [W_{j_k} : W_{j_k}(\lambda)]$ . By 7.5 the tensor product of these algebras is tame if and only if  $[W_{j_1} : W_{j_1}(\lambda)] = [W_{j_2} : W_{j_2}(\lambda)] = 2$ . This is equivalent to Condition 3 in Theorem 5.3.

In Case 2 the blocks  $\mathcal{B}_{\chi_i, \lambda_i}(\tilde{\mathfrak{g}}_i)$  are simple and  $W(\lambda_i) = W_i$  for  $i \neq j$ . By Corollary 3.3 if  $\xi_j$  is regular then the block  $\mathcal{B}_{\xi_j, \lambda_j}(\tilde{\mathfrak{g}}_j)$  has either finite or tame representation type. Thus the pair  $(W_j, W_j(\lambda_j))$  belongs to one of the cases of Theorem 7.2 and Theorem 7.3. Suppose  $\chi_j$  is itself not regular. Then applying Corollary 3.3 once again we deduce that  $\mathcal{B}_{\zeta_j, \lambda_j}(\tilde{\mathfrak{g}}_j)$  has finite or tame representation type for subregular  $\zeta_j$ . In particular  $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\zeta_j, \lambda_j) \leq 2$ . The calculations in 8.6 - 8.13 show that  $W_j$  is of type of  $A_n$  and  $W_j(\lambda_j)$  is of type  $A_{n-1}$ . If  $\chi_j$  is not itself subregular then 8.5 shows that  $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j, \lambda_j) \geq 3$ , contradicting tameness.  $\square$

As a consequence it only remains to show that Condition 2 of Theorem 5.3 yields a tame block. This will be done in Section 9.

**8.18. Proof of Theorem 5.4.** We assume that Theorem 5.3 has been proved. It is proved in [35, Proposition 5.3] that  $U_\chi$  has finite representation type if and only if one of the conditions of Part 1 is satisfied. It is also a consequence of Theorem 5.2, see [29, 4.3].

We want to describe  $\chi$  for which all blocks  $\mathcal{B}_{\chi, \lambda}$  are tame. It is straightforward to check using Theorems 5.2 and 5.3 that all the algebras mentioned in Theorem 5.4.2 are tame. We need to discount all other possibilities. It follows from Theorems 5.2 and 5.3 that if  $\mathcal{B}_{\chi, \lambda}$  is tame then  $\chi$  must be regular or subregular.

Suppose first  $\chi_j$  is subregular and assume  $p \neq 2$ . Then  $\tilde{\mathfrak{g}}_j$  is of type  $A_n$  by Theorem 5.3.2 and  $\text{rank } W_j(\lambda_j) \geq \text{rank } W_j - 1$  for all  $\lambda \in \Lambda/W$ . If  $n \geq 3$  then the parabolic subgroup of  $W$  corresponding to any weight whose restriction to  $T \cap G^{(1)}$  equals  $\varpi_{n-1} + \varpi_n$  contains at least  $\text{rank } W_j - 2$  simple reflections. If  $p = 2$  and  $n \geq 3$  then the parabolic subgroup corresponding to 0 is reducible as a Coxeter group. But then  $U_\chi$  has a wild block by Theorem 5.3. The only case remaining (apart from  $A_1$ ) is  $A_2$ . Since the Coxeter number is 3, for  $p \geq 3$  there are  $\lambda$  such that  $W(\lambda) = 1$ , by [21,



II.6.2]. We deduce that  $\tilde{\mathfrak{g}}_j$  is of type  $A_1$  (any  $p$ ) or  $A_2$  ( $p = 2$ ). In both cases  $\dim \mathcal{V}_{\tilde{\mathfrak{g}}_j}(\chi_j, \lambda_j) = 2$  is achieved and so by (15)  $\mathcal{V}_{\tilde{\mathfrak{g}}_i}(\chi_i, \lambda_i) = 0$  for all  $\lambda_i$  and all  $i \neq j$ . This implies  $W_i(\lambda_i) = W_i$  for all  $i \neq j$  and all  $\lambda_i$ . Therefore  $W_i$  is trivial for all  $i \neq j$ . This yields cases 2(a) and 2(b).

Suppose that  $\chi_j$  is regular for  $1 \leq j \leq m$ . If  $\tilde{\mathfrak{g}}_j$  is of type  $A_n$  then the argument of the previous paragraph shows that either  $n \leq 2$  or  $n = 3$  and  $p = 2$  (and if  $n = 1$  then we have finite representation type). If  $\tilde{\mathfrak{g}}_j$  has type  $B_n$  or  $C_n$  then  $G_j = G_j^{(1)}$  and the parabolic subgroup of  $W$  corresponding to  $\varpi_n$  is of type  $A_{n-1}$ . Therefore  $n \leq 3$ . If  $n = 3$  the Coxeter number is 6 so for  $p \geq 7$  there exists  $\lambda$  such that  $W_j(\lambda_j) = 1$ . If  $p = 5$  then the weight  $\varpi_2 + \varpi_3$  provides a wild block, whilst if  $p = 3$  the weight  $\varpi_2$  yields a wild block. If  $\tilde{\mathfrak{g}}_j$  has type  $D_n$  then again  $G_j = G_j^{(1)}$  and the weight  $\varpi_n$  yields a parabolic subgroup of type  $A_{n-1}$ , and so yields a wild block unless  $n = 4$ . In case  $D_4$  the Coxeter number is 6 so we need to consider the cases  $p = 3$  and  $p = 5$  only. In both cases the weight  $\varpi_2$  yields a wild block. By Theorem 5.3 there are no tame blocks in types  $E$  and  $F_4$ . If  $G^{(1)}$  has more than two simple components then (15) shows that  $\dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) \geq 3$  for some  $\lambda$ , implying wildness. If there are two simple components then  $\dim \mathcal{V}_{\mathfrak{g}}(\chi, \lambda) \geq 3$  for some  $\lambda$  unless the components belong to the finite representation list of Theorem 5.4. It follows from 7.5 and Theorem 7.2 that both components must be of type  $A_1$ . The theorem follows.

## 9. THE SUBREGULAR TAME CASE

9.1. In this section we show that the blocks occurring in Theorem 5.3.2 have tame representation type. Thanks to Theorem 7.3 and Proposition 8.17 this completes the proof of Theorem 5.3. The proof relies heavily on the results of [25].

9.2. Let  $G = SL_{n+1}(K)$  if  $\text{char } K = p$  does not divide  $n + 1$  and  $G = GL_{n+1}(K)$  if  $\text{char } K = p$  divides  $n + 1$ . Let  $\mathfrak{g} = \text{Lie}(G)$ , so that  $\mathfrak{g} = \mathfrak{sl}_{n+1}(K)$ , respectively  $\mathfrak{g} = \mathfrak{gl}_{n+1}(K)$ . Let  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow K$  be the non-degenerate  $G$ -invariant bilinear form defined by  $B(x, y) = \text{tr}(xy)$  for  $x, y \in \mathfrak{g}$ . Choose root vectors  $e_\alpha$  as in 8.4 and let  $\chi \in \mathfrak{g}^*$  be the subregular nilpotent element defined by  $\chi(x) = B(x, e)$ , where  $e = e_{\alpha_1} + \dots + e_{\alpha_{n-1}}$ .

Let  $\lambda \in \Lambda/W$  be such that the group  $W(\lambda)$  is the standard parabolic subgroup of  $W$  generated by the simple reflections  $s_1, \dots, s_{n-1}$ .

9.3. We will be concerned with the category of finite dimensional  $U_\chi$ -modules,  $U_\chi$ -mod, or more specifically the subcategory of  $\mathcal{B}_{\chi,\lambda}$ -modules. These categories have graded analogues which we introduce now.

If  $G = SL_{n+1}(K)$  let

$$(16) \quad T_0 = \bigcap_{i=1}^{n-1} \ker(\alpha_i),$$

whilst if  $G = GL_{n+1}(K)$  let

$$(17) \quad T_0 = \{\tau E_{1,1} + \dots + \tau E_{n,n} + E_{n+1,n+1} : \tau \in K^*\}.$$

Thus, in either case,  $T_0$  is a one-parameter subgroup of the torus  $T \subset G$  such that  $\chi(\text{Ad}(t)x) = \chi(x)$  for all  $x \in \mathfrak{g}$  and  $t \in T_0$ . As a result the adjoint action of  $T_0$  on  $U$  passes to an action on the quotient  $U_\chi$ .

We define a  $U_\chi$ - $T_0$ -module to be a finite dimensional vector space  $V$  over  $K$  that has a structure both as a  $U_\chi$ -module and as a rational  $T_0$ -module such that the following compatibility conditions hold:

1. We have  $t(xv) = (\text{Ad}(t)x)tv$  for all  $x \in \mathfrak{g}, t \in T_0$  and  $v \in V$ ;
2. The restriction of the  $\mathfrak{g}$ -action on  $V$  to  $\text{Lie}(T_0) \subseteq \mathfrak{h}$  is equal to the derivative of the  $T_0$ -action on  $V$ .

We obtain the category,  $U_\chi$ - $T_0$ -mod, with objects the  $U_\chi$ - $T_0$ -modules and morphisms  $T_0$ -equivariant  $U_\chi$ -module homomorphisms. Since  $\mathcal{B}_{\chi,\lambda} = e_\lambda U_\chi$  for some  $G$ -invariant element  $e_\lambda$ , the full subcategory  $\mathcal{B}_{\chi,\lambda}$ - $T_0$ -mod of the category  $U_\chi$ - $T_0$ -mod is well-defined. Its objects are  $\mathcal{B}_{\chi,\lambda}$ -modules with a compatible rational  $T_0$ -action.

*Remark.* In [25] the case  $G = SL_{n+1}(K)$  where  $p$  does not divide  $n+1$  is considered and the category  $U_\chi$ - $T_0$ -mod is studied where  $T_0$  is as in (16). The results of [25, Section 2] continue to hold for  $G = GL_{n+1}(K)$  where  $p$  divides  $n+1$ , if we choose  $T_0$  as in (17). The proofs can be repeated almost verbatim. The only difference occurs in character formulae where appearances of  $(n+1)$  in [25] should be replaced by 1. This is due essentially to the following fact: if  $\phi$  is the cocharacter of  $T$  corresponding to  $T_0$  in (16)

(respectively (17)) then  $\langle \alpha_n, \phi \rangle = n + 1$  (respectively  $\langle \alpha_n, \phi \rangle = 1$ ). From now on we will use the results in [25] in both cases without further comment.

9.4. Let  $F : U_\chi\text{-}T_0\text{-mod} \longrightarrow U_\chi\text{-mod}$  denote the functor which forgets the  $T_0$ -structure. The objects of  $U_\chi\text{-mod}$  which are in the image of  $F$  are called *gradable*. Suppose  $M$  is gradable, that is there exists a  $U_\chi\text{-}T_0$ -module  $V$  such that  $F(V) = M$ . Then, by [24, Remark 1.5], we have  $F(\text{soc } V) = \text{soc } M$  and  $F(\text{rad } V) = \text{rad } M$ .

It follows from [16, Corollary 3.4] and [24, Corollary 1.4.1] that the simple  $U_\chi$ -modules and their projective covers are gradable. Moreover any lift of a simple  $U_\chi$ -module is simple in  $U_\chi\text{-}T_0\text{-mod}$  and any lift of a projective indecomposable  $U_\chi$ -module is projective indecomposable in  $U_\chi\text{-}T_0\text{-mod}$ .

9.5. The category  $U_\chi\text{-}T_0\text{-mod}$  has shift functors  $[\pm 1] : U_\chi\text{-}T_0\text{-mod} \longrightarrow U_\chi\text{-}T_0\text{-mod}$ . These send a given  $U_\chi\text{-}T_0$ -module  $V$  to the object having the same  $U_\chi$ -module structure but with  $t \in T_0$  acting by  $t.v = (\pm p\alpha_n)(t)tv$  for all  $v \in V$ . Given  $i \in \mathbb{N}$  we write  $[i]$  (respectively  $[-i]$ ) for the  $i$ -fold composition of  $[1]$  (respectively  $[-1]$ ).

Given  $U_\chi\text{-}T_0$ -modules  $V_0$  and  $V_1$  we have a natural isomorphism

$$(18) \quad \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{U_\chi\text{-}T_0}(V_0[i], V_1) \cong \text{Hom}_{U_\chi}(F(V_0), F(V_1)).$$

Moreover, if  $V_0 = V_1$  then the left hand side of (18) acquires an algebra structure through the identification

$$\text{Hom}_{U_\chi\text{-}T_0}(V_0[i+j], V_1[j]) \cong \text{Hom}_{U_\chi\text{-}T_0}(V_0[i], V_1),$$

for all  $i, j \in \mathbb{Z}$ . With this (18) becomes an isomorphism of algebras.

9.6. Thanks to [25, Theorem 2.6] there are exactly two non-isomorphic simple  $\mathcal{B}_{\chi, \lambda}$ -modules. We let  $S_0$  and  $S_1$  be lifts of these to  $\mathcal{B}_{\chi, \lambda}\text{-}T_0\text{-mod}$  and let  $P(S_0)$  and  $P(S_1)$  be lifts of their projective covers, projecting onto  $S_0$  and  $S_1$  respectively. By [16, Theorem 4.1] and [24, Theorem 1.4.2] we have that  $\{S_0[i], S_1[i] : i \in \mathbb{Z}\}$  is a complete set of representatives of mutually non-isomorphic simple  $\mathcal{B}_{\chi, \lambda}\text{-}T_0$ -modules. Then  $P(S_0)[i]$  (respectively  $P(S_1)[i]$ ) is the projective cover of  $S_0[i]$  (respectively  $S_1[i]$ ) in  $\mathcal{B}_{\chi, \lambda}\text{-}T_0\text{-mod}$ .

9.7. The category  $U_\chi\text{-}T_0\text{-mod}$  admits a contravariant self-equivalence,  $D$ , whose square is the identity functor. By [25, Proposition 2.16] we have for all  $i \in \mathbb{Z}$

$$(19) \quad D(S_0[i]) \cong S_0[i], \quad D(S_1[i]) \cong S_1[i].$$

Since  $U_\chi$  is a symmetric algebra, [39] and [10, 1.2],  $P(S_0)$  and  $P(S_1)$  are both projective and injective. So we deduce from (19) that

$$(20) \quad D(P(S_0)) \cong P(S_0), \quad D(P(S_1)) \cong P(S_1).$$

9.8. Besides the simples and their projective covers there are two other families of distinguished objects in  $\mathcal{B}_{\chi,\lambda}\text{-}T_0\text{-mod}$ . The first we denote by  $Z(S_0)$  and  $Z(S_1)$ . These are indecomposable  $\mathcal{B}_{\chi,\lambda}\text{-}T_0\text{-modules}$  whose underlying  $U_\chi\text{-modules}$   $F(Z(S_0))$  and  $F(Z(S_1))$  are the baby Verma modules in  $U_\chi\text{-mod}$  with heads  $F(S_0)$  and  $F(S_1)$  respectively. We have, by [25, Theorem 2.6], short exact sequences in  $\mathcal{B}_{\chi,\lambda}\text{-}T_0\text{-mod}$

$$(21) \quad 0 \longrightarrow S_1 \longrightarrow Z(S_0) \longrightarrow S_0 \longrightarrow 0,$$

and

$$(22) \quad 0 \longrightarrow S_0[-1] \longrightarrow Z(S_1) \longrightarrow S_1 \longrightarrow 0.$$

To define the second objects we need a little notation. Let  $\Phi' \subset \Phi$  be the root system generated by  $\alpha_1, \dots, \alpha_{n-1}$ . Let  $\mathfrak{p}$  be the parabolic subalgebra of  $\mathfrak{g}$  spanned by  $\mathfrak{b}^+$  and all  $\mathfrak{g}_\alpha$  for  $\alpha \in \Phi'$ . Then  $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$  where  $\mathfrak{u}$  is the unipotent radical of  $\mathfrak{p}$  and  $\mathfrak{l}$  is the Levi subalgebra of  $\mathfrak{g}$  which equals the direct sum of  $\mathfrak{h}$  and all  $\mathfrak{g}_\alpha$  for  $\alpha \in \Phi'$ . Since  $\chi(\mathfrak{u}) = 0$  any  $U_\chi(\mathfrak{l})\text{-module}$   $M$  can be extended to a  $U_\chi(\mathfrak{p})\text{-module}$  by letting  $\mathfrak{u}$  act trivially. Thus we obtain a  $U_\chi(\mathfrak{g})\text{-module}$  by *Harish-Chandra induction*,

$$\text{Ind}(M) = U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p})} M.$$

Let  $\mu \in X(T)$  and let  $Z_\chi^\mathfrak{l}(\mu)$  be the corresponding baby Verma module over  $\mathfrak{l}$ . Let  $Q_\chi^\mathfrak{l}(\mu)$  be the projective cover of  $Z_\chi^\mathfrak{l}(\mu)$  as a  $U_\chi(\mathfrak{l})\text{-module}$ . Then  $\text{Ind}(Z_\chi^\mathfrak{l}(\mu)) = Z_\chi(\mu)$  and we write  $\Delta_\chi(\mu)$  for  $\text{Ind}(Q_\chi^\mathfrak{l}(\mu))$ . We furnish these modules with a compatible  $T_0\text{-structure}$  by concentrating  $1 \otimes Z_\chi^\mathfrak{l}(\mu)$  (respectively  $1 \otimes Q_\chi^\mathfrak{l}(\mu)$ ) in the weight space given by the restriction of  $\mu$  to  $T_0$  and letting  $T_0$  act on  $U_\chi(\mathfrak{g})$  via the adjoint action. This is well-defined since

the  $T_0$ -action on  $U_\chi(l)$  is by construction trivial. For further details see [23, Sections 10 and 11] and [24, Section 2].

Using the above construction we see that for  $i = 1, 2$  there exists  $\mu_i \in X(T)$  such that  $Z(S_i) \cong \text{Ind}(Z_\chi^l(\mu_i))$  as a  $\mathcal{B}_{\chi,\lambda}\text{-}T_0$ -module. We define  $\Delta(S_i) := \Delta_\chi(\mu_i)$  as  $\mathcal{B}_{\chi,\lambda}\text{-}T_0$ -modules. By [23, 11.18] and [24, 2.9] we have  $\Delta(S_0) \cong Z(S_0)$ , whilst the module  $\Delta(S_1)$  has a filtration in  $\mathcal{B}_{\chi,\lambda}\text{-}T_0\text{-mod}$

$$(23) \quad 0 = \Delta_0(S_1) \subset \Delta_1(S_1) \subset \cdots \subset \Delta_n(S_1) = \Delta(S_1),$$

such that  $\Delta_i(S_1)/\Delta_{i-1}(S_1) \cong Z(S_1)$  for  $1 \leq i \leq n$ . By [23, Proposition 11.18] and [24, Proposition 2.9] we have short exact sequences

$$(24) \quad 0 \longrightarrow \Delta(S_1)[1] \longrightarrow P(S_0) \longrightarrow \Delta(S_0) \longrightarrow 0,$$

and

$$(25) \quad 0 \longrightarrow \Delta(S_0) \longrightarrow P(S_1) \longrightarrow \Delta(S_1) \longrightarrow 0.$$

In particular, both  $\Delta(S_0)$  and  $\Delta(S_1)$  are indecomposable with simple heads  $S_0$  and  $S_1$  respectively.

9.9. We are able to calculate some extension groups in  $U_\chi\text{-}T_0\text{-mod}$ .

**Lemma.** *Let  $V$  be a simple  $U_\chi\text{-}T_0$ -module. Then*

$$\text{Ext}_{U_\chi\text{-}T_0}^1(S_0, V) = \begin{cases} K & \text{if } V \cong S_1, S_1[1] \\ 0 & \text{otherwise.} \end{cases}$$

*Similarly, we have*

$$\text{Ext}_{U_\chi\text{-}T_0}^1(S_1, V) = \begin{cases} K & \text{if } V \cong S_0, S_0[-1] \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Thanks to [25, Proposition 2.19] it only remains to prove that  $S_0$  and  $S_1$  have no self-extensions.

We have an exact sequence induced from (24)

$$\text{Hom}_{U_\chi\text{-}T_0}(\Delta(S_1)[1], S_0) \longrightarrow \text{Ext}_{U_\chi\text{-}T_0}^1(\Delta(S_0), S_0) \longrightarrow \text{Ext}_{U_\chi\text{-}T_0}^1(P(S_0), S_0).$$

Since  $P(S_0)$  is injective and since the head of  $\Delta(S_1)[1]$  is  $S_1[1]$  we deduce that  $\text{Ext}_{U_\chi\text{-}T_0}^1(\Delta(S_0), S_0) = 0$ . We have an exact sequence from (21)

$$\text{Hom}_{U_\chi\text{-}T_0}(S_1, S_0) \longrightarrow \text{Ext}_{U_\chi\text{-}T_0}^1(S_0, S_0) \longrightarrow \text{Ext}_{U_\chi\text{-}T_0}^1(\Delta(S_0), S_0).$$

It follows that  $\text{Ext}_{U_\chi-T_0}^1(S_0, S_0) = 0$ .

To deal with  $S_1$  consider the following exact sequence induced from (25)

$$\text{Hom}_{U_\chi-T_0}(S_1, \Delta(S_1)) \longrightarrow \text{Ext}_{U_\chi-T_0}^1(S_1, \Delta(S_0)) \longrightarrow \text{Ext}_{U_\chi-T_0}^1(S_1, P(S_1)).$$

Since  $P(S_1)$  is projective and the socle of  $\Delta(S_1)$  is  $S_0[-1]$  it follows that  $\text{Ext}_{U_\chi-T_0}^1(S_1, \Delta(S_0)) = 0$ . The exact sequence induced from (21)

$$\text{Hom}_{U_\chi-T_0}(S_1, S_0) \longrightarrow \text{Ext}_{U_\chi-T_0}^1(S_1, S_1) \longrightarrow \text{Ext}_{U_\chi-T_0}^1(S_1, \Delta(S_0)).$$

now shows that  $\text{Ext}_{U_\chi-T_0}^1(S_1, S_1) = 0$ , as required.  $\square$

9.10. A module is called *uniserial* if it has a unique composition series. Given a  $U_\chi$ - $T_0$ -module  $V$  we define the radical series of  $V$  as follows: set  $\text{rad}^0 V = V$  and for  $i > 0$  let  $\text{rad}^i V = \text{rad}(\text{rad}^{i-1} V)$ .

**Lemma.** *Suppose  $V$  is a  $U_\chi$ - $T_0$ -module such that  $\text{rad}^i V / \text{rad}^{i+1} V$  is simple for all  $i$ . Then  $V$  is uniserial.*

*Proof.* It is sufficient to prove that any  $U_\chi$ - $T_0$ -submodule  $M$  of  $V$  equals  $\text{rad}^i V$  for some  $i \geq 0$ . We prove this by downward induction on the number of composition factors of  $M$ . Let  $M$  be a proper  $U_\chi$ - $T_0$ -submodule of  $V$  and let  $M' \supseteq M$  be such that  $M'/M$  is simple. By induction  $M' = \text{rad}^i V$  for some  $i \geq 0$  and so  $\text{rad}^{i+1} V = \text{rad}(M') \subseteq M$ . Therefore  $M'/M$  is a factor of  $\text{rad}^i V / \text{rad}^{i+1} V$ , a simple module, implying equality.  $\square$

9.11. The following lemma is the crucial result in this section.

**Lemma.** *The modules  $\Delta(S_0)$  and  $\Delta(S_1)$  are uniserial.*

*Proof.* The module  $\Delta(S_0)$  is trivially uniserial. We will show that the module  $\text{rad}^i \Delta(S_1) / \text{rad}^{i+1} \Delta(S_1)$  is simple in  $U_\chi$ - $T_0$ -mod for all  $i \geq 0$  and then apply Lemma 9.10. Suppose for a contradiction that  $i$  is minimal such that  $\text{rad}^i \Delta(S_1) / \text{rad}^{i+1} \Delta(S_1)$  is not simple. Since  $\Delta(S_1)$  has a simple head we have  $i \geq 1$ . There is a short exact sequence

$$(26) \quad 0 \longrightarrow \frac{\text{rad}^i \Delta(S_1)}{\text{rad}^{i+1} \Delta(S_1)} \longrightarrow \frac{\text{rad}^{i-1} \Delta(S_1)}{\text{rad}^{i+1} \Delta(S_1)} \longrightarrow \frac{\text{rad}^{i-1} \Delta(S_1)}{\text{rad}^i \Delta(S_1)} \longrightarrow 0.$$

By hypothesis the term on the right hand side of (26) is simple. The module  $\Delta(S_1)$  has only two isomorphism classes of composition factors,  $S_1$  and  $S_0[-1]$ . We assume that the right hand side of (26) is isomorphic to  $S_0[-1]$ . The other case is treated similarly.

The left hand side of (26) is semisimple so isomorphic to a direct sum of copies of  $S_0[-1]$  and  $S_1$ . By Lemma 9.9  $\text{Ext}_{U_{\chi-T_0}}^1(S_0[-1], S_0[-1]) = 0$ , so we deduce that this direct sum can contain no copies of  $S_0[-1]$ .

Let us write  $V = \text{rad}^{i-1} \Delta(S_1) / \text{rad}^{i+1} \Delta(S_1)$ , a module with head isomorphic to  $S_0[-1]$  and socle a number of copies of  $S_1$ . By [19, Proof of Theorem 9] we have that

$$(27) \quad [\text{rad } P(S_0)[-1] / \text{rad}^2 P(S_0)[-1] : S_1] = \dim \text{Ext}_{U_{\chi-T_0}}^1(S_0[-1], S_1).$$

By Lemma 9.9 the right hand side of (27) equals one. We have a commutative diagram

$$\begin{array}{ccc} P(S_0)[-1] & \longrightarrow & S_0[-1] \\ & \searrow f & \uparrow \\ & & V \end{array}$$

where  $f$  exists by the projectivity of  $P(S_0)[-1]$ . This induces a homomorphism

$$\tilde{f} : \frac{\text{rad } P(S_0)[-1]}{\text{rad}^2 P(S_0)[-1]} \longrightarrow \text{rad } V.$$

If  $\tilde{f}$  is not surjective then we have  $V / \text{im } \tilde{f} \neq 0$  and so there is a copy  $S_1$  in the head of  $V$ , a contradiction. Therefore  $\tilde{f}$  is surjective and so  $\text{rad } V = S_1$  as required.  $\square$

9.12. Let  $A$  be a finite dimensional  $K$ -algebra. Given any  $A$ -module  $M$  with simple head we define the *heart of  $M$*  to be

$$H(M) = \frac{\text{rad } M}{\text{soc } M}.$$

The algebra  $A$  is called *biserial* if every nonuniserial projective indecomposable left or right  $A$ -module  $P$  contains two uniserial submodules whose sum is the unique maximal submodule of  $P$  and whose intersection is either zero or simple. It is easily seen that if  $A$  is symmetric and the heart of any projective indecomposable  $A$ -module is a direct sum of two uniserial modules then  $A$  is biserial. According to [4] any biserial algebra  $A$  is tame.

**Theorem.** *Let  $\chi \in \mathfrak{g}^*$  and  $\lambda \in \Lambda/W$  be as in 9.2. Then  $\mathcal{B}_{\chi, \lambda}$  is a biserial algebra. In particular  $\mathcal{B}_{\chi, \lambda}$  is tame.*

*Proof.* By 9.4 the radical and socle of a  $\mathcal{B}_{\chi,\lambda}$ - $T_0$ -module  $V$  agrees with the radical and socle of  $F(V)$ , so it is enough to prove that the heart of the projective indecomposables in  $\mathcal{B}_{\chi,\lambda}$ - $T_0$ -mod are direct sums of two uniserial modules.

We claim that

$$H(P(S_0)) \cong \frac{\Delta(S_1)[1]}{S_0} \oplus S_1,$$

and

$$H(P(S_1)) \cong \text{rad } \Delta(S_1) \oplus S_0.$$

The uniseriality of these summands follows from Lemma 9.11.

Thanks to (24) we have a short exact sequence

$$0 \longrightarrow \frac{\Delta(S_1)[1]}{S_0} \longrightarrow H(P(S_0)) \longrightarrow S_1 \longrightarrow 0.$$

Using (24), (19), (20) and (21) we also have a composition

$$0 \longrightarrow S_1 \cong D(S_1) \cong \frac{D(\Delta(S_0))}{D(S_0)} \longrightarrow \frac{D(P(S_0))}{D(S_0)} \cong \frac{P(S_0)}{S_0},$$

showing that there is a copy of  $S_1$  in the head and the socle of  $H(P(S_0))$ . By (23) and (24)  $S_1 = S_1[0]$  occurs only once as a composition factor of  $P(S_0)$  in  $\mathcal{B}_{\chi,\lambda}$ - $T_0$ -mod. So we deduce that  $S_1$  is a direct summand of  $H(P(S_0))$ , proving the first half of the claim.

For the second half of the claim we have, thanks to (25) and Lemma 9.11, a short exact sequence

$$0 \longrightarrow S_0 \longrightarrow H(P(S_1)) \longrightarrow \text{rad } \Delta(S_1) \longrightarrow 0.$$

By (25), (19) and (20) we have a composition of maps

$$\text{rad } P(S_1) \cong \text{rad } D(P(S_1)) \longrightarrow D(S_0) \cong S_0 \longrightarrow 0.$$

As above we deduce that  $S_0$  is a direct summand of  $H(P(S_1))$ , as required.  $\square$

**9.13. Quiver and relations for  $\mathcal{B}_{\chi,\lambda}$ .** Using the results of the previous subsections we will determine the quiver and relations for the algebra  $\mathcal{B}_{\chi,\lambda}$ . The basic algebra Morita equivalent to  $\mathcal{B}_{\chi,\lambda}$  is simply  $\text{End}_{\mathcal{B}_{\chi,\lambda}}(F(P(S_0)) \oplus F(P(S_1)))$ . To ease notation set  $P_0 = P(S_0)$  and  $P_1 = P(S_1)$ .



By (18) we have an algebra isomorphism

$$(28) \quad \text{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1)) \cong \bigoplus_{i \in \mathbb{Z}} \text{End}_{\mathcal{B}_{\chi,\lambda}\text{-}T_0}(P_0[i] \oplus P_1[i], P_0 \oplus P_1).$$

We construct four elements in the right hand side of (28), show that these generate the basic algebra and then describe the relations between these elements.

Let  $A$  be the unique (up to scalars) map in  $\mathcal{B}_{\chi,\lambda}\text{-}T_0\text{-mod}$  of degree zero, defined as the composition  $P_0 \twoheadrightarrow \Delta(S_0) \hookrightarrow P_1$ . Similarly, let  $\tilde{B} : P_1[1] \twoheadrightarrow \Delta(S_1)[1] \hookrightarrow P_0$  be the unique map of degree 1.

By the proof of Theorem 9.12  $\text{rad } P_1$  has head isomorphic to  $S_0 \oplus S_0[-1]$ . So we have

$$\begin{array}{ccc} P_0[-1] & \longrightarrow & S_0[-1] \\ & \searrow f & \uparrow \\ & & \text{rad } P_1 \end{array}$$

Composing  $f$  with the inclusion  $\text{rad } P_1 \subset P_1$  we obtain  $\tilde{A} : P_0[-1] \twoheadrightarrow P_1$ , a map of degree  $-1$ . Similarly, using the proof of Theorem 9.12, we define  $B$  as

$$\begin{array}{ccccccc} P_1 & \longrightarrow & S_1 & \longrightarrow & H(P_0) & \longrightarrow & P_0/S_0 \\ & & & & & & \uparrow \\ & & & & & & P_0 \end{array}$$

$\searrow B$

By construction  $B$  is a map of degree zero.

**Lemma.** *The elements  $A, \tilde{A}, B$  and  $\tilde{B}$  generate  $\text{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1))$  as an algebra.*

*Proof.* Consider the map induced by  $\tilde{A}$ ,

$$\bar{A} : \frac{P_0[-1]}{\bar{A}^{-1}(S_1)} \longrightarrow \frac{P_1}{S_1}.$$

Using Theorem 9.12 we see the image of  $\bar{A}$  is  $\text{rad } \Delta(S_1)$ . Since  $S_1 = \text{soc } P_1 \subseteq \ker \tilde{B}$  we deduce that

$$(29) \quad \text{im}(\tilde{B} \circ \bar{A}) = \text{im}(\tilde{B} \circ \bar{A}) = \text{rad } \Delta(S_1)[1].$$

Write  $X = \tilde{B}\bar{A}$ . By (29)  $X$  induces an endomorphism of  $\Delta(S_1)[1] \subset P_0$  which sends the head of  $\Delta(S_1)[1]$  onto  $\text{rad}^2 \Delta(S_1)[1]$ . Since  $\Delta(S_1)[1]$  is

uniserial it follows that  $X$  sends  $\text{rad}^i \Delta(S_1)[1]$  to  $\text{rad}^{i+2} \Delta(S_1)[1]$ . Hence the image of  $X^n : P_0 \longrightarrow P_0$  equals  $S_0 = \text{soc } P_0$ .

Let  $Y = \tilde{A}\tilde{B}$ . We have

$$\tilde{B} \circ Y^{n-1} : P_1 \xrightarrow{\tilde{B}} P_0[-1] \xrightarrow{(X[-1])^{n-1}} P_0[-1].$$

It follows from the definition of  $\tilde{B}$  and the previous paragraph that the image of  $\tilde{B} \circ Y^{n-1}$  equals  $\text{rad}^{2n-2} \Delta(S_1)$ , a module with composition factors  $S_0[-1]$  and  $S_0$  by Lemma 9.11. Using (21), (22), (23), (24) and (25) we see that the kernel of  $\tilde{A}$  has composition factors  $S_0[-1]$  and  $S_1[-1]$ . It follows that  $Y^n$  has image  $S_1 = \text{soc } P_1$ .

It is clear that  $\text{im}(B \circ A) = \text{soc } P_0$  and  $\text{im}(A \circ B) = \text{soc } P_1$ . Thus, possibly after rescaling,  $X^n = B \circ A$  and  $Y^n = A \circ B$ .

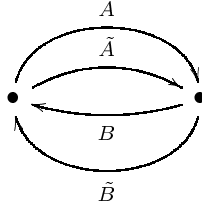
Let  $\tilde{A}_i = Y^i \circ \tilde{A}$  and  $\tilde{B}_i = X^i \circ \tilde{B}$ . The elements  $X^i, Y^i, \tilde{A}_i$  and  $\tilde{B}_i$  are linearly independent in  $\text{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1))$  for  $0 \leq i \leq n$ . On the other hand, by (21), (22), (23), (24) and (25), we have  $[F(P_0) : F(S_0)] = [F(P_0) : F(S_1)] = [F(P_1) : F(S_0)] = [F(P_1) : F(S_1)] = n + 1$ . We deduce, by [1, Lemma 1.7.6], that

$$\dim \text{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1)) = \sum_{i,j=0}^n [F(P_i) : F(S_j)] = 4(n+1),$$

as required.  $\square$

9.14. It is now straightforward to obtain the quiver and relations.

**Theorem.** *Let  $\chi$  be subregular and  $\lambda \in \Lambda/W$  be such that  $W(\lambda)$  has type  $A_{n-1}$ . Then  $\mathcal{B}_{\chi,\lambda}$  is Morita equivalent to the quiver*



with relations  $A\tilde{B} = \tilde{B}A = \tilde{A}B = B\tilde{A} = 0$ ,  $AB = (\tilde{A}\tilde{B})^n$  and  $BA = (\tilde{B}\tilde{A})^n$ .

*Proof.* We have shown in Lemma 9.13 that  $A, \tilde{A}, B$  and  $\tilde{B}$  generate the  $4(n+1)$ -dimensional algebra  $\text{End}_{\mathcal{B}_{\chi,\lambda}}(F(P_0) \oplus F(P_1))$ . The first set of relations holds thanks to the grading on  $P_0$  and  $P_1$ : the module  $P_0$  (respectively  $P_1$ ) has no composition factors isomorphic to  $S_0[i]$  (respectively  $S_1[i]$ ) for

$i$  non-zero. The second set of relations already appeared in the proof of Lemma 9.13. Therefore the endomorphism ring is a quotient of the above path algebra. The path algebra has a basis consisting of  $A$ ,  $B$ ,  $e_1$ ,  $e_2$  and alternating products of  $\tilde{A}$  and  $\tilde{B}$  of length no more than  $2n$ . Consequently the algebras are isomorphic.  $\square$

*Remark.* In the case  $n = 1$  there is a unique subregular nilpotent element,  $\chi = 0$ , and  $\lambda$  is a regular weight. The above description of the basic algebra of  $\mathcal{B}_{0,\lambda}$  was given in [7].

*Remark.* It follows from Theorem 9.14 that the basic algebra of  $\mathcal{B}_{\chi,\lambda}$  is *special* biserial (see [5, Definition II.1.1]). All indecomposable representations of such algebras are classified, [5, II.3]. The classification is based on [14] and [38]. It would be interesting to pull the information provided by [5, II.3] back to  $\mathfrak{g}$  to obtain an *explicit* description of all indecomposable representations of the tame subregular blocks.

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