SURGERY ON SMALL VOLUME HYPERBOLIC 3-ORBIFOLDS

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ABSTRACT

We consider closed hyperbolic 3-orbifolds obtained by surgery on the smallest hyperbolic 3-orbifolds with a non-rigid cusp, and coverings of these orbifolds by hyperbolic 3-manifolds obtained by surgery on links. We relate surgeries on the orbifolds to surgeries on the links. Applying the program SnapPea to the links the volumes of these hyperbolic 3-orbifolds can be computed, among them various of the hyperbolic orbifolds of smallest known volumes.

Keywords: Hyperbolic 3-orbifolds of small volume, surgery on 3-orbifolds and links, Picard orbifold, Borromean rings.

Introduction

Computations of volumes of hyperbolic 3-orbifolds whose singular sets are links in the 3-sphere, or which are obtained by surgeries on links, are possible due to J. Weeks' computer program *SnapPea*. If the singular set of a 3-orbifold is a graph which is not a link, there is no such general procedure and the computation of volumes is more difficult, in general, and has to be done case by case. The small volume hyperbolic 3-manifolds have been studied extensively. The smallest known manifold $\mathcal{M}_1$, of volume 0.942707, was found independently by Fomenko and Matveev, and by Weeks, and the ten smallest volume manifolds are discussed in [HW] and [MV1] and are obtained by surgery on the hyperbolic knots and links of smallest known volumes. Thus it seems to be useful to study also the closed 3-orbifolds obtained by surgery on the smallest cusped hyperbolic 3-orbifolds.

In the present paper we are interested in closed hyperbolic 3-orbifolds obtained by surgery on the smallest cusped hyperbolic 3-orbifolds, and in coverings of these orbifolds by hyperbolic 3-manifolds obtained by surgery on links. We relate surgeries on the orbifolds to surgeries on the links. Applying the program SnapPea to the links the volumes of these small hyperbolic 3-orbifolds can be computed. We note

that any closed hyperbolic 3-orbifold can be obtained by surgery on a hyperbolic orbifold with a *non-rigid* cusp (i.e. a cusp on which surgery, or Dehn filling, can be performed).
As an example, the smallest orbifold with a non-rigid cusp is the *Picard orbifold* (the quotient of the hyperbolic 3-space by the Picard group) which is covered by the complement of the Borromean rings, and the orbifolds obtained by surgery on the Picard orbifold are covered by manifolds (resp. cone-manifolds, in general) obtained by suitable surgeries on the Borromean rings.

We shall consider only orientable 3-orbifolds; our main references for the theory of orbifolds are [Th] and [DM].

As in the case of hyperbolic 3-manifolds, the volumes of hyperbolic 3-orbifolds form a well-ordered non-discrete subset of $\mathbb{R}$ of order type ω^ω, and each volume occurs only for finitely many orbifolds. In particular, there is a hyperbolic 3-orbifold of smallest volume (which is still not known), and also of smallest limit volume. The singular set of the smallest known hyperbolic 3-orbifold is shown in Figure 0.1 (the underlying space is the 3-sphere, labels 2 on edges are omitted), its volume is (approximately) 0.039050.

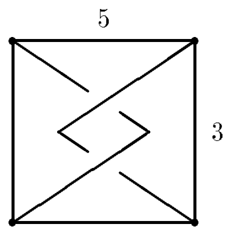

Figure 0.1. The smallest volume orbifold.

By [A], the unique smallest hyperbolic 3-orbifold with a non-rigid cusp is the *Picard orbifold*, the quotient of hyperbolic 3-space by the Picard group $PSL(2,\mathbb{Z}[i])$), of volume 0.305321 which is thus the smallest limit volume. By Thurston's hyperbolic surgery theorem, the volume of a cusped orbifold is the upper limit of the volumes of the orbifolds obtained by surgery on its cusp. As all hyperbolic orbifolds whose volumes are bounded by some constant are obtained by surgery on one of finitely many cusped orbifolds ([DM]), it follows that all but finitely many hyperbolic 3-orbifolds whose volumes are smaller than that of the Picard orbifold are in fact obtained by surgery on the Picard orbifold.

The smallest cusped hyperbolic 3-orbifolds were found by Meyerhoff and Adams: they all have exactly one rigid cup, so their volumes are not limit volumes (see [NR] for various descriptions of these orbifolds).

The three smallest hyperbolic 3-orbifolds with a non-rigid cusp were found by Adams [A], their volumes are 0.305321, 0.444451 and 0.457982; we will call these orbifolds the *Adams orbifolds* and denote them by $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$; in particular, $\mathcal{A}_1$ is the Picard orbifold $\mathbb{H}^3/PSL(2,\mathbb{Z}[i])$. The underlying space of $\mathcal{A}_1$ is the 3-sphere minus one point $S^3 \setminus \{\infty\}$, its singular set is shown in Figure 0.2. We denote by $\mathcal{A}_1(p,q)$ the orbifold shown in Figure 0.2 obtained by (p,q)-surgery on the cusp of $\mathcal{A}_1$ (see section 1 for definitions).

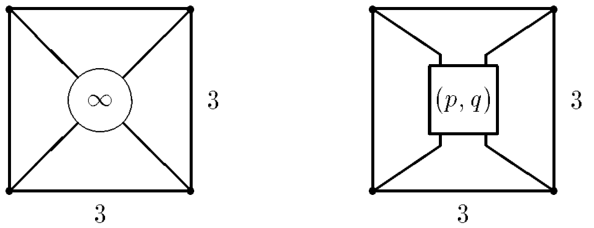

Figure 0.2. The orbifolds $\mathcal{A}_1$ and $\mathcal{A}_1(p,q)$.

It is well-known (see [A], [BFLW] or [H1]) that $\mathcal{A}_1$ is 24-fold covered by the complement of the *Borromean rings* which is the 3-component link 6^3_2 in Rolfsen's notations [R]. In section 2 we will show that the manifold $6^3_2(p,q)$ (or orbifold, or cone-manifold) obtained by (p,q)-surgery on all three components of the Borromean rings is a regular 24-fold covering of the cone-orbifold $\mathcal{A}_1(p-2q, p+2q)$ obtained by $(p-2q, p+2q)$-surgery on the cusp of the Picard orbifold. Using SnapPea (see e.g. [AHW]) this allows to compute the volumes of the orbifolds obtained by surgery on the Picard orbifold.

The second Adams orbifold can be obtained as the quotient $\mathcal{A}_2 = \mathbb{H}^3/PGL(2, O_7)$, where for a positive square free integer d we denote by O_d the ring of integers of the field $\mathbb{Q}(\sqrt{-d})$. In particular, also $\mathcal{A}_2$ is arithmetic. The underlying space of $\mathcal{A}_2$ is $S^3 \setminus \{\infty\}$, and its singular set is shown in Figure 0.3 (see [FF] where a picture of $\mathbb{H}^3/PSL(2, O_7)$ is presented, or [H2]).

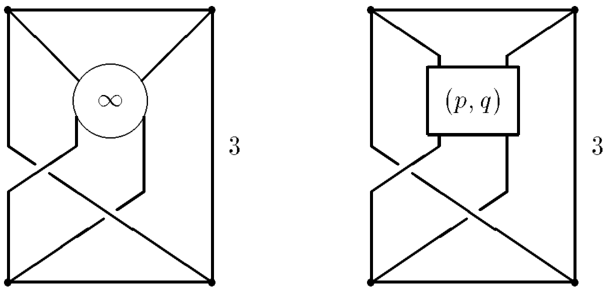

Figure 0.3. The orbifolds $\mathcal{A}_2$ and $\mathcal{A}_2(p,q)$.

It is remarked by Adams [A] that $\mathcal{A}_2$ is 12-fold covered by the complement of the 3-component link 6^3_1 in S^3. Some link complements commensurable with $\mathcal{A}_2$ are given in [GH] and [H1]. In section 3, we shall obtain the exact correspondence between surgery parameters for the orbifold $\mathcal{A}_2$ and the link 6^3_1. Finally, the orbifold $\mathcal{A}_3$ is 16-fold covered by the complement of the 4-component link 8^4_2. and the exact correspondence between surgeries on $\mathcal{A}_3$ and on 8^4_2 will be studied in section 4.

1. Preliminaries

For rational numbers p and q, there is a unique presentation $(p,q) = d(p',q')$ where p' and q' are coprime integers and d is a positive rational number.

By definition, the result of (p,q)-surgery on a knot in S^3 is the cone-manifold whose space is the 3-manifold obtained by the usual p'/q'-surgery on the knot, and whose singular set, labelled by d, is the central curve of the surgered solid torus, with a cone angle of $2\pi/d$ around it (see [Ke] or [Ko] for cone-manifolds); note that this cone-manifold is a manifold (i.e. with empty singular set) or an orbifold exactly if α is equal to one or an integer, resp.

A rational (p,q)-tangle is an orbifold given by a rational tangle of slope p'/q' whose two arcs are labelled by two, with an additional arc labelled by d of cone-angle $2\pi/d$ (which, if $d = 1$, is not present, see [D] for details and pictures); some illustrating examples are shown in Figure 1.1. Then, the result of (p,q)-surgery on a non-rigid cusp of a 3-orbifold is the cone-orbifold obtained by gluing a (p,q)-tangle to the cusp as indicated in Figure 0.2; the cone-orbifold is an orbifold exactly if d is an integer.

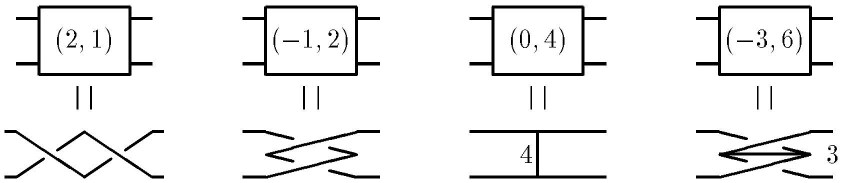

Figure 1.1. Examples of tangles.

Also, the well-known properties (see [C]) of rational tangles presented in Figure 1.5 will be used below.

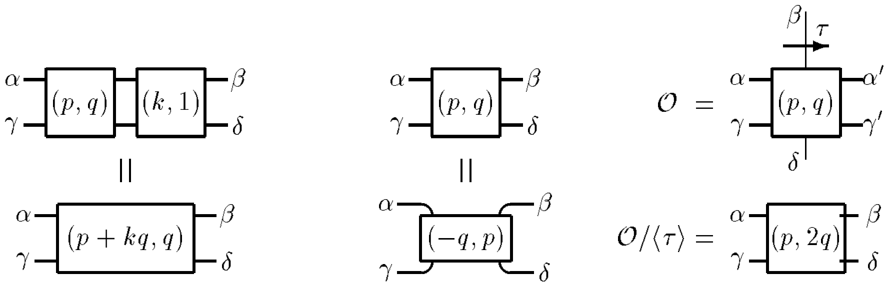

Figure 1.2. Properties of tangles.

2. The Borromean rings, the Picard orbifold and the Wolcott θ-graphs

In this section we consider the class of closed 3-manifolds obtained by surgery on the Borromean rings 6^3_2 shown in Figure 2.1.

Recall that 6_2^3 has symmetries which exchange each two of its components. We denote by $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))$ the cone-manifold obtained by surgery on the three components of the Borromean rings 6_2^3, with surgery coefficients (p_1,q_1), (p_2,q_2) and (p_3,q_3). We note that all manifolds obtained by surgery on the Whitehead link 5_1^2 and the figure-eight knot 4_1 belong to the above class: these are exactly the manifolds $6_2^3((p_1,q_1),(p_2,q_2),(-1,1)) = 5_1^2((p_1,q_1),(p_2,q_2))$ respectively $6_2^3((p_1,q_1),(-1,1),(1,1)) = 4_1(p_1,q_1)$. The Borromean rings 6_2^3 is a hyperbolic link [Th], so most of these manifolds are also hyperbolic.

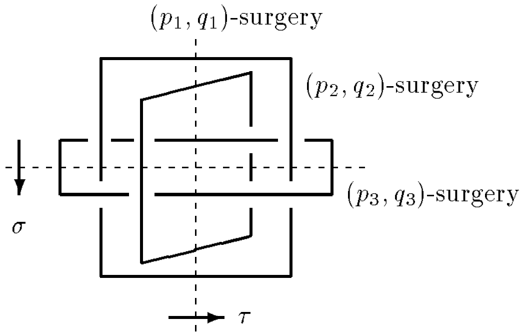

Figure 2.1. The Borromean rings 6_2^3.

The orientation-preserving symmetry group of 6_2^3 is isomorphic to the symmetric group $\mathbb{S}_4$ of order 24 (see e.g. [BoZ] or [Z1]), and this symmetry group can be realized by an orthogonal action of $\mathbb{S}_4$ on the 3-sphere leaving invariant the link. We will consider first the case of $\mathbb{S}_4$-equivariant surgery on 6_2^3. This means that the surgery coefficients for the three components of 6_2^3 are the same, say (p,q). We will denote the corresponding cone-manifold by $6_2^3(p,q)$ in this case. The action of $\mathbb{S}_4$ on the complement of 6_2^3 induces an action of $\mathbb{S}_4$ on $6_2^3(p,q)$.

Theorem 2.1. *For rational numbers p and q, the cone-manifold $6_2^3(p,q)$ obtained by $\mathbb{S}_4$-equivariant (p,q)-surgery on the Borromean rings is a regular $\mathbb{S}_4$-covering of the cone-orbifold $\mathcal{A}_1(p-2q,p+2q)$ obtained by $(p-2q,p+2q)$-surgery on the cusp of the Picard orbifold.*

Proof. Consider the cone-manifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))$ obtained by surgery on the Borromean rings (see Figure 2.1).
The axes of two involutions τ and σ from the symmetry group $\mathbb{S}_4$ are pictured by dashed lines. The involution τ is a strong inversion for two components of 6_2^3. The singular set of the orbifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))/\langle\tau\rangle$ (quotient-orbifold) is presented in Figure 2.2.
The involution σ of 6_2^3 induces an involution of this quotient-orbifold. The singular set of the orbifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))/\langle\tau,\sigma\rangle$ is presented in Figure 2.3.
This singular set is a generalization of a spatial θ-graph in the sense of [W]. In particular, if $i = 2q_1/p_1$, $j = 2q_2/p_2$ and $k = 2q_3/p_3$ are integers, then the corresponding singular set coincides with the Wolcott θ-graph $\mathcal{W}(i,j,k)$.

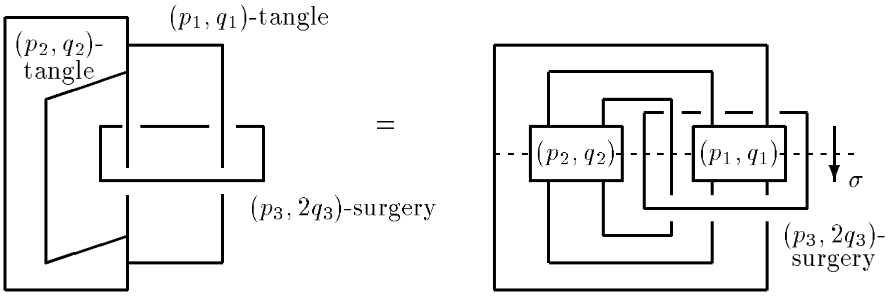

Figure 2.2. The orbifold $6_2^3((p_1, q_1), (p_2, q_2), (p_3, q_3))/\langle\tau\rangle$.

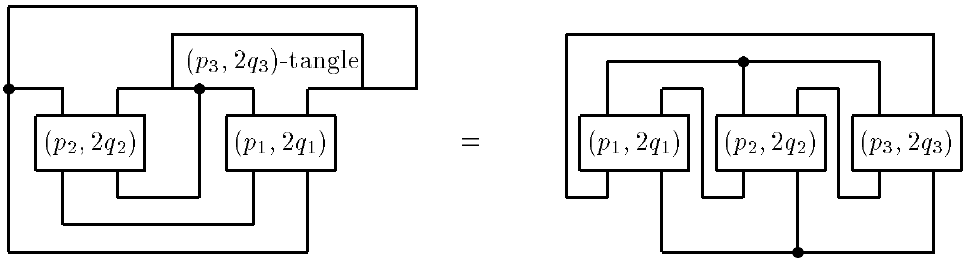

Figure 2.3. The orbifold $6_2^3((p_1, q_1), (p_2, q_2), (p_3, q_3))/\langle\tau, \sigma\rangle$.

From now on, we consider $\mathbb{S}_4$-equivariant surgery, that is $(p_1, q_1) = (p_2, q_2) = (p_3, q_3) = (p, q)$. Then, obviously, the singular set from Figure 2.3 has a symmetry ρ of order 3 that exchanges tangles and leaves vertices of the graph fixed. The singular set of the quotient-orbifold $(6_2^3(p, q)/\langle\tau, \sigma\rangle)/\langle\rho\rangle$ is shown in Figure 2.4 (see also Figure 1.2).

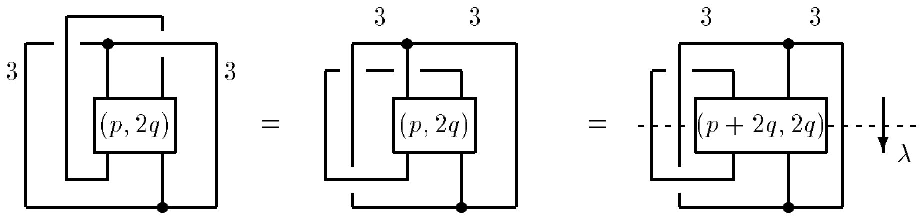

Figure 2.4 The orbifold $(6_2^3(p, q)/\langle\tau, \sigma\rangle)/\langle\rho\rangle$.

Obviously, the singular set in Figure 2.4. has a symmetry λ of order 2. The singular set of the quotient-orbifold $((6_2^3(p, q)/\langle\tau, \sigma\rangle)/\langle\rho\rangle)/\langle\lambda\rangle$ is presented in Figure 2.5.

As we see, this quotient orbifold coincides with the orbifold $\mathcal{A}_1(p - 2q, p + 2q)$ obtained by $(p - 2q, p + 2q)$-filling on the cusp of the Picard orbifold $\mathcal{A}_1$.

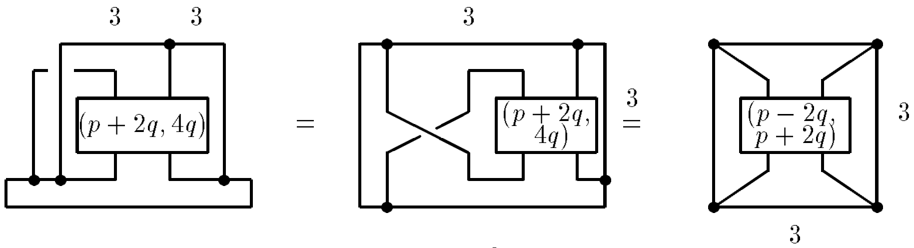

Figure 2.5 The orbifold $((6_2^3(p,q)/\langle\tau,\sigma\rangle)/\langle\rho\rangle)/\langle\lambda\rangle$.

This finishes the proof of Theorem 2.1.

Let $\mathbb{D}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be the normal subgroup of $\mathbb{S}_4$ generated by the involutions τ and σ in the proof of Theorem 2.1. As it was remarked above, the singular sets of $6_2^3((1,i),(1,j),(1,k))/\mathbb{D}_2$ are the Wolcott θ-graphs $\mathcal{W}(i,j,k)$ considered in [W]. They have the property to be non-planar (not homeomorphic in S^3 to the planar θ-graph), but all three constituent knots formed by any two of the three edges of the θ-graph are trivial. It follows from [N] that the unique $\mathbb{D}_2$-covering of such a θ-graph is a homology 3-sphere. In this case the 2-fold branched covering of S^3 along any of these three constituent knots is S^3 again, and the preimage of the third edge is a knot in S^3 whose 2-fold branched covering is the $\mathbb{D}_2$-covering of the θ-graph. Note that the manifold $6_2^3((p_1,q_1),(p_2,q_2),(p_3,q_3))$ is a homology 3-sphere if and only if it is of the from $6_2^3((1,i),(1,j),(1,k))$.
For integers i,j,k, consider the knots $\mathcal{K}(i,j,k)$ introduced in [Z4] and shown in Figure 2.6 where j and k denote numbers of half-twists on two strings, and i denotes the number of full twists on three strings.

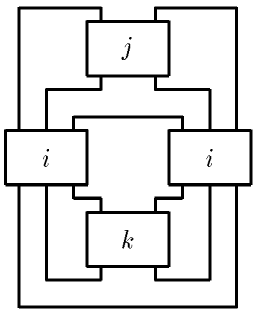

Figure 2.6 The knot $\mathcal{K}(i,j,k)$.

In general, the three knots $\mathcal{K}(i,j,k)$, $\mathcal{K}(j,k,i)$ and $\mathcal{K}(k,i,j)$ are pairwise non-equivalent (see [Z4]).

Theorem 2.2. *The homology 3-sphere $6_2^3((1,i),(1,j),(1,k))$ is the regular branched $\mathbb{D}_2$-covering of the Wolcott θ-graph $\mathcal{W}(i,j,k)$, and also the 2-fold branched covering of each of the three knots $\mathcal{K}(i,j,k)$, $\mathcal{K}(j,k,i)$ and $\mathcal{K}(k,i,j)$.*

Proof. The first statement follows from the first part of the proof of Theorem 2.1.

Regarding the second statement it was shown in [Z4] that, in the 2-fold covering of any of the three constituent knots of the θ-graph $\mathcal{W}(i,j,k)$, the preimage of the remaining edge is one of the three knots $\mathcal{K}(i,j,k)$, $\mathcal{K}(j,k,i)$ or $\mathcal{K}(k,i,j)$, respectively. This finishes the proof.

Another interesting case occurs when one choses surgeries of types $1/n$ and also $2/n$ for the components of the Borromean rings (where n is odd in the second case). In this case the singular set of the quotient orbifold contains some extra arcs of branching index two. Again the quotients of the three involutions in $\mathbb{D}_2$ give the 3-sphere but now the branch sets will be links in general (see also [MV2]).

Remarks.

a) Hyperbolic volume.
By Theorem 2.1, an orbifold $\mathcal{A}_1(x,y)$ obtained by surgery on the Picard orbifold has a 24-fold covering by the cone-manifold $6^3_2((y+x)/2,(y-x)/4)$ obtained by surgery on the Borromean rings. Applying SnapPea to the Borromean rings one obtains the volumes of these cone-manifolds (if hyperbolic) and, dividing by 24, of the corresponding orbifolds. The smallest volumes we found are 0.040890 of $\mathcal{A}_1(4,1)$ (covered by $6^3_2(5/2,-3/4)$) and 0.052654 of $\mathcal{A}_1(3,2)$ (covered by $6^3_2(5/2,-1/4)$). These seem to be the second and third smallest known volumes of hyperbolic 3-orbifolds, after the volume 0.039050 of the orbifold shown in Figure 0.1.
We remark that independent computations of the volumes for small surgeries on the Picard orbifold have recently been obtained by V. Petrov [P], by explicitly constructing orbifold fundamental polyhedra.

b) Manifolds of equal volumes.
By Theorem 2.1, the cone-manifolds $6^3_2(2p,q)$ and $6^3_2(2q,p)$ are $\mathbb{S}_4$-coverings of the orbifolds $\mathcal{A}_1(2p-2q,2p+2q)$ and $\mathcal{A}_1(2q-2p,2p+2q)$ which are homeomorphic (by a reflection). It follows that $6^3_2(2p,q)$ and $6^3_2(2q,p)$, if hyperbolic, have the same volume.

c) Non-hyperbolic manifolds.
The manifold $6^3_2(1,1)$ is the Poincaré homology 3-sphere, with a spherical structure, so also the quotient $6^3_2(1,1)/\mathbb{S}_4=\mathcal{A}_1(-1,3)$ has a spherical structure.
The 2-fold branched covering of the Borromean rings, or equivalently, of the orbifold $6^3_2(2,0)$, is the Hantzsche-Wendt manifold M which has a euclidean structure, so also the quotient $6^3_2(2,0)/\mathbb{S}_4=\mathcal{A}_1(2,2)=M/(\mathbb{Z}_2\times\mathbb{S}_4)$ has a euclidean structure. Here $\mathbb{Z}_2\times\mathbb{S}_4$ is the orientation-preserving isometry group of the Hantzsche-Wendt manifold M, see [Z1].
The manifold $6^3_2(0,1)$ is the 3-torus so the quotient $6^3_2(0,1)/\mathbb{S}_4=\mathcal{A}_1(-2,2)$ is euclidean.

d) Heegaard genus.
As the Borromean rings are a 3-bridge link, any 3-manifold obtained by surgery on the Borromean rings has Heegaard genus at most three. For integer co-prime surgeries with $p>1$, the manifolds $6^3_2(p,q)$ have Heegaard genus three because their

first homology is the 3-generator group $(\mathbb{Z}_p)^3$. On the other hand, all manifolds obtained by surgery on the Whitehead link and the figure-eight knot (which are 2-bridge links) are also obtained by surgery on the Borromean rings and have Heegaard genus at most two, among these the ten hyperbolic 3-manifolds of smallest known volumes (see [MV1]).

By Theorem 2.2, the manifolds $6_2^3((1,i),(1,j),(1,k))$ are 2-fold branched coverings of the 3-bridge knots $\mathcal{K}(i,j,k)$ (see Figure 2.6), so the Heegaard genus of the homology 3-spheres $6_2^3((1,i),(1,j),(1,k))$ is at most two.

e) Maximally symmetric manifolds and equivariant Heegaard genus.

Some of the manifolds $6_2^3(p,q)$ are maximally symmetric $\mathbb{S}_4$-manifolds ([Z2]): they admit a Heegaard splitting of genus three invariant under the $\mathbb{S}_4$-action which realizes the maximal order $12(g-1)$ for finite group actions on handlebodies of genus g.

By Theorem 2.1, the manifolds $6_2^3(1+2q,q)$ and $6_2^3(1-2q,q)$ cover the orbifolds $\mathcal{A}_1(1,1+4q)$ and $\mathcal{A}_1(1-4q,1)$, respectively. These orbifolds admit a decomposition along an embedded 2-sphere into two handlebody-orbifolds (see [Z2]). The preimage of this 2-sphere gives a Heegaard splitting of genus three of the corresponding 3-manifold such that the covering group maps each handlebody of the Heegaard splitting to itself. So the manifolds $6_2^3(1\pm 2q,q)$ are maximally symmetric $\mathbb{S}_4$-manifolds of genus three; note that also the usual Heegaard genus of these manifolds is three if q is different from ∓ 1.

Using SnapPea the smallest volume we found for hyperbolic 3-manifolds of type $6_2^3(p,q)$ is 2.468232 for $6_2^3(3,1)$. This manifold is a maximally symmetric $\mathbb{S}_4$-manifold of (equivariant and usual) Heegaard genus three, and there is some evidence that this might be the smallest volume of any hyperbolic 3-manifold admitting an $\mathbb{S}_4$-action (and maybe also of Heegaard genus three). We note that also the Fomenko-Matveev-Weeks manifold $\mathcal{M}_1 = 6_2^3((-5,1),(-5,2),(-1,1))$ is a maximally symmetric $\mathbb{D}_6$-manifold of (equivariant and usual) Heegaard genus two ([MV1], see also the next section; here $\mathbb{D}_6$ denotes the dihedral group of order 12. Apart from manifolds of Heegaard genus two and three, the only other maximally symmetric hyperbolic 3-manifold for which the equivariant and the usual Heegaard genus coincide is the manifold constructed in [Z3], of genus 11, with an $(\mathbb{A}_5\times\mathbb{Z}_2)$-invariant Heegaard decomposition of genus 11.

3. The link 6_1^3, the orbifold $\mathcal{A}_2$, and the Takahashi manifolds

We consider the n-component alternating links L_n defined as in Figure 3.1 for $n=3$ and $n=4$; in particular, $L_3 = 6_1^3$ and $L_4 = 8_1^4$.

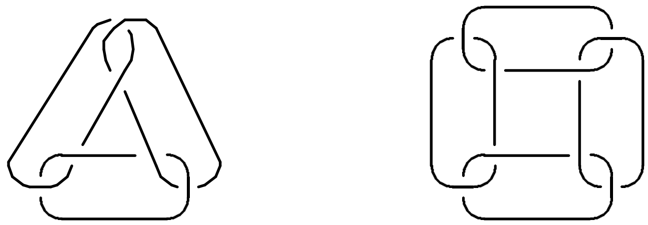

Figure 3.1. The links $L_3 = 6_1^3$ and $L_4 = 8_1^4$.

Remark that by [H1], $\pi_1(S^3 \setminus 6_1^3)$ is commensurable with $PGL(2, O_7)$ and $\pi_1(S^3 \setminus 8_1^4)$ is commensurable with $PGL(2, O_3)$.
Let $L_n(p, q) = L_n((p, q), \ldots, (p, q))$ be the cone-manifold obtained by surgery with coefficients (p, q) for each component of L_n, so $6_1^3(p, q) = L_3(p, q)$ and $8_1^4(p, q) = L_4(p, q)$. It was shown in [Th] that, for $n \geq 3$, the link complement $S^3 \setminus L_n$ is hyperbolic, so also almost all of the manifolds $L_n(p/q)$ are hyperbolic.
Denote by $\mathcal{A}_2^n(p, q)$ the 3-orbifold with underlying space S^3 and singular set the spatial graph shown in Figure 3.2, so one of its edges has singularity index n and all other edges have singularity index 2. For $n = 3$ we get the orbifolds $\mathcal{A}_2^3(p, q) = \mathcal{A}_2(p, q)$ obtained by surgery on the second Adams orbifold.

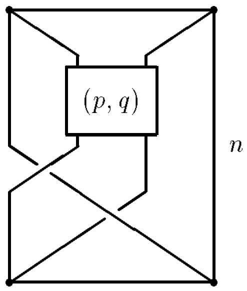

Figure 3.2. The singular set of $\mathcal{A}_2^n(p, q)$.

Theorem 3.1. *The cone-manifold $L_n(p, q)$ is a regular $4n$-fold covering of the cone-orbifold $\mathcal{A}_2^n(p, p + 2q)$.*

Proof. By the Kirby calculus, the manifolds $L_n(p/q)$ can be obtained also by surgery on the $2n$-component links $\mathcal{L}_{2n}$, see Figure 3.3. More precisely, the manifold $L_n(p, q) = L_n((p, q), \ldots, (p, q))$ coincides with $\mathcal{L}_{2n}((1, 1), (p + 2q, q), \ldots, (1, 1), (p + 2q, q))$. The manifolds obtained by surgery on $\mathcal{L}_{2n}$ are usually refered to as *Takahashi manifolds* as nice presentations of their fundamental groups were found in [Ta].
The link $\mathcal{L}_{2n}$ has a strong inversion (involution) τ whose axis is indicated by a dotted circle in Figure 3.3. This involution induces an involution, also denoted by τ, of the

manifold $L_n(p,q)$. The singular set of the quotient-orbifold $\mathcal{O}(n,p,q) = L_n(p,q)/\tau$ is shown in Figure 3.4, where we use $\alpha = (p+2q,q)$ to simplify notations (compare also [KV], [RS]).

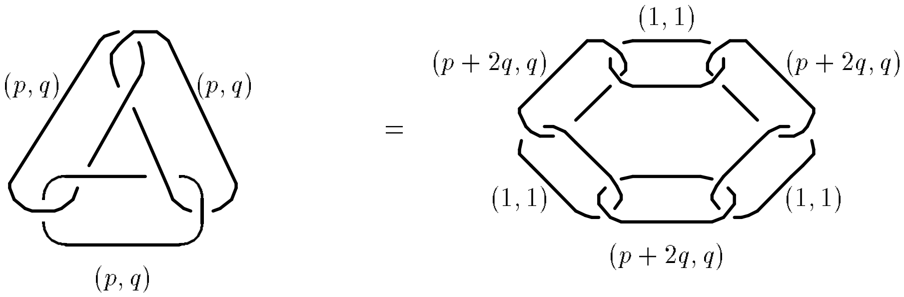

Figure 3.3. The links L_3 and $\mathcal{L}_6$.

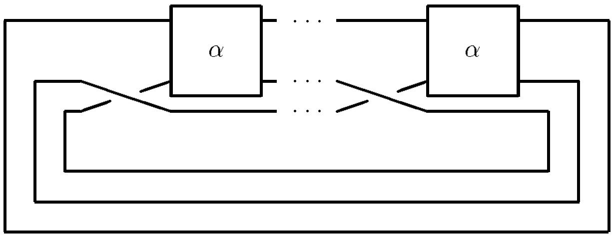

Figure 3.4. The singular set of $\mathcal{O}(n,p,q)$.

In notations from [KV] and [RS], the singular set of $\mathcal{O}(n,p,q)$ is the closure of the rational 3-strings braid $(\sigma_1^{p/q+2}\sigma_2)^n$. For example, if $p/q = -3/2$ then the closure of $(\sigma_1^{1/2}\sigma_2)^3$ is the knot 9_{49} (see [BuZ], p. 265).
The orbifold $\mathcal{O}(n,p,q)$ (or its singlar set) has an obvious cyclic symmetry ρ of order n permuting the tangles. The singular set of the quotient-orbifold $\mathcal{O}(n,p,q)/\langle\rho\rangle$ is shown in Figure 3.5.

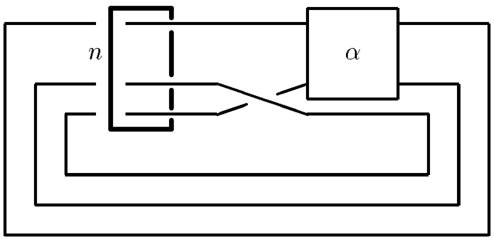

Figure 3.5. The singular set of $\mathcal{O}(n,p,q)/\langle\rho\rangle$.

Let us redraw the singular set as in Figure 3.6.

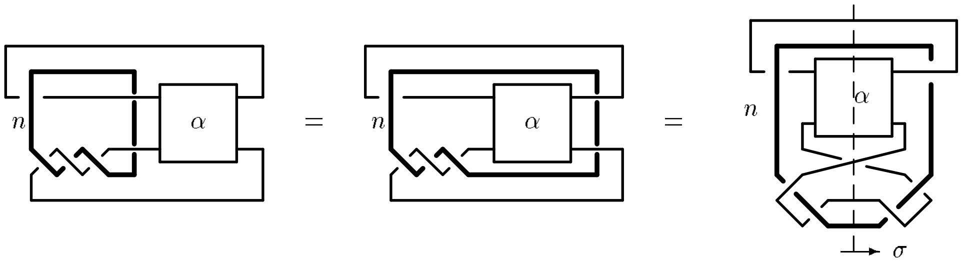

Figure 3.6. The singular set of $\mathcal{O}(n,p,q)/\langle\rho\rangle$.

From Figure 3.6 we see that the orbifold $\mathcal{O}(n,p,q)/\langle\rho\rangle$ admits an involution σ (its axis is given by the dashed line). The singular set of the quotient-orbifold $(\mathcal{O}(n,p,q)/\langle\rho\rangle)/\langle\sigma\rangle$ is shown in Figures 3.7.

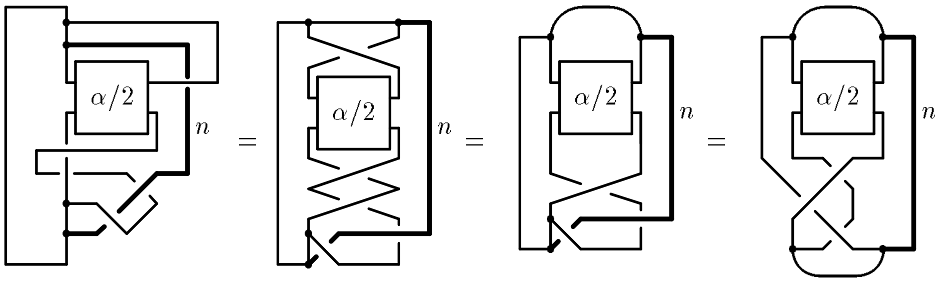

Figure 3.7. The singular set of $(\mathcal{O}(n,p,q)/\langle\rho\rangle/)\langle\sigma\rangle$.

Using tangle operations as in Figure 1.2, the singular set is equivalent to that of the orbifold $\mathcal{A}_2^n(p,p+2q)$ (see Figure 3.2).
This finishes the proof of Theorem 3.1.

The theorem admits to compute the volumes of the orbifolds $\mathcal{A}_2(x,y)$ using surgeries on 6_1^3. The smallest volume we found is 0.065965 for the orbifold $\mathcal{A}_2(3,2)$ (covered by $6_1^3(-3,1/2)$) which seems to be the fourth smallest volume known for hyperbolic 3-orbifolds. The second smallest volume we found is 0.078559 of the orbifold $\mathcal{A}_2(-3,1)$ which is covered by the Fomenko-Matveev-Weeks manifold $\mathcal{M}_1 = 6_1^3(-3,2)$.

As in section 2, the manifolds $6_1^3(p,(\pm 1 - p)/2)$ (where p is odd) are maximally symmetric $\mathbb{D}_6$-manifolds of Heegaard genus two, among them the manifold $\mathcal{M}_1$.

We remark that the Fibonacci manifolds uniformized by the Fibonacci groups $F(2,2n)$ can be obtained as $L_n(-3,1)$, so the Fibonacci manifold $\mathbb{H}^3/F(2,2n)$ is a covering of $\mathcal{A}_2^n(3,1)$.

4. The link 8^4_2 and the orbifold $\mathcal{A}_3$

Denote by $8^4_2(p,q)$ the cone-manifold obtained by (p,q)-surgery on all four components of the link 8^4_2 (see Figure 4.1). Let $\mathcal{A}_3(p,q)$ be the orbifold whose singular set is presented in Figure 4.1.

Figure 4.1. The link 8^4_2 and the orbifold $\mathcal{A}_3(p,q)$

By the Kirby calculus, the cone-manifold $8^4_2(p,q)$ can be obtained by surgery on the link $\mathcal{L}_8$ that belongs to the series of links considered in the previous section; in fact, $8^4_2(p,q) = \mathcal{L}_8((p+2q,q),(1,1),(p+2q,q),(1,1),(p,q),(-1,1),(p,q),(1,1))$ is a Takahashi manifold.
Using the strong inversion (involution) of $\mathcal{L}_8$ considered in the previous section, we obtain $8^4_2(p,q)$ as the 2-fold branched covering of the closure of a generalized 3-strings braid.
After three further steps of involutions, we get the following result.

Theorem 4.1. *The cone-manifold $8^4_2(p,q)$ obtained by (p,q)-surgery on the components of the link 8^4_2 is a regular 16-fold covering of the cone-orbifold $\mathcal{A}_3(p,2q)$ obtained by $(p,2q)$-surgery on the third Adams orbifold $\mathcal{A}_3$.*

The smallest volume of an orbifold of type $\mathcal{A}_3(x,y)$ that we found by SnapPea is 0.117838 of $\mathcal{A}_3(3,2)$, covered by the manifold $8^4_2(3,1)$. We note that $\mathcal{A}_3(3,2)$ is a π-orbifold, i.e. all singularity indices are equal to two; in fact it is the smallest π-orbifold that we know. The smallest known π-orbifold whose singular set is a knot or link is the π-orbifold whose singular set is the knot 9_{49}, of volume 0.471354, whose 2-fold branched covering is the Fomenko-Matveev-Weeks manifold $\mathcal{M}_1$.
We remark that the volume of $\mathcal{M}_1$ is eight times that of the orbifold $\mathcal{A}_3(3,2)$. The orbifold $\mathcal{A}_3(3,2)$, whose singular set is a spatial handcuff (or pince-nez) graph, has no regular covering of order less or equal to eight by a manifold.

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