UNIFORM APPROXIMATIONS IN THE CLT FOR BALLS IN EUCLIDIAN SPACES

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1. Introduction and Main Results

Let $X, X_1, X_2, \ldots$ be independent identically distributed random elements with values in a real separable Hilbert space H. Let (x, y) for $x, y \in H$ denote the inner product in H and put $|x| = (x, x)^{1/2}$. We assume that $\mathbb{E}|X_1|^2 < \infty$ and denote by V a covariance operator of X_1

$$(Vx, y) = \mathbb{E}(X_1 - \mathbb{E}X_1, x)(X_1 - \mathbb{E}X_1, y).$$

Let $\sigma_1^2 \geq \sigma_2^2 \geq \ldots$ be the eigenvalues of V and let $e_1, e_2, \ldots$ be the corresponding eigenvectors which we assume to be orthonormal. We define

$$S_n = n^{-1/2}\sigma^{-1}\sum_{i=1}^{n}(X_i - \mathbb{E}X_i),$$

where $\sigma^2 = \mathbb{E}|X_1 - \mathbb{E}X_1|^2$. Without loss of generality we may assume that $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1|^2 = 1$. The general case can be reduced to this one when we consider $(X_i - \mathbb{E}X_i)/\sigma$ instead of X_i, $i = 1, 2, \ldots$. For any integer $k > 0$ we put

$$c_k(V) = \prod_1^k \sigma_i^{-1}, \quad \overline{c}_k(V) = (\prod_1^k \sigma_i^{-1})^{(k-1)/k}. \tag{1.1}$$

In what follows we use c and $c(\cdot)$, with or without indices, to denote generic constants and constants depending on parameters in brackets. Except for $c_i(V)$ and $\overline{c}_i(V)$ the same symbol may be used for various constants. Let $Y, Y_1, Y_2, \ldots$ be independent H-valued Gaussian $(0, V)$ random elements. Put for any $a \in H$

$$F(x) = P\{|S_n - a|^2 \leq x\}, \quad F_0(x) = P\{|Y - a|^2 \leq x\},$$
$$\delta_n(a) = \sup_x |F(x) - F_0(x)|.$$

It is known (see e.g. Sazonov (1968) and Bentkus (1986)) that in the case $H = \mathbb{R}^d$, $d < \infty$, i.e. in the finite dimensional case, we have

$$\delta_n(a) \leq c\,\mathbb{E}|X_1|^3\,\sigma_d^{-3}\,n^{-1/2}$$

and the bound is optimal with respect to the dependence on moments, eigenvalues as well as n.

However in the infinite dimensional case the situation essentially changes. Here we have (see e.g. Sazonov (1981), ch.2)

$$\sup_a \delta_n(a) \geq 1/2.$$

The development of bounds for $\delta_n(a)$ in the infinite dimensional case can be divided roughly into three phases: proving bounds with optimal

– dependence on n;
– moment conditions;
– dependence on the eigenvalues of V.

The first phase started in the middle of 60-s with bounds of logarithmic order for $\delta_n(a)$ (see Kandelaki (1965)) and ended with the result:

$$\delta_n(a) = \mathcal{O}(n^{-1/2}),$$

due to Götze (1979), which was based on a Weyl type symmetrization inequality, which since then has been successfully applied and developed by a number of authors.

The second phase finishes with a paper by Yurinskii (1982) proving

$$\delta_n(a) \leq \frac{c(V)}{\sqrt{n}} \, (1 + |a|^3) \, \mathbb{E}|X_1|^3,$$

where $c(V)$ is denotes a constant depending on V only.

At the end of the third phase it was proved (see Sazonov, Ulyanov and Zalesskii (1988b), Nagaev (1989), Senatov (1989))

$$\delta_n(a) \leq \frac{c \, c_6(V)}{\sqrt{n}} \, (1 + |a|^3) \, \mathbb{E}|X_1|^3, \tag{1.2}$$

where $c_6(V)$ is defined in (1.1). It is known that for any $c_0 > 0$ and for any given eigenvalues $\sigma_1^2, \ldots, \sigma_6^2$ of a covariance operator V there exist (see example 3 in Senatov (1985)) a vector $a \in H$, $|a| > c_0$, and a sequence $X_1, X_2, \ldots$ of i.i.d. random elements in H with zero mean and covariance operator V such that

$$\liminf_{n\to\infty} \sqrt{n} \, \delta_n(a) \geq c \, c_6(V) \, (1 + |a|^3) \, \mathbb{E}|X_1|^3. \tag{1.3}$$

Thus, bound (1.2) is the best possible in case of finite third moment of $|X_1|$. For further refinements see Senatov (1997).

At the same time better approximations for $F(x)$ are available using an additional term, say $F_1(x)$, of its asymptotic expansion. This term $F_1(x)$ is defined as the unique function satisfying $F_1(-\infty) = 0$ with Fourier-Stieltjes transform equal to

$$\hat{F}_1(t) = -\frac{2t^2}{3\sqrt{n}} \mathbb{E} e\{t|Y-a|^2\} \left(3(X, Y-a)|X|^2 + 2it(X, Y-a)^3\right). \quad (1.4)$$

Here and in the following we write $e\{x\} = \exp\{ix\}$.

Introduce the error

$$\Delta_n(a) = \sup_x |F(x) - F_0(x) - F_1(x)|.$$

Note that $\hat{F}_1(t) = 0$ and hence $F_1(x) = 0$ when $a = 0$ or X has a symmetric distribution, i.e. when X and $-X$ are identically distributed. Therefore, we get

$$\Delta_n(0) = \delta_n(0).$$

Similar to the developments of bounds for $\delta_n(a)$ the first task has been to derive bounds for $\Delta_n(a)$ with optimal dependence on n. Starting with a seminal paper by Esseen (1945) for finite dimensional spaces $H = \mathbb{R}^d$, $d < \infty$ who proved

$$\Delta_n(0) = \mathcal{O}(n^{-d/(d+1)}), \quad (1.5)$$

a comparable bound

$$\Delta_n(0) = \mathcal{O}(n^{-\gamma})$$

with $\gamma = 1 - \varepsilon$ for any $\varepsilon > 0$. was finally proved in Götze(1979, 1984), based on Weyl type inequalities mentioned above. Further refinements and generalizations in the case $a \neq 0$ and $\gamma < 1$ are due to Bentkus and Zalesskii (1985), Nagaev and Chebotarev (1986), Sazonov, Ulyanov and Zalesskii (1988a). Note however, that the results in the infinite dimensional case did not even yield (1.5) as corollary when $\sigma_{d+1} = 0$. Fifty years after Essen's result the optimal bounds (in n)

$$\Delta_n(0) \leq \frac{c(9, V)}{n} \, \mathbb{E}|X_1|^4, \quad (1.6)$$

$$\Delta_n(a) \leq \frac{c(13, V)}{n} \, (1 + |a|^6) \, \mathbb{E}|X_1|^4, \quad (1.7)$$

where $c(i, V) \leq \exp\{c\sigma_i^{-2}\}, \;\; i = 9, 13.$ were finally established in Bentkus and Götze (1997), using new techniques which allowed to prove optimal bounds in classical lattice point problems as well.

The bounds (1.6) and (1.7) are optimal with respect to the dependence on n (Götze 1998b) and on moments. The bound (1.6) improves as well Esseen's result (1.5) for Euclidean spaces $\mathbb{R}^d$ with $d > 8$. However the dependence on covariance operator V in (1.6), (1.7) can be improved. Nagaev and Chebotarev

(1999) considered the case $a = 0$ and got a bound of type (1.6) replacing $c(9, V)$ by the following function $c(V)$:

$$c(V) = c\ \left(\overline{c}_{13}(V) + (c_4(V))^{4/9}\sigma_9^{-6}\right),$$

where $\overline{c}_{13}(V)$ and $c_9(V)$ are defined by (1.1). This improves the dependence on the eigenvalues of V (compared to (1.6)) but still requires that $\sigma_{13} > 0$ instead of the weaker condition $\sigma_9 > 0$ in (1.6).

The aim of the present paper is to derive a bound for $\Delta_n(a)$ in the general case $a \neq 0$ depending on *twelve* largest eigenvalues of V only (see Remarks after Corollary 1.4).

Theorem 1.1. *There exist absolute constants c, c_1, c_2 such that for any $a \in H$*

$$\begin{aligned}\Delta_n(a) \ &\leq\ \frac{c}{n} \cdot c_{12}(V) \cdot \big(\mathbb{E}|X_1|^4 + \mathbb{E}(X_1, a)^4\big)\big(1 + (Va, a)\big) \\ &\quad + c \int_{N_0 \leq |t| \leq N} \frac{|f_n(t)|}{|t|}\, dt, \end{aligned} \tag{1.8}$$

where $f_n(t) = \mathbb{E}e\{t|S_n - a|^2\}$ and

$$N_0 = \left(\frac{c_1\, n\, \sigma_9^4}{(\mathbb{E}|X_1|^3)^2}\right)^{4/9}, \quad N = \frac{c_2\, \sigma_9^8\, n}{\sigma_1^4\, (c_9(V))^{2/9}\, \mathbb{E}|X_1|^4}.$$

Theorem 1.2. *There exists an absolute constant c such that*

$$\int_{N_0 \leq |t| \leq N} \frac{|f_n(t)|}{|t|}\, dt \leq \frac{c}{n}\, \mathbb{E}|X_1|^4\, \left(\sigma_1^4\, (c_9(V))^{4/9}\, \sigma_9^{-8}\ +\ (c_9(V))^{1/2}\, \sigma_9^{-4}\right).$$

Remark. The proof of Theorem 1.2 is based on Lemma 2.12. See the examples following Lemma 2.12 explaining why the bound of Theorem 1.2 depends on first *nine* eigenvalues of V only. Theorems 1.1 and 1.2 together imply

Corollary 1.3. *There exists an absolute constant c such that*

$$\begin{aligned}\Delta_n(a) \ &\leq\ \frac{c}{n}(c_{12}(V) + \sigma_1^4\, \sigma_9^{-8}\, (c_9(V))^{4/9} + \sigma_9^{-4}\, (c_9(V))^{1/2}) \\ &\quad \times \big(\mathbb{E}|X_1|^4 + \mathbb{E}(X_1, a)^4\big)\big(1 + (Va, a)\big).\end{aligned}$$

Corollary 1.4. *If*

$$c_{12}(V) \geq \sigma_1^4\, \sigma_9^{-8}\, (c_9(V))^{1/2}, \tag{1.9}$$

then there exists c such that

$$\Delta_n(a) \leq \frac{c}{n}\, c_{12}(V)\, (1 + |a|^6)\, \mathbb{E}|X_1|^4. \tag{1.10}$$

Remark. Condition (1.9) is not very restrictive. For example, it is satisfied when $\sigma_1 = \cdots = \sigma_9$ and arbitray $\sigma_i,\ i = 10, 11, \ldots$.

It follows from Lemma 2.6 (see below) that for any given eigenvalues $\sigma_1^2, \ldots, \sigma_{12}^2 > 0$ of a covariance operator V there exist $a \in H,\ |a| > 1$, and a sequence $X_1, X_2, \ldots$ of i.i.d. random elements in H with zero mean and covariance operator V such that

$$\liminf_{n\to\infty} n\ \Delta_n(a) \geq c\ c_{12}(V)\ (1 + |a|^6)\ \mathbb{E}|X_1|^4.$$

Hence, (1.10) is best possible in the sense that it is impossible that $\Delta_n(a)$ is of order $\mathcal{O}(n^{-1})$ *uniformly* for distributions of X_1 with arbitrary eigenvalues $\sigma_1^2, \sigma_2^2, \ldots$. This means any explicit bound in terms of eigenvalues has to depend on the first 12 eigenvalues of V.

For earlier versions of this result on the optimality of 12 eigenvalues and a detailed discussion of the connection of the rate problems in the central limit theorem with classical lattice point problems in analytic number theory, see the ICM-1998 Proceedings paper by Götze (1998b), and also ibid. (1998a).

Note however that in special 'symmetric' cases of the distribution of X_1 or of the center, say a, of the ball, the number of eigenvalues which are necessary for optimal bounds may well decrease below 12. For example, when $\mathbb{E}(X, b)^3 = 0$ for all $b \in H$, by Corollary 2.7 (see below) for any given eigenvalues $\sigma_1^2, \ldots, \sigma_8^2 > 0$ of a covariance operator V there exists a center $a \in H,\ |a| > 1$, and a sequence $X, X_1, X_2, \ldots$ of i.i.d. random elements in H with zero mean and covariance operator V such that

$$\liminf_{n\to\infty} n\ \Delta_n(a) \geq c\ c_8(V)\ (1 + |a|^4)\ \mathbb{E}|X_1|^4.$$

Hence, in this case an upper bound of order $\mathcal{O}(n^{-1})$ for $\Delta_n(a)$ has to involve at least the *eight* largest eigenvalues of V.

Furthermore, lower bounds for $n\Delta_n(a)$ in the case $a = 0$ are not available. A conjecture, see Götze (1998b), says that in this case the five first eigenvalues of V suffice. This conjecture is motivated by the special case where the sums of random vectors have independent coordinates where indeed we have $\Delta_n(a) = \mathcal{O}(n^{-1})$ provided that $\sigma_5 > 0$ only, see Bentkus and Götze (1996).

The proofs of the Theorems are given in Section 3. They are based on the Lemmas from Section 2.

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2. Auxiliary Results

Let Z_j for $j = 1, 2, \ldots$ be i.i.d. random elements with distribution Q such that $Q = (Q_1 + Q_2)/2$, where Q_1, Q_2 are probability measures, $Q_1(B_L) = 1$ for a ball $B_L = \{x \in H : |x|^2 < L\}$ with some $L > 0$ and a covariance operator V' of Q_1 has a trace $\mathrm{tr} V' \le 2$. Let Y be a Gaussian random element with parameters $(0, \alpha V)$, $\alpha > 0$. Assume that Y, Z_0, Z_{0j}, Z_j for $j = 1, 2, \ldots$ are independent random elements. Let l, m, n_0, n be positive integers such that $l \le m$, $l + m \le n$, $n_0 \le n$. Put

$$U_1 = n^{-1/2} \sum_1^{l+m} Z_j, \;\; U_2 = n^{-1/2} \sum_1^{n_0} Z_{0j}.$$

Lemma 2.1. *(see Lemma 4 in Sazonov, Ulyanov and Zaleskii (1991)) For any $A > 0$, any integer $r \ge 0$, $l > L^2$, and t satisfying*

$$|t| \le c(A) L^{-1} n (l \; ln(L^{-2} l))^{-1/2}$$

as well as for arbitrary t if $l \le L^2$, we have

$$\begin{aligned} &|\mathbb{E} \; exp \; \{i\, t |U_1 + U_2 + Z_0|^2\} (x, U_1 + U_2)^r| \\ &\le c_1 K_1 (exp\{-c_2 l\} + c(A)(L^2/l)^A + h^{1/2}(c_3 t_1^2 lm/n^2, V')), \end{aligned}$$

where $c_j = c_j(r)$, $j = 1, 2, 3$,

$$K_1 = \mathbb{E} |(x, U_1 + U_2)^r| \;\; when \;\; l < r + 1,$$

and for $l \ge r + 1$

$$K_1 = \sum_{j=1}^{r+1} \mathbb{E} |(x, U_{1j} + U_{2j})^r|$$

with $U_{1j} = n^{-1/2} \sum_{k=k_{j-1}}^{kj} Z_k$, $k_j = j[(l+m)/(r+1)], j = \overline{0, r}$, $k_{r+1} = l + m$ and U_{2j} for $j = \overline{1, r+1}$, form a partition of a sum U_2 into $r + 1$ parts (some of which may be empty);

$$t_1 = \min\{|t|, L^{-1} n (m \; ln(m/L^2))^{-1/2}\},$$

$$h(s, V') = \prod_{j=1}^{\infty} (1 + 2s(\sigma_j')^4)^{-1/2}$$

and $(\sigma_1')^2 \ge (\sigma_2')^2 \ge \ldots$ denote the eigenvalues of V'.

Moreover, for any t we have

$$|\mathbb{E} \; exp \; \{i\, t |Y + U_2 + Z_0|^2\} (x, Y + U_2)^r| \le c_4 K_2 h^{1/2}(c_5 t^2 \alpha, V),$$

where $c_j = c_j(r),\ j = 4, 5$ *and*

$$K_2 = \sum_{j=1}^{r+1} \mathbb{E}|(x, Y_j + U_{2j})|^r,$$

where $Y_j,\quad j = \overline{1, r+1}$ *are i.i.d. Gaussian random elements in the representation* $Y = \sum_1^{r+1} Y_j$ *and* $U_{2j},\quad j = \overline{1, r+1}$, *form a partition of a sum* U_2 *into* $r+1$ *parts some of which may be empty.*

We shall use the following *Rosenthal type inequality* (see e.g. Pinelis (1980) and de Acosta (1981)):

for independent random elements $X_1, \dots, X_n$ *in* H *with mean zero and for any* $q \geq 2$ *we have*

$$\mathbb{E}|\sum_{i=1}^n X_i|^q \leq c(q)\ \big(\sum_{i=1}^n \mathbb{E}|X_i|^q + (\sum_{i=1}^n \mathbb{E}|X_i|^2)^{q/2}\big).$$

Lemma 2.2. *Let* $T > 0,\ b \in \mathbb{R}^1,\ b \neq 0,\ l$ *be an integer,* $l \geq 1,\ Y = (Y_1, \dots, Y_{2l})$ *be a Gaussian random vector with values in* $\mathbb{R}^{2l},\ Y_1, \dots, Y_{2l}$ *are independent and* $\mathbb{E}Y_i = 0,\quad \mathbb{E}Y_i^2 = \sigma_i^2$ *for* $i = 1, 2, \dots, 2l;\ \sigma_1^2 \geq \sigma_2^2 \geq \dots \geq \sigma_{2l}^2 > 0$ *and* $a \in \mathbb{R}^{2l}$. *Then there exists a positive constant* $c = c(l)$ *such that*

$$\begin{aligned} I \quad &\equiv \quad \Big|\int_{-T}^{T} s^{l-1}\mathbb{E}\exp\{is|Y+a|^2\}e^{ibs}ds\Big| \\ &\leq c\prod_{j=1}^{2l}\sigma_j^{-1}. \end{aligned} \tag{2.1}$$

Proof. It is known that

$$\mathbb{E}\exp\{is|Y+a|^2\} = \prod_{j=1}^{2l}(1 - 2is\sigma_j^2)^{-1/2}\exp\{ia_j^2 s/(1 - 2is\sigma_j^2)\}.$$

Using Euler's formula for complex exponentials we get for positive g and any $d \in \mathbb{R}^1$

$$g + id = \sqrt{g^2 + d^2}e^{i\zeta} \ \text{ with } \ \zeta = \arcsin\frac{d^2}{\sqrt{g^2 + d^2}}.$$

Therefore, for I we have

$$\begin{aligned}
I &= \Big|\int_{-T}^{T} s^{l-1} e^{isb} \prod_{j=1}^{2l} (1+4s^2\sigma_j^4)^{-1/4} \\
&\quad \times \exp\left(-\frac{2a_j^2 s^2 \sigma_j^2}{1+4s^2\sigma_j^4}\right) \cdot \exp\left(\frac{ia_j^2 s}{1+4s^2\sigma_j^4} + \frac{i\varphi_j}{2}\right) ds\Big| \\
&= \Big|\int_{-T}^{T} s^{l-1} \mathrm{trig}\left(sb + \sum_{j=1}^{2l}\left(\frac{a_j^2 s}{1+4s^2\sigma_j^4} + \frac{\varphi_j}{2}\right)\right) \\
&\quad \times \exp\left(-\sum_{j=1}^{2l} \frac{2a_j^2 s^2 \sigma_j^2}{1+4s^2\sigma_j^4}\right) \prod_{j=1}^{2l} (1+4s^2\sigma_j^4)^{-1/4} ds\Big|,
\end{aligned}$$

where $\varphi_j = \arcsin(2s\sigma_j^2/(1+4s^2\sigma_j^4)^{1/2})$ and

$$\mathrm{trig}(x) = \begin{cases} \cos x \ \text{ if } \ l \ \text{ is odd,} \\ \sin x \ \text{ if } \ l \ \text{ is even.} \end{cases}$$

This decomposition has been suggested to us by N. Blagoveshchenskii. We may write

$$\begin{aligned}
I &= 2\prod_{j=1}^{2l} \sigma_j^{-1} \Big| \int_0^T \frac{1}{s} \sin\left(sb + \sum_{j=1}^{2l}\left(\frac{a_j^2 s}{1+4s^2\sigma_j^4} + \frac{1}{2}(\varphi_j - \frac{\pi}{2})\right)\right) \\
&\quad \times \exp\left(-\sum_{j=1}^{2l} \frac{2a_j^2 s^2 \sigma_j^2}{1+4s^2\sigma_j^4}\right) \prod_{j=1}^{2l} \left(\frac{s^2\sigma_j^4}{1+4s^2\sigma_j^4}\right)^{1/4} ds\Big|. \qquad (2.2)
\end{aligned}$$

Note that

$$\prod_{j=1}^{2l} \left(\frac{s^2\sigma_j^4}{1+4s^2\sigma_j^4}\right)^{1/4} \leq \sqrt{|s|}\sigma_{2l}.$$

Therefore it follows from (2.2) that in order to prove (2.1) it is enough to show

$$\begin{aligned}
&\Big|\int_{1/\sigma_{2l}^2}^{T} \frac{1}{s} \sin\left(sb + \sum_{j=1}^{2l}\left(\frac{a_j^2 s}{1+4s^2\sigma_j^4} + \frac{1}{2}(\varphi_j - \frac{\pi}{2})\right)\right) \\
&\times \exp\left(-\sum_{j=1}^{2l} \frac{2a_j^2 s^2 \sigma_j^2}{1+4s^2\sigma_j^4}\right) \prod_{j=1}^{2l} \left(\frac{s^2\sigma_j^4}{1+4s^2\sigma_j^4}\right)^{1/4} ds\Big| \leq c. \qquad (2.3)
\end{aligned}$$

We shall use the following obvious inequalities

$$0 < 1 - \left(\frac{4s^2\sigma_j^4}{1+4s^2\sigma_j^4}\right)^{1/4} \le \frac{1}{4s^2\sigma_{2l}^4} \quad \text{for } j = 1, 2, \dots, 2l;$$
$$|\sin(x+y) - \sin x| \le |y| \text{ and } |x|e^{-|x|} \le 1 \text{ for any } x, y \in \mathbb{R}^1;$$
$$0 \le \frac{\pi}{2} - \arcsin(1-z) \le 2^{3/2}z^{1/2} \quad \text{for } z : 0 \le z \le 1.$$

In view of these inequalities the bound (2.3) follows from

$$\left|\int_{1/\sigma_{2l}^2}^{T} \frac{1}{s}\sin(sb)\exp\left(-\sum_{j=1}^{2l}\frac{2a_j^2s^2\sigma_j^2}{1+4s^2\sigma_j^4}\right)ds\right| \le c. \tag{2.4}$$

The inequality (2.4) is a conequence of the second mean-value formula for integrals (Bonnet's theorem) since the function

$$\exp\left(-\sum_{j=1}^{2l}\frac{2a_j^2s^2\sigma_j^2}{1+4s^2\sigma_j^4}\right)$$

is a bounded and monotone function of s for $s > 0$ and for any positive a and b we have

$$\left|\int_a^b \frac{\sin x}{x}dx\right| \le 2\int_0^\pi \frac{\sin x}{x}dx$$

The Lemma is proved.

Lemma 2.3. *Let $Y_1, \dots, Y_d$, $Y_1', \dots, Y_d'$ be i.i.d. Gaussian $(0, C)$ random elements in a Hilbert space H. Let $\sigma_1^2 \ge \sigma_2^2 \ge \dots$ be the eigenvalues of C. Put $W = (detA)^2$, where $A = \{a_{ij}\}_{i,j=1}^d$ and $a_{ij} = (Y_i, Y_j')$. Then*

$$\mathbb{E}W = (d!)^2 \sum_{1\le i_1<\dots<i_d<\infty} \sigma_{i_1}^4 \dots \sigma_{i_d}^4, \tag{2.5}$$

$$(\mathbb{E}W^2)^{1/2} \le d^2 9^{d+1}\mathbb{E}W. \tag{2.6}$$

Proof. Without loss of generality we may assume that an orthonormal base $\{e_i\}$ in H consists from the eigenvectors of C. By the assumptions of the Lemma we have

$$Y_i = \sum_{j=1}^\infty Y_{ij}e_j, \quad Y_i' = \sum_{j=1}^\infty Y_{ij}'e_j,$$

where $\{Y_{ij}\}$, $\{Y_{ij}'\}$ are independent real normal random variables

$$\mathbb{E}Y_{ij} = \mathbb{E}Y_{ij}' = 0, \ \mathbb{E}Y_{ij}^2 = \mathbb{E}(Y_{ij}')^2 = \sigma_j^2, \ i = 1, \dots, d; \ j = 1, 2, \dots$$

According to the extension of the Lagrange identity for the determinants to Hilbert spaces, (see e.g. Greub (1978), ch. 5, §5.7, problem 4 or Smirnov (1964), ch. 1,§1, sect. 7) we can write

$$\det A = \sum_{<} \det Y(i) \cdot \det Y'(i), \tag{2.7}$$

where $Y(i)$ and $Y'(i)$ are the matrices $\{Y_{ki_l}\}_{k,l=1}^d$ and $\{Y'_{ki_l}\}_{k,l=1}^d$ respectively and the symbol $<$ indicates that the indices $(i_1, \dots, i_d)$ are subject to the condition $1 \le i_1 < \dots < i_d < \infty$. Note that $\mathbb{E} \det Y(i) = 0$,

$$\mathbb{E}(\det Y(i))^2 = d! \sigma_{i_1}^2 \cdot \dots \cdot \sigma_{i_d}^2$$

and $\mathbb{E} \det Y(i) \cdot \det Y(j) = 0$ for any $i = (i_1, \dots, i_d)$ with $1 \le i_1 < \dots < i_d < \infty$ and $i \ne j$, i.e. there exists $l \in \{1, \dots, d\}$ for which $i_l \ne j_l$. Using (2.7) we get

$$\mathbb{E}W = \sum_{<} \mathbb{E}(\det Y(i))^2 \cdot \mathbb{E}(\det Y'(i))^2,$$

which implies (2.5).

The inequality (2.6) is a direct corollary of the logarithmic inequalities for Gaussian measures (see e.g. Bogachev (1998), Corollary 5.5.9, p. 228).

Let $X, X_1, \dots, X_n$ denote i.i.d. random elements in a Hilbert space H with $\mathbb{E}X = 0$, $\operatorname{cov}(X) = C$ and $\sigma_1^2 \ge \sigma_2^2 \ge \dots$ be the eigenvalues of C.

Put $S_m = m^{-1/2}(X_1 + \dots + X_m)$. Let $S_{m1}, S'_{m1}, S_{m2}, \dots, S'_{md}$ be $2d$ independent copies of S_m. Put $W_m(X) = (\det A_m(X))^2$, where $A_m(X) = \{a_{ij}(X)\}_{i,j=1}^d$ and $a_{ij}(X) = (S_{mi}, S'_{mj})$. Denote by W the same variable as in Lemma 2.3.

Lemma 2.4. *There exist the constants c_{1d} and c_{2d} such that for any integer m satisfying*

$$\frac{\mathbb{E}|X|^4}{m\sigma_d^8} \left(\sigma_1^4 + \frac{\mathbb{E}|X|^4}{m} \right) \le c_{1d}, \tag{2.8}$$

we have

$$|\mathbb{E}\, W^2 - \mathbb{E} W_m^2(X)| \le c_{2d} (\mathbb{E}\, W)^2. \tag{2.9}$$

Proof. Let $Y, Y_1, Y'_1, \dots, Y_d, Y'_d$ denote independent Gaussian $(0, C)$ random elements in H.

Since

$$\mathbb{E}\, X = \mathbb{E}\, Y = 0, \quad \operatorname{cov}(X) = \operatorname{cov}(Y),$$

for any $b_1, b_2, b_3, b_4 \in H$ we obtain

$$\mathbb{E} \prod_{i=1}^4 (S_m, b_i) = \mathbb{E} \prod_{i=1}^4 (Y, b_i) + \frac{1}{m} \left(\mathbb{E} \prod_{i=1}^4 (X, b_i) - \mathbb{E} \prod_{i=1}^4 (Y, b_i) \right). \tag{2.10}$$

Applying (2.10) sequentially to variables $S_{m1}, S^{'}_{m1}, S_{m2}, \ldots, S^{'}_{md}$ we get

$$\mathbb{E}\, W_m^2(X) = \mathbb{E}\, W^2 + \sum_{i=1}^{d} m^{-2i+1} \sum_1 + \sum_{i=1}^{d} m^{-2i} \sum_2, \tag{2.11}$$

where $\sum_1$ and $\sum_2$ are sums of 2^{2i-1} resp. 2^{2i} summands of type

$$c\, \mathbb{E}\, W^2(Z_1, Z_1^{'}, Z_2, Z_2^{'}, \ldots Z_i, Y_i^{'}, Y_{i+1}, Y_{i+1}^{'}, Y_{i+1}^{'}, \ldots, Y_d, Y_d^{'})$$

resp.

$$c\, \mathbb{E}\, W^2(Z_1, Z_1^{'}, \ldots, Z_i, Z_i^{'}, Y_{i+1}, Y_{i+1}^{'}, \ldots, Y_d, Y_d^{'});$$

here the cofactors c are real numbers uniformly bounded by a constant c_d, i.e.

$$|c| \le c_d$$

and c_d depends on d only; Z_i resp. $Z_i^{'}$ are either equal to X_i or Y_i resp. $X_i^{'}$ or $Y_i^{'}$; and

$$W(Z_1, Z_1^{'}, \ldots, Z_i, Z_i^{'}, Y_{i+1}, Y_{i+1}^{'}, \ldots, Y_d, Y_d^{'}) = (\det B)^2$$

where $B = \{b_{kj}\}_{k,j=1}^{d}$ with $b_{kj} = (T_k, T_j^{'})$ and

$$T_k = \begin{cases} Z_k & \text{for} \quad k \le i, \\ Y_k & \text{for} \quad k > i, \end{cases} \qquad T_k^{'} = \begin{cases} Z_k^{'} & \text{for} \quad k \le i, \\ Y_k^{'} & \text{for} \quad k > i. \end{cases}$$

Moreover, it is easy to see that for a non-increasing sequence $\{a_i\}$ of non-negative numbers such that $\sum a_i < \infty$, we have for any integer $d \ge 2$

$$\sum_{1 \le i_1 < \cdots < i_d < \infty} a_{i_1} \ldots a_{i_d} \ge 2^{-d}\, a_d \sum_{1 \le i_1 < \cdots < i_{d-1} < \infty} a_{i_1} \ldots a_{i_{d-1}} \tag{2.12}$$

Using an expansion of the determinant in the elements of rows and cofactors, together with Lemma 2.3, (2.11) and (2.12) we arrive at (2.9) provided that m satisfies (2.8).

Lemma 2.5. *(Cf. Lemma 4.7 in Bentkus, Götze (1997)). Let A be a non-degenerate $d \times d$ matrix. Let $X \in \mathbb{R}^d$ be a random vector with a covariance matrix B. Assume that there exists a constant c_d such that*

$$P\{|X| \le c_d\} = 1, \quad |A| \le c_d, \quad |B^{-1}| \le c_d.$$

Let U and V be independent random vectors which are sums of n independent copies of X. Then

$$|\mathbb{E}e\{t(AU, V)\}| \le c(d)\; |\det A|^{-1} \mathcal{M}^{2d}(t; n) \;\; \textit{for} \;\; |t| > 0,$$

where

$$\mathcal{M}(t; n) = 1/\sqrt{|t| n} + \sqrt{|t|} \;\; \textit{for} \;\; |t| > 0.$$

Proof. It is enough to repeat the proof of Lemma 4.7 in Bentkus, Götze (1997) with minor modifications using

$$\int_{|x|\le 1} \exp\{-|Ax|^2\}\, dx \le 2\, |\det A|^{-1}.$$

Let $X, X_1, X_2, \dots, X_n$ denote i.i.d. random vectors in $\mathbb{R}^d$ with zero mean and covariance operator V and eigenvalues $\sigma_1^2 \ge \sigma_2^2 \ge \cdots \ge \sigma_d^2$. Put $S_n = n^{-1/2}(X_1 + \cdots + X_n)$ and

$$F(x) = P\{|S_n - a|^2 < x\}.$$

Let Y be a Gaussian vector distributed according to $N(0, V)$ in $\mathbb{R}^d$. Put

$$F_0(x) = P\{|Y - a|^2 < x\}$$

and denote by $F_1(x)$ the function of bounded variation satisfying $F_1(-\infty) = 0$ with Fourier-Stieltjes transform given by

$$\hat{F}_1(t) = \frac{-2t^2}{3\sqrt{n}} \mathbb{E}e\{t|Y - a|^2\} \left(3(X, Y - a)|X|^2 + 2it(X, Y - a)^3\right).$$

In fact (see e.g. Bentkus, Götze (1997)) $F_1(x)$ can be written as a signed measure μ of a ball $B(a, x) = \{y \in \mathbb{R}^d : |y - a|^2 < x\}$. We have

Lemma 2.6. *In the Euclidean spaces $\mathbb{R}^d, d = 2, 3, \dots, 13$, there exists a distribution P of X and balls $B(a, r)$ with $|a| > 1$ such that for given values $\sigma_1^2, \dots, \sigma_{d-1}^2$ of the eigenvalues of the covariance operator V and all sufficiently small σ_d*

$$\begin{aligned} &\liminf_{n\to\infty} n \sup_r |P\{|S_n - a| < r\} - P\{|Y - a| < r\} - \mu\{B(a, r)\}| \\ &\ge c(P)|a|^{(d-1)/2}(\sigma_1 \cdot \dots \cdot \sigma_{d-1})^{-1} \left[\sigma_d^{(d-13)/2}\alpha_3^2 + \sigma_d^{(d-9)/2}(\alpha_4 - 3\alpha_2^2)\right], \quad (2.13) \end{aligned}$$

where $\alpha_i = \mathbb{E}X_d^i$ and $X = (X_1, \dots, X_d)$.

Corollary 2.7. *Assume that $\alpha_3 = 0$ and that the conditions of Lemma 2.6 are satisfied . Then we have for all $d = 2, \dots, 9$*

$$\begin{aligned} &\liminf_{n\to\infty} n \sup_r |P\{|S_n - a| < r\} - P\{(Y - a) < r\} - \mu\{B(a, r)\}| \\ &\qquad \ge c(P)|a|^{(d-1)/2}(\sigma_1 \cdot \dots \cdot \sigma_{d-1})^{-1}\sigma_d^{(d-9)/2}(\alpha_4 - 3\alpha_2^2). \quad (2.14) \end{aligned}$$

Proof. We shall follow the arguments of Example 3 in Senatov (1985).

We choose a distribution P in $\mathbb{R}^d$ such that the projections onto vectors of an orthonormal basis $e_1, \dots, e_{d-1}$ in $\mathbb{R}^d$ are normal distributed with zero mean and variances $\sigma_1^2, \dots, \sigma_{d-1}^2$. Furthermore we may choose P such that the projection onto e_d has zero mean, $\mathbb{E}(X, e_d)^2 = \sigma_d^2$, $\alpha_3 > 0$, $\alpha_4 - 3\alpha_2 > 0$ and it satisfies a Cramer condition of smoothness. Moreover we assume that the components of this random vector X under P are independent. Let G denote the distribution function of the projection of P onto e_d. Obviously, the projections of distribution of S_n onto $e_1, \ldots, e_{d-1}$ for any n will be normal with zero means and variances $\sigma_1^2, \ldots, \sigma_{d-1}^2$ while the distribution function G_n of the projection of distribution of S_n onto e_d has the following asymptotic behaviour (this follows from classical results on Edgeworth approximations of sums of i.i.d. vectors, see e.g. Petrov (1987))

$$R_n(x) = G_n(x) - \Phi(x) = \frac{Q_1(x)}{\sqrt{n}} + \frac{Q_{21}(x) + Q_{22}(x)}{n} + o(n^{-1}) \text{ as } n \to \infty,$$

where $\Phi(x)$ is the distribution function of the normal law with zero mean and variance σ_d^2,

$$\begin{aligned}
Q_1(x) &= -\frac{\alpha_3}{6\sqrt{2\pi}\sigma_d^3}\big(\frac{x^2}{\sigma_d^2} - 1\big)\exp(-x^2/(2\sigma_d^2)), \\
Q_{21}(x) &= -\frac{\alpha_3^2}{72\sqrt{2\pi}\sigma_d^6}\left(\big(\frac{x}{\sigma_j}\big)^5 - 10\big(\frac{x}{\sigma_d}\big)^3 + 15\big(\frac{x}{\sigma_d}\big)\right)\exp(-x^2/(2\sigma_d^2)), \\
Q_{22}(x) &= -\frac{(\alpha_4 - 3\alpha_2^2)}{24\sqrt{2\pi}\sigma_d^4}\left(\big(\frac{x}{\sigma_d}\big)^3 - 3\frac{x}{\sigma_d}\right)\exp(-x^2/(2\sigma_d^2)).
\end{aligned}$$

Consider the ball $B = \{y \in \mathbb{R}^d : |y - ae_d| < R\}$, where a is a positive number depending on R to be determined later.

We have

$$I \equiv \Phi_1(B) - P_n(B) = \sum_{i=1}^{2} \int_{b_R(0)} (R_{ni}(x_1) - R_{ni}(x_2))\widetilde{\Phi}(du) + o(n^{-1}),$$

where $P_n(B) = P\{(S_n - a| < R\}$, $\Phi_1(B) = P\{|Y - a| < R\} + \mu(B)$, $R_{ni}(x) = -Q_{2i}(x)/n$, $\widetilde{\Phi}$ is the projection of P onto $\mathbb{R}^{d-1}$ which coincides with the corresponding projection of the distribution of Y, $b_R(0)$ is the ball in $\mathbb{R}^{d-1}$ with center at 0 and radius R, $x_1 = a - r$, $x_2 = a + r$, $r^2 = R^2 - |u|^2$ and $|u|^2 = u_1^2 + \cdots + u_{d-1}^2$.

It is easy to see that there exists c_0 such that for $x > c_0\sigma_d$ the functions $R_{ni}(x)$ are positive and decreasing. Put $a = R + c_0\sigma_d$. Then for any $u \in b_R(0)$ the differences $R_{ni}(x_1) - R_{ni}(x_2)$ are non-negative.

The arguments are similar for the first two summands in the expression for I. Therefore we shall give a proof for

$$I_1 = \int_{b_R(0)} \big(R_{n1}(x_1) - R_{n1}(x_2)\big)\widetilde{\Phi}(du)$$

only. There exists a constant c_1 such that for all $v \geq c_0\sigma_d$

$$R_{n1}(v) \geq 2R_{n1}(v + c_1\sigma_d).$$

Since $x_2 - x_1 = 2\sqrt{R^2 - |u|^2}$, there exists c_2 such that for $u: \ |u| \leq \sqrt{R\sigma_d}$ and $R \geq c_2\sigma_d$ we have

$$R_n(x_1) \geq 2R_n(x_2).$$

Therefore for $R \geq c_2\sigma_d$ we get

$$I_1 \geq \frac{1}{2}\int_{b_{\sqrt{R\sigma_d}}(0)} R_{n1}(x_1)\widetilde{\Phi}(du)$$

and, since $c_3\sigma_d \leq x_1 \leq c_4\sigma_d$,

$$I_1 \geq c\alpha_3^2\sigma_d^{-6}n^{-1}\widetilde{\Phi}\big(b_{\sqrt{R\sigma_d}}(0)\big).$$

Since for sufficiently small σ_d we have

$$\widetilde{\Phi}\big(b_{\sqrt{R\sigma_d}}(0)\big) \geq c\prod_{i=1}^{d-1}\min(1, \sqrt{R\sigma_d}/\sigma_i) \geq c(R\sigma_d)^{(d-1)/2}(\sigma_1 \cdot \cdots \cdot \sigma_{d-1})^{-1}$$

with some constant c depending only on d, we arrive at the first summand on the right-hand side of (2.13) as a lower bound. Repeating similar arguments for

$$\int_{b_R(0)} (R_{n2}(x_1) - R_{n2}(x_2))\widetilde{\Phi}(du)$$

concludes the proof of the Lemma.

Lemma 2.8. *(See Sazonov (1981), p.85). Let X be a random element with values in H such that $\mathbb{E}X = 0$, $\mathbb{E}|X|^3 < \infty$. Denote by V the covariance operator of X. Let Y denote a Gaussian $(0, V/2)$ distributed random element and let ζ denote a real random variable such that*

$$\mathbb{E}\zeta = 0, \ \mathbb{E}\zeta^2 = 1/2, \ \mathbb{E}\zeta^3 = 1.$$

Suppose that X, Y and ζ are independent. Then $Z = \zeta X + Y$ has mean zero, covariance operator V and for any $h_1, h_2, h_3 \in H$ we have

$$\mathbb{E}(X, h_1)(X, h_2)(X, h_3) = \mathbb{E}(Z, h_1)(Z, h_2)(Z, h_3).$$

Lemma 2.9. *Let k and n be natural numbers, $n \geq k$, and $X_1, \dots, X_{kn}$ be $k \cdot n$ i.i.d. random elements with values in H. Let $f(x_1, \dots x_k)$ be a functional of $x_1, \dots x_k \in H$ with values in $\mathbb{R}^1$ such that it is linear with respects to each variable. Assume that $g : H \to \mathbb{R}$ is such that $\mathbb{E} f(X_1, \dots, X_k) g(T)$ exists, where $T = X_1 + X_2 + \dots + X_{kn}$. Put $S = \mathbb{E} g(T) \sum f(X_{i_1}, \dots, X_{i_k})$, where the summation is taken over all values of $i_1, \dots, i_k \in \{1, 2 \dots, kn\}$ which are pairwise different. Then S can be splitted into $m = m(k)$ summands of type $\mathbb{E} g(T) f(W_1, \dots, W_k)$, where $W_i = \sum_{j \in N(i)} X_j$, $N(1) \cup \dots \cup N(k) = \{1, 2, \dots, kn\}$, $N(i) \cap N(j) = \emptyset$ for $i \neq j$ and $N(i)/n \to 1$ as $n \to \infty$ for all $i \in \{1, 2, \dots, k\}$.*

Proof. We prove the Lemma in the case $k = 2$ and $g = 1$. In this case it is enough to show that

$$S \equiv \sum_{\substack{i,j=1 \\ i \neq j}}^{2n} \mathbb{E} f(X_i, X_j) = 2(\mathbb{E} f(T_1, T_2^{'}) + \mathbb{E} f(T_1, T_2)), \tag{2.15}$$

where

$$T_1 = \sum_{i=1}^{n} X_i, \;\; T_2^{'} = \sum_{j=n+2}^{2n} X_j, \;\; T_2 = X_{n+1} + T_2^{'}.$$

We have

$$S = S_1 + S_2, \tag{2.16}$$

where

$$S_1 = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{2n} \mathbb{E} f(X_i, X_j).$$

It follows from the conditions of the Lemma that

$$\begin{aligned} \sum_{j=2}^{2n} \mathbb{E} f(X_1, X_j) &= \sum_{j=2}^{n} \mathbb{E} f(X_1, X_j) + \mathbb{E} f(X_1, T_2) \\ &= \sum_{j=2}^{n} \mathbb{E} f(X_1, X_{n+j} + \mathbb{E} f(X_1, T_2) \\ &= \mathbb{E} f(X_1, T_2^{'}) + \mathbb{E} f(X_1, T_2). \end{aligned} \tag{2.17}$$

Since $X_1, X_2, \dots X_{2n}$ are identically distributed we get similarly to (2.17) for any $i \in \{1, \dots, n\}$

$$\sum_{\substack{j=1 \\ j\neq i}}^{2n} \mathbb{E}f(X_i, X_j) = \mathbb{E}f(X_i, T_2^{'}) + \mathbb{E}f(X_i, T_2) \tag{2.18}$$

and for any $i \in \{n+1, n+2, \dots, 2n\}$

$$\sum_{\substack{j=1 \\ j\neq i}}^{2n} \mathbb{E}f(X_i, X_j) = \mathbb{E}f(X_{i-n}, T_2^{'}) + \mathbb{E}f(X_{i-n}, T_2). \tag{2.19}$$

Combining (2.16) - (2.19) we obtain (2.15). Thus the Lemma is proved in the case $k = 2$ and $g = 1$. It is easy to see that similar but more tedious arguments prove the Lemma in the general case.

Let $Z, \overline{Z}, Z_1, \dots Z_n$ be i.i.d. random elements,

$$Z_1 = \zeta_1 X_1 + Y_1, \tag{2.20}$$

where ζ_1 and X_1 have the same distribution as ζ and X resp. in Lemma 2.8. Let Y, $\overline{Y}$ be the independent Gaussian $(0, V)$ random elements. According to Lemma 2.8 we have

$$\mathbb{E}Z_1 = \mathbb{E}Y = 0, \ \mathrm{cov}(Z_1) = \mathrm{cov}(Y) = V, \tag{2.21}$$

where $\mathrm{cov}(Z_1)$ denotes the covariance operator of Z_1.

Put $T_n = n^{-1/2}(Z_1 + \cdots + Z_n)$,

$$I_0 = \mathbb{E}e\{t|T_n - a|^2\}(\mathbb{E}_Z L(Z) - \mathbb{E}_Y L(Y))(\mathbb{E}_{\overline{Z}}(\overline{Z}) - \mathbb{E}_{\overline{Y}} L(\overline{Y})),$$

where

$$L(u) = e\{t(|u/\sqrt{n}|^2 + 2(T_n - a, u/\sqrt{n}))\}$$

and $\mathbb{E}_X f(X, Y)$ means that we consider expectation with respect to X only.

Lemma 2.10. *If*

$$n^{-1}(\mathbb{E}|X|^4 + \mathbb{E}(X, a)^4)(1 + (Va, a)) \le 1, \tag{2.22}$$

then for any $T > 0$ and $x > 0$

$$\Big|\int_{-T}^{T} \frac{e\{-xt\}}{t}\, I_0 \, dt\Big| \le \frac{c_{12}(V)}{n^2}(\mathbb{E}|X|^4 + \mathbb{E}(X, a)^4)(1 + (Va, a)). \tag{2.23}$$

Proof. In the following we say that a function $f(n, t)$ belongs to a class of functions $\mathcal{F}$ iff for any $T > 0$ and $x > 0$ the quantity

$$\Big|\int_{-T}^{T} \frac{e\{-xt\}}{t}\, f(n, t)\, dt\Big|$$

can be bounded from above by the right-hand side of (2.23), provided that (2.22) holds. Thus we have to show that $I_0 \in \mathcal{F}$.

Below we shall use often the Taylor formula for a smooth function $f(s)$

$$f(s) = \sum_{j=0}^{m-1} f^{(j)}(0)s^j/j! + ((m-1)!)^{-1}\int_0^s f^{(m)}(\lambda)(s-\lambda)^{m-1}d\lambda. \qquad (2.24)$$

Using (2.24) we can write

$$\begin{aligned} e\{t(|u|^2+(b,u))\} &= e\{t(b,u)\}(1+it|u|^2+(it)^2|u|^4\mathbb{E}_\tau e\{t\tau|u|^2\}(1-\tau)) \\ &= 1+it(b,u)+it|u|^2+\sum_{j=0}^{2}K_j(b,u), \qquad (2.25) \end{aligned}$$

where τ is a random variable uniformly distributed on $[0;1]$ and

$$\begin{aligned} K_0(b,u) &= -t^2|u|^4\mathbb{E}_\tau e\{t((b,u)+\tau|u|^2)\}(1-\tau), \\ K_1(b,u) &= -t^2|u|^2(b,u)\mathbb{E}_\tau e\{t\tau(b,u)\}, \\ K_2(b,u) &= -t^2(b,u)^2\mathbb{E}_\tau e\{t\tau(b,u)\}(1-\tau). \end{aligned}$$

It follows from (2.21) and (2.25) that

$$\begin{aligned} I_0 &= \mathbb{E}e\{t|T_n-a|^2\}\sum_{j,l=0}^{2}(K_l(2T_n-2a,Z/\sqrt{n})-K_l(2T_n-2a,Y/\sqrt{n})) \\ &\quad \times(K_j(2T_n-2a,\overline{Z}/\sqrt{n})-K_j(2T_n-2a,\overline{Y}/\sqrt{n})) \\ &= \sum_{j,l=0}^{2} I_{jl} = I_{22}+I_{12}+I_{21}+R_0(t). \qquad (2.26) \end{aligned}$$

Note that up to absolute constants $R_0(t)$ is a sum of terms of the type

$$t^4\mathbb{E}\left|\frac{U}{\sqrt{n}}\right|^{2\alpha_1}\left|\frac{\overline{U}}{\sqrt{n}}\right|^{2\alpha_2}\left(T_n-a,\frac{U}{\sqrt{n}}\right)^{\alpha_3}\left(T_n-a,\frac{\overline{U}}{\sqrt{n}}\right)^{\alpha_4} e\{t\left|T_n+b_1\right|^2+b_2t\},$$

where U (resp. $\overline{U}$) is either Z or Y (resp. either $\overline{Z}$ or $\overline{Y}$); random elements $U,\overline{U},T_n,b_1$ and b_2 are independent; $P\{b_2=0\}=0$; $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are non-negative integers such that

$$\alpha_1 \le 2, \quad \alpha_2 \le 2, \quad \alpha_3+\alpha_4 \le 2,$$

$$2\alpha_1+\alpha_3 \le 4, \quad 2\alpha_2+\alpha_4 \le 4,$$

$$2\alpha_1+2\alpha_2+\alpha_3+\alpha_4 \ge 6.$$

Using the wellknown techniques of splitting sums and conditioning on corresponding sets of random variables, together with Rosenthal type inequality, Lemmas 2.1 and 2.2 and Fubini's theorem we get

$$R_0(t) \in \mathcal{F}. \tag{2.27}$$

Now we consider I_{22}. We have

$$\begin{aligned} I_{22} &= -4t^4 \mathbb{E} e\{t|T_n - a|^2\} \sum_{j,l=0}^{2} (K_{2j}(T_n, Z/\sqrt{n}) - K_{2j}(T_n, Y/\sqrt{n})) \\ &\quad \times (K_{2l}(T_n, \overline{Z}/\sqrt{n}) - K_{2l}(T_n, \overline{Y}/\sqrt{n})) \\ &= \sum_{j,l=0}^{2} L_{jl}. \end{aligned} \tag{2.28}$$

where for $j = 0, 1, 2$

$$K_{2j}(b, u) = \binom{2}{j} (b, u)^j (a, u)^{2-j} \mathbb{E}_\tau e\{t\tau(2b - 2a, u)\}(1 - \tau). \tag{2.29}$$

The further arguments are based on the following ideas:

a) Since the covariance operators of $Y, Z, \overline{Y}$ and $\overline{Z}$ coincide we obtain for $j = 0, 1, 2$ and any $a, b \in H$ that

$$\begin{aligned} \mathbb{E}\ (b, Z)^j (a, Z)^{2-j} &= \mathbb{E}\ (b, Y)^j (a, Y)^{2-j} = \mathbb{E}\ (b, \overline{Z})^j (a, \overline{Z})^{2-j} \\ &= \mathbb{E}\ (b, \overline{Y})^j (a, \overline{Y})^{2-j} \end{aligned} \tag{2.30}$$

Therefore, applying the Taylor expansion (2.24) with respect to s for a function

$$f(s, u) = e\{st\tau(2b - 2a, u)\}$$

(see (2.29)) with $m = 1$ we have to bound the remainder terms in (2.28) only.

b) While using Taylor expansions to show that $I_{22} \in \mathcal{F}$, we have to make sure that the error terms occuring in our arguments involve moments of random elements of order at most four.

This tedious but straightfoward procedure has been described in detail in similar problems in Bentkus, Götze (1999), Lemma 8.4. Therefore we restrict ourselves to necessary remarks which are specific to our task only.

Applying (2.24) for $f(s, U)$ and $f(s, \overline{U})$ we can write $\sum_{0 \le j+l \le 2} L_{jl}$ as a sum of terms of the type

$$c\, \mathbb{E} \left(T_n, \frac{U}{\sqrt{n}}\right)^{\beta_1} \left(a, \frac{U}{\sqrt{n}}\right)^{\beta_2} \left(T_n, \frac{\overline{U}}{\sqrt{n}}\right)^{\beta_3} \left(a, \frac{\overline{U}}{\sqrt{n}}\right)^{\beta_4} e\{t\, |T_n + b_1|^2 + b_2 t\},$$

where $U, \overline{U}, T_n, b_1, b_2$ are independent and $\beta_1, \beta_2, \beta_3, \beta_4$ are non-negative integers such that

$$\beta_1 + \beta_2 = \beta_3 + \beta_4 = 3, \quad \beta_1 + \beta_3 \leq 4.$$

Similarly to (2.27) we can show that

$$\sum_{0 \leq j+l \leq 2} L_{jl} \in \mathcal{F}.$$

Now we show that

$$L_{22} \in \mathcal{F} \tag{2.31}$$

We have

$$\begin{aligned}
L_{22} = & -4t^4 \mathbb{E}\{t|T_n - a|^2\}(1-\tau_1)\,(1-\tau_2) \\
& \times \sum_{i,j,k,l=1}^{n} \Big[(Z_i/\sqrt{n}, Z/\sqrt{n})(Z_j/\sqrt{n}, Z/\sqrt{n}) e\{t\tau_1(2T_n - 2a, Z/\sqrt{n})\} \\
& \qquad - (Z_i/\sqrt{n}, Y/\sqrt{n})(Z_j/\sqrt{n}, Y/\sqrt{n})\; e\{t\tau_1(2T_n - 2a, Y/\sqrt{n})\}\Big] \\
& \qquad \times \Big[(Z_k/\sqrt{n}, \overline{Z}/\sqrt{n})(Z_l/\sqrt{n}, \overline{Z}/\sqrt{n}) e\{t\tau_2(2T_n - 2a, \overline{Z}/\sqrt{n})\} \\
& \qquad - (Z_k/\sqrt{n}, \overline{Y}/\sqrt{n})(Z_l/\sqrt{n}, \overline{Y}/\sqrt{n}) e\{t\tau_2(2T_n - 2a, \overline{Y}/\sqrt{n})\}\Big] \\
= & \sum_{m=1}^{3} D_m,
\end{aligned} \tag{2.32}$$

where

D_1 represents the summands with $i = j = k = l$;
D_2 represents those summands where three of four indices i, j, k and l coincide;
D_3 represents all remaining summands.

Since D_1 is a sum of n terms of type

$$\begin{aligned}
& c\, t^4\, \mathbb{E}\; e\{t\,|T_n - a|^2 + t\left(2T_n - 2a, \left(\tau_1 U + \tau_2 \overline{U}\right)/\sqrt{n}\right)\} \\
& \times (1-\tau_1)(1-\tau_2)\left(\frac{Z_1}{\sqrt{n}}, \frac{U}{\sqrt{n}}\right)^2 \left(\frac{Z_1}{\sqrt{n}}, \frac{\overline{U}}{\sqrt{n}}\right)^2,
\end{aligned}$$

it follows from Fubini's theorem and Lemmas 2.1 and 2.2 that

$$D_1 \in \mathcal{F}.$$

In order to bound D_2 it is enough to consider without loss of generality the part D_{21} of D_2 which corresponds to the case when $i = j = k \neq l$. We shall use (2.24) for a function $f(s) = f(s, \overline{U})$ with $m = 1$. Then we get that D_{21} can be written as a sum of n terms of the type

$$c\, t^5\, \mathbb{E}\, e\{t|T_n - a|^2 + 2t(T_n - a, (\tau_1 U + \tau_2\tau_3\overline{U})/\sqrt{n})\}$$

$$\times \tau_2(1-\tau_1)(1-\tau_2)\left(\frac{Z_1}{\sqrt{n}}, \frac{U}{\sqrt{n}}\right)^2 \left(\frac{Z_1}{\sqrt{n}}, \frac{\overline{U}}{\sqrt{n}}\right)$$

$$\times \left(T_{n(1)}, \frac{\overline{U}}{\sqrt{n}}\right)\left(\frac{Z_1}{\sqrt{n}} + T_{n(1)} - a, \frac{\overline{U}}{\sqrt{n}}\right),$$

where $T_{n(1)} = T_n - Z_1/\sqrt{n}$.

Similar to (2.27) we get $D_{21} \in \mathcal{F}$ and therefore

$$D_2 \in \mathcal{F}.$$

In order to obtain bounds for D_3 we argue in a similar way using (2.24) for both functions $f(s, U)$ and $f(s, \overline{U})$ and additionally apply Lemma 2.9. Thus, (2.31) holds.

Similar arguments show that

$$I_{12} + I_{21} \in \mathcal{F}.$$

The Lemma is proved.

Let $X, X_1, \ldots, X_n$ be i.i.d. random elements satisfying the conditions of Theorem 1.1. Let $Z, Z_1, \ldots Z_n$ be i.i.d. random elements constructed in Lemma 2.8. Put $T_n = n^{-1/2}(W_1 + \cdots + W_n)$, where $W_1, \ldots,\ W_n$ are independent random elements and W_i is either X_i or Z_i.

Lemma 2.11. *If*

$$n^{-1}(\mathbb{E}|X|^4 + \mathbb{E}(X,a)^4)(1 + (Va, a)) \le 1$$

then

$$\int_{-N_0}^{N_0} \frac{1}{|t|}\, |\mathbb{E}e\{t|T_n + X/\sqrt{n} - a|^2\} -\ \mathbb{E}\, e\{t|T_n + Z/\sqrt{n} - a|^2\}|\, dt$$

$$\le \frac{\overline{c}_9(V)}{n^2}(\mathbb{E}|X|^4 + \mathbb{E}(X,a)^4),$$

where

$$N_0 = \left(c\,\frac{n\sigma_9^4}{(\mathbb{E}|X_1|^3)^2}\right)^{4/9}.$$

Proof. The arguments are similar to those used in Lemma 2.10. Just note that instead of (2.25) we use

$$e\{t(|u|^2+(b,u))\} = 1+i\,t(b,u)-\frac{1}{2}t^2(b,u)^2+it|u|^2-t^2|u|^2(b,u)+\sum_{j=0}^{2}K_j'(b,u),$$

where $K_0'(b,u)=K_0(b,u)$ with $K_0(b,u)$ from (2.25),

$$\begin{aligned} K_1'(b,u) &= -it^3|u|^2\mathbb{E}_\tau e\{t\tau(b,u)\}, \\ K_2'(b,u) &= -\frac{i}{2}t^3(b,u)^3\mathbb{E}_\tau e\{t\tau(b,u)\}(1-\tau)^2. \end{aligned}$$

We shall apply Lemmas 2.1 and 2.2 as well.

Lemma 2.12. *(See Lemma 3.2, Theorem 10.1 and formulas (10.7)–(10.8) in Bentkus and Götze (1999)). Let $\varphi(t),\ t\ge 0$ denote a continuous function such that $0\le\varphi\le 1$. Assume that*

$$\varphi(t)\,\varphi(t+\tau)\le\theta\mathcal{M}^d(\tau,N) \tag{2.33}$$

for all $t\ge 0$ and $\tau\ge 0$ with some $\theta\ge 1$ independent of t and τ. Then for any $0<B\le 1$ and $N\ge 1$

$$\int_{B/\sqrt{N}}^{1}\frac{\varphi(t)}{t}\,dt\le c(s)\,\theta\,(N^{-1}+(B\sqrt{N})^{-d/2})\quad \textit{for}\quad d>8. \tag{2.34}$$

Examples. Assuming condition (2.33) the bound (2.34) has an optimal dependence on N in the following sense. There are two examples of sequences of functions $\varphi_N(t)$ and $\psi_N(t)$, $N=1,2,\dots$ satisfying (2.33) with $\theta=1$ and such that

$$\begin{aligned} \int_{N^{-1/4}}^{1}\frac{\varphi_N(t)}{t}\,dt &\ge \frac{\ln N}{4N}\quad \text{for}\quad d=8, N=2,3,\dots, \\ \int_{N^{-2/d}}^{1}\frac{\psi_N(t)}{t}\,dt &\ge \frac{c(d)}{N}\quad \text{for}\quad d\ge 9, N\ge\frac{3}{2^{2/d}-1}. \end{aligned}$$

See Seleznev (1999). In fact, it is enough to take

$$\begin{aligned} \varphi_N(t) &= \begin{cases} 1-Nt & \text{for}\quad 0\le t\le N^{-1}, \\ 0 & \text{for}\quad N^{-1}\le t\le N^{-1/4}-N^{-2}, \\ N(t-N^{-1/4})+N^{-1} & \text{for}\quad N^{-1/4}-N^{-2}\le t\le N^{-1/4}, \\ N^{-1} & \text{for}\quad N^{-1/4}\le t, \end{cases} \\ \psi_N(t) &= \begin{cases} 1-Nt & \text{for}\quad 0\le t\le N^{-1}, \\ 0 & \text{for}\quad N^{-1}\le t\le 1-N^{-1}, \\ N(t-1+N^{-1})/2 & \text{for}\quad 1-N^{-1}\le t, \end{cases} \\ c(d) &= (4-(2^{2/d}-1)^2)^{-1}. \end{aligned}$$

3. Proofs of the Theorems

Proof of Theorem 1.1 We write

$$\Delta_n(a) \le \Delta_{n1} + \Delta_{n2}, \tag{3.1}$$

where

$$\begin{aligned} \Delta_{n1} &= \sup_x |F(x) - G(x)|, \\ \Delta_{n2} &= \sup_x |G(x) - F_0(x) - F_1(x)|, \\ G(x) &= P\{|(Z_1 + \cdots + Z_n)/\sqrt{n} - a|^2 \le x\} \end{aligned}$$

and $Z_1, Z_2, \ldots, Z_n$ are i.i.d. random elements constructed in Lemma 2.8 when we take $X = X_1$. Denote

$$\hat{G}(t) = \mathbb{E}e\{t|(Z_1 + \cdots + Z_n)/\sqrt{n} - a|^2\}.$$

Using Theorem 2 in Petrov (1987), ch.5, sect.1. we get

$$\Delta_{n1} \le c_1 \int_{-N}^{N} \frac{|\hat{F}(t) - \hat{G}(t)|}{|t|} dt + \frac{c_2}{N} \sup_{x>0} |\int_{-\infty}^{\infty} e\{-tx\}\, \hat{G}(t)\, dt|. \tag{3.2}$$

According to the construction of Z_1 it follows from Lemma 2.2 that

$$\sup_{x>0} |\int_{-\infty}^{\infty} e\{-tx\}\, \hat{G}(t)\, dt| \le c \cdot c_2(V). \tag{3.3}$$

We write

$$\int_{-N}^{N} \frac{|\hat{F}(t) - \hat{G}(t)|}{|t|} dt \le I_1 + I_2 + I_3, \tag{3.4}$$

where

$$\begin{aligned} I_1 &= \int_{-N_0}^{N_0} \frac{|\hat{F}(t) - \hat{G}(t)|}{|t|} dt, \\ I_2 &= \int_{N_0<|t|\le N} \frac{|\hat{F}(t)|}{|t|} dt, \\ I_3 &= \int_{N_0<|t|\le N} \frac{|\hat{G}(t)|}{|t|} dt. \end{aligned}$$

It follows from Lemma 2.1 that

$$I_3 \le c \cdot \frac{c_5(V)}{N_0}. \tag{3.5}$$

Using the coincidence of the moments of the first three orders for the distributions of X_1 and Z_1 (see Lemma 2.8) and applying Lemma 2.11 and standard arguments (see e.g. Sazonov, Ulyanov and Zalesskii (1988a, 1991)) we get

$$I_1 \leq c \cdot \frac{\overline{c}_9(V)}{n}(\mathbb{E}|X_1|^4 + \mathbb{E}(X_1, a)^4). \tag{3.6}$$

We now consider Δ_{n2}.

We shall use the following approximation formulae for the Fourier inversion. A smoothing inequality of Prawitz (1972) implies (see Bentkus and Götze (1996), Section 4) that

$$F(x) = \frac{1}{2} + \frac{i}{2\pi} V.P. \int_{-K}^{K} e\{-xt\} \hat{F}(t) \frac{dt}{t} + R, \tag{3.7}$$

for any $K > 0$ and any distribution F with characteristic function $\hat{F}$, where

$$|R| \leq \frac{1}{K} \int_{-K}^{K} |\hat{F}(t)| dt.$$

Here V.P. $\int f(t)dt = \lim_{\varepsilon \to 0} \int_{|t|>\varepsilon} f(t)dt$ denotes the principal value of the integral.

For any function $F : \mathbb{R} \to \mathbb{R}$ of bounded variation such that $F(-\infty) = 0$ and $2F(x) = F(x+) + F(x-)$ for all $x \in \mathbb{R}$ the following Fourier-Stieltjes inversion formula holds (see e.g. Chung (1974))

$$F(x) = \frac{1}{2}F(\infty) + \frac{i}{2\pi} \lim_{M\to\infty} V.P. \int_{|t|\leq M} e\{-xt\}\ \hat{F}(t)\ \frac{dt}{t}. \tag{3.8}$$

The formula is well-known for the distribution functions. For functions of bounded variation it extends by linearity arguments.

Using (3.7) we can write

$$G(x) - F_0(x) = \frac{i}{2\pi} V.P. \int_{-n}^{n} e\{-xt\}(\hat{G}(t) - \hat{F}_0(t))\frac{dt}{t} + R_1, \tag{3.9}$$

where similarly to (3.5) we obtain

$$|R_1| \leq c \cdot \frac{\overline{c}_3(V)}{n}. \tag{3.10}$$

Applying (3.8) to $F_1(x)$ and using (3.9) we get

$$\begin{aligned} &G(x) - F_0(x) - F_1(x) \\ &= \frac{i}{2\pi} V.P. \int_{-n}^{n} \frac{e\{-xt\}}{t}(\hat{G}(t) - \hat{F}_0(t) - \hat{F}_1(t))dt + R_2, \end{aligned} \tag{3.11}$$

where by Lemma 2.1 and (3.10) we have for R_2

$$|R_2| \le c \cdot \frac{\overline{c}_5(V)}{n} \,(\mathbb{E}|X_1|^3 + \mathbb{E}|(X_1, a)|^3). \tag{3.12}$$

By Lemmas 2.2 and 2.10 combining (3.11) and (3.12) we get

$$\Delta_{n2} \le c \cdot \frac{c_{12}(V)}{n}(\mathbb{E}|X_1|^4 + \mathbb{E}(X_1, a)^4)(1 + (Va, a)). \tag{3.13}$$

Combining (3.1)-(3.6) and (3.13) we arrive at (1.8).

Proof of Theorem 1.2. Let m_0 denote an integer satisfying (2.8) with $d = 9$. Put $S_{m_0} = m_0^{-1/2}(\overline{X}_1 + \cdots + \overline{X}_{m_0})$, where $\overline{X}_i$ denotes the symmetrization of X_i.

Since for any real non-negative random variable Z such that $\mathbb{E}Z = 1,\ \mathbb{E}Z^2 \le A$ we have

$$\mathrm{P}\{Z > 0.5\} \ge 0.25\, A^{-2},$$

it follows from Lemmas 2.3 and 2.4 that

$$\mathrm{P}\{W_{m_0}(\overline{X}) > \sigma_1^4 \cdot \cdots \cdot \sigma_9^4\} \ge c. \tag{3.14}$$

Replacing the nondegeneracy condition $\mathcal{N}(p, \delta, s, Z)$ (see (3.2) in Bentkus and Götze (1999)) by the condition (3.14) and applying Lemmas 6.3, 6.7 and 7.1 from Bentkus and Götze (1999) and our Lemma 2.5 we get

$$\int_{N_0 \le t \le N} \frac{|f_n(t)|}{|t|}\, dt \le c \left(\frac{m_0}{n} + \int_{N_0/N}^{1} \frac{\varphi(t)}{t}\, dt \right), \tag{3.15}$$

where $\varphi(t)$ is a continuous function such that $0 \le \varphi \le 1$ and

$$\varphi(t)\ \varphi(t + \tau) \le c\, c_9(V)\ \mathcal{M}^9(\tau N m_0/n,\ n/m_0). \tag{3.16}$$

Since for any $\varepsilon > 0$

$$\mathcal{M}(\tau\varepsilon,\ n) = \sqrt{\varepsilon}\ \mathcal{M}(\tau,\ n\epsilon^2),$$

an application of Lemmas 2.12, (3.15) and (3.16) concludes the proof of Theorem 1.2.

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