FROM STABLE EQUIVALENCES TO DERIVED EQUIVALENCES FOR TWISTED MULTIFOLD TRIVIAL EXTENSIONS OF PIECEWISE HEREDITARY ALGEBRAS

HIDETO ASASHIBA

Dedicated to Professor Yukio Tsushima on the occasion of his 60-th birthday

Throughout this paper k denotes an algebraically closed field. All algebras considered here are basic connected finite-dimensional k-algebras. The stable category $\underline{\mathrm{mod}}\,\Lambda$ of the category $\mathrm{mod}\,\Lambda$ of finite-dimensional right modules over a selfinjective algebra Λ has the canonical structure of a triangulated category. We say that selfinjective algebras Λ and Π are *stably equivalent* in case the stable module categories $\underline{\mathrm{mod}}\,\Lambda$ and $\underline{\mathrm{mod}}\,\Pi$ are equivalent as triangulated categories. It is well-known that if two algebras are derived equivalent, then they are stably equivalent (Keller-Vossieck [12, 2.3], Rickard [14, Theorem 2.1]). We are interested in the cases when the converse statement holds. In [2] we classified representation-finite selfinjective algebras, and as a corollary we have obtained that two such algebras are derived equivalent if and only if they are stably equivalent, namely the converse statement holds for these algebras.

Here recall that the *frequency* of a representation-finite selfinjective algebra $\Lambda(\neq k)$ is equal to the rational number s/n where s is the number of isoclasses of simple Λ-modules and n the number of vertices of the tree class of Λ (see [2, 2.1]). In this paper the converse statement for algebras with integral frequency will be generalized to a wider class containing representation-infinite algebras.

Let A be an algebra and n an integer ≥ 1. Then an algebra of the form $T^n_\phi(A) := \hat{A}/\langle\hat{\phi}\nu_A^n\rangle$ for some automorphism ϕ of A, is called a *twisted n-fold trivial extension* of A, where $\hat{A}$ denotes the repetition (= the repetitive algebra in Hughes and Waschbüsch [10]) of A, $\hat{\phi}$ is the automorphism of $\hat{A}$ induced by ϕ in an obvious way, and ν_A is the Nakayama automorphism of $\hat{A}$ (see [2, 2.3] in which $\hat{A}$ and $\hat{\phi}$ were denoted by $A^{\mathbb{Z}}$ and $\phi^{\mathbb{Z}}$, respectively). A twisted m-fold trivial extension of A for some integer $m \geq 1$ is called a *twisted multifold trivial extension* of A, and it turns out to be a selfinjective algebra. Note that if ϕ is the identity automorphism $\mathbb{1}$ of A and $n = 1$, then $T^1_{\mathbb{1}}(A)$ is isomorphic to the trivial extension $T(A) := A \ltimes DA$ of A by the A-A-bimodule $DA := \mathrm{Hom}_k(A, k)$, from which the terminology was taken. Using this notion representation-finite selfinjective algebras with integral frequency are characterized as follows (see [2, 6.1]):

Proposition. *Let Λ be an algebra and n an integer ≥ 1. Then Λ is a representation-finite selfinjective algebra with frequency n if and only if it is* (*isomorphic to*) *a twisted n-fold trivial extension of an algebra which is tilted from a hereditary algebra kQ defined by a Dynkin quiver Q.*

An algebra A is called *piecewise hereditary* if it is derived equivalent to a hereditary algebra H (Happel [8, IV.1]). Note here that the ordinary quiver Q of H has no oriented cycles and $H \cong kQ$. This Q is called a *type* of A (type is uniquely determined up to "reflections"). We are now in a position to state our main result in this paper.

Main Theorem. *Let Λ and Π be twisted multifold trivial extensions of piecewise hereditary algebras A and B, respectively. Assume that a type of A is an oriented tree. Then Λ and Π are derived equivalent if and only if they are stably equivalent.*

Note that a type of a piecewise hereditary algebra A is an oriented tree if and only if the first Hochschild cohomology $H^1(A) := H^1(A, A)$ of A vanishes (see, e.g., Happel [9, Theorem 2.2]).

In section 2 we recall fundamental facts about the repetition of an algebra, the repetition of an isomorphism and right modules over the repetition of an algebra; and prepare Lemmas 2.5 and 2.6 to investigate the action of an automorphism of a hereditary algebra A on the stable Auslander-Reiten quiver of the repetition of A. In section 3 we collect necessary facts on derived equivalences from [2]. We make full use of Proposition 3.2 (= [2, Proposition 5.4.3]) to deduce derived equivalences between twisted multifold trivial extensions. In section 4 we first recall facts about trees, and then give a special orientation on a tree, which is needed in later sections. Section 5 is devoted to a reduction of the problem on piecewise hereditary algebras to hereditary tree algebras. In section 6 we show that the underlying graphs and folding numbers are invariant under stable equivalences (Lemma 6.2). In section 7 we show that in many cases stably equivalent twisted multifold trivial extensions of a hereditary tree algebra are isomorphic. Finally in section 8 we give a proof of Main Theorem.

1. Preliminaries

In the sequel every tree considered here is assumed to be *finite and connected.* For a quiver Q we denote by $\overline{Q}$, by Q_0, by Q_1 and by kQ the underlying graph of Q, the set of vertices of Q, the set of arrows of Q and the path-category defined by Q, respectively. An algebra A is called a *tree* algebra if its ordinary quiver is an oriented tree.

For an additive category $\mathcal{A}$, we denote by $\mathcal{H}(\mathcal{A})$ and $\mathcal{H}^{\mathrm{b}}(\mathcal{A})$ the homotopy category of differential complexes and the homotopy category of bounded differential complexes in $\mathcal{A}$, respectively; and when $\mathcal{A}$ is an exact category, we denote by $\mathcal{D}(\mathcal{A})$ and by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ the corresponding derived categories.

Recall from Gabriel-Roiter [7] that a small category A is called a *spectroid* (= a *locally finite-dimensional category* in [6]) if the following three conditions are satisfied:

(i) Distinct objects of A are not isomorphic;
(ii) Every object of A has a local endomorphism algebra; and
(iii) The space $A(x, y)$ is finite-dimensional for every $x, y \in A$.

For a spectroid A, A is called *finite* if A has only a finite number of objects; and A is called *locally bounded* if for every $x \in A$, there are only finitely many $y \in A$ such that $A(x, y) \neq 0$ or $A(y, x) \neq 0$.

For a spectroid A, we denote by $\operatorname{Mod} A$ the category of all (right) A-modules (= contravariant functors from A to the category $\operatorname{Mod} k$ of k-vector spaces); by $\operatorname{mod} A$ the full subcategory of $\operatorname{Mod} A$ consisting of finitely presented objects; and by $\operatorname{pro} A$ the full subcategory of $\operatorname{Mod} A$ consisting of finitely generated projective objects. In addition, $\underline{\operatorname{mod}} A$ denotes the stable category of $\operatorname{mod} A$. By $D := \operatorname{Hom}_k(\text{-}, k)$, by Ω_A, by τ_A, by Γ_A and by ${}_{\mathrm{s}}\Gamma_A$ we denote the usual selfduality of A, the loop functor $\underline{\operatorname{mod}} A \to \underline{\operatorname{mod}} A$, the Auslander-Reiten translation of A, the Auslander-Reiten quiver of A and the stable Auslander-Reiten quiver of A, respectively.

As in [1] we regard every algebra A as a finite spectroid, namely fixing a set $\operatorname{obj}(A) := \{e_1, \dots, e_p\}$ of orthogonal local idempotents of A with $e_1 + \cdots + e_p = 1$, A is identified with the finite spectroid $c(A)$ defined as follows: the set of objects of $c(A)$ is $\operatorname{obj}(A)$ and $c(A)(x, y) := yAx$ for all $x, y \in \operatorname{obj}(A)$ and the composition of $c(A)$ is given by the multiplication of A. Therefore, in particular, automorphisms of A are required to preserve the set $\operatorname{obj}(A)$. The expression $x \in A$ stands for $x \in \operatorname{obj}(A)$.

By $\operatorname{Aut}(X)$ we denote the group of automorphisms of a quiver, a graph, a translation quiver or a spectroid X.

2. Repetitions

First we recall the definition of the repetition of an algebra (from [2, 1.2] to fix the notation) and the way how to compute repetitions for tree algebras from a more general construction in [2, 1.3, 1.5].

Definition. Let A be an algebra.

(1) A k-category $\hat{A}$, called the *repetition* of A, is defined as follows (Cf. [10], [7]). Objects are the pairs $x^{[n]} := (x, n)$ with $x \in A$ and $n \in \mathbb{Z}$.

$$\hat{A}(x^{[n]}, y^{[m]}) := \begin{cases} \{f^{[n]} := (f, n) | f \in A(x, y)\} & \text{if } m = n; \\ \{\varphi^{[n]} := (\varphi, n) | \varphi \in DA(y, x)\} & \text{if } m = n + 1; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The composition

$$\hat{A}(y^{[m]}, z^{[l]}) \times \hat{A}(x^{[n]}, y^{[m]}) \to \hat{A}(x^{[n]}, z^{[l]})$$

is given as follows:

(i) If $m = n, l = m$, then this is the composition of A:

$$A(y, z) \times A(x, y) \to A(x, z).$$

(ii) If $m = n, l = m + 1$, then this is given by the right A-module structure of $DA(-, ?)$:

$$DA(z, y) \times A(x, y) \to DA(z, x).$$

(iii) If $m = n + 1, l = m$, then this is given by the left A-module structure of $DA(-, ?)$:

$$A(y, z) \times DA(y, x) \to DA(z, x).$$

(iv) Otherwise the composition is zero.

(2) For each $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $\hat{A}$ formed by $x^{[n]}$ with $x \in A$, and by $\mathbb{1}^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow \hat{A}, x \mapsto x^{[n]}$, the embedding functor.

(3) The Nakayama automorphism ν_A of $\hat{A}$ is defined by

- $\nu_A(x^{[n]}) := x^{[n+1]}$;
- $\nu_A(f^{[n]}) := f^{[n+1]}$; and
- $\nu_A(\varphi^{[n]}) := \varphi^{[n+1]}$

for all $x \in A$ and for all $f \in A(x, y)$, $\varphi \in DA(y, x)$ with $x, y \in A$.

Note that $\hat{A}$ is a locally bounded spectroid.

A path μ from y to x in a quiver with relations (Q, I) is called *maximal* if $\mu \notin I$ but $\alpha\mu, \mu\beta \in I$ for each arrow α of Q with tail x and each β with head y. (This was called *complete* in [2].) For a k-vector space V with a basis $\{v_1, \dots, v_n\}$ we denote by $\{v_1^*, \dots, v_n^*\}$ the basis of DV dual to the basis $\{v_1, \dots, v_n\}$. In particular if $\dim_k V = 1$, $v^* \in DV$ is defined for all $v \in V \backslash \{0\}$.

Lemma 2.1. *Let A be a tree algebra and $\Phi\colon kQ \to A$ a display-functor* ([7, 8.1, 8.3a]) *with $I := \operatorname{Ker}\Phi$. Then*

(1) *Φ uniquely induces the following display-functor $\hat{\Phi}\colon k\hat{Q} \to \hat{A}$ for $\hat{A}$:*
  (i) *$\hat{Q}$ is obtained from the disjoint union $Q \times \mathbb{Z}$ of copies $Q^{[i]}$ of Q by adding arrows $\alpha_{y,x}^{[i]}\colon x^{[i]} \to y^{[i+1]}$ for all $i \in \mathbb{Z}$, whenever there is a maximal path μ from y to x in (Q, I) (we sometimes denote this $\alpha_{y,x}^{[i]}$ by $\mu^{*[i]}$).*
  (ii) *$\hat{\Phi}$ is defined as follows:*
    - *$\hat{\Phi}(x^{[i]}) := (\Phi x)^{[i]}$;*
    - *$\hat{\Phi}(\alpha^{[i]}) := (\Phi\alpha)^{[i]}$; and*
    - *$\hat{\Phi}(\alpha_{y,x}^{[i]}) := (\Phi(\mu)^*)^{[i]}$*

  for all $i \in \mathbb{Z}$, all vertices x of Q, all arrows α of Q and all pairs (x, y) of vertices of Q such that there is a maximal path μ from y to x in (Q, I).

(2) *$\operatorname{Ker}\hat{\Phi}$ is equal to the ideal $\hat{I}$ defined by the full commutativity relations on $\hat{Q}$ and the zero relations $\mu = 0$ for those paths μ of $\hat{Q}$ for which there is no path from the head of μ to the $x^{[i+1]}$ with $x^{[i]}$ the tail of μ. (Therefore note that if a path $\alpha_n \dots \alpha_1$ is in I, then $\alpha_n^{[i]} \dots \alpha_1^{[i]}$ is in $\hat{I}$ for all $i \in \mathbb{Z}$.)*

To recall the repetition of an isomorphism we cite the following from [2, 2.3] without proof.

Lemma 2.2. *Let $\psi\colon A \to B$ be an isomorphism of algebras. Denote by $\psi_x^y\colon A(y, x) \to B(\psi y, \psi x)$ the isomorphism defined by ψ for all $x, y \in A$. Define $\hat{\psi}\colon \hat{A} \to \hat{B}$ as follows.*

For each $x^{[i]} \in \hat{A}$, $\hat{\psi}(x^{[i]}) := (\psi x)^{[i]}$;
For each $f^{[i]} \in \hat{A}(x^{[i]}, y^{[i]})$, $\hat{\psi}(f^{[i]}) := (\psi f)^{[i]}$; and
For each $\varphi^{[i]} \in \hat{A}(x^{[i]}, y^{[i+1]})$, $\hat{\psi}(\varphi^{[i]}) := (D((\psi_x^y)^{-1})(\varphi))^{[i]} = (\varphi \circ (\psi_x^y)^{-1})^{[i]}$.

Then

(1) *$\hat{\psi}$ is an isomorphism.*
(2) *Given an isomorphism $\rho\colon \hat{A} \to \hat{B}$, the following are equivalent.*
  (a) *$\rho = \hat{\psi}$;*
  (b) *ρ satisfies the following.*
    (i) *$\rho\nu_A = \nu_B\rho$;*
    (ii) *$\rho(A^{[0]}) = A^{[0]}$;*

(iii) *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle 1^{[0]}}\big\downarrow & & \big\downarrow{\scriptstyle 1^{[0]}} \\ A^{[0]} & \xrightarrow[\rho]{} & B^{[0]} \end{array}$$

is commutative; and

(iv) $\rho(\varphi^{[0]}) = (\varphi \circ (\psi_x^y)^{-1})^{[0]}$ *for all* $x, y \in A$ *and all* $\varphi \in DA(y, x)$.

Lemma 2.3. *Let* $\rho : A \to B$ *and* $\sigma : B \to C$ *be isomorphisms of algebras. Then we have* $(\sigma\rho)\hat{} = \hat{\sigma}\hat{\rho}$.

Proof. This is straightforward by the definition of the repetition of isomorphisms. □

We recall a presentation of right modules over the repetition of an algebra. For an algebra A define a category $\mathcal{E} = \mathcal{E}_A$ as follows (cf. [8, 2.1]) (note that we are dealing with right modules, not with left modules as in [8]). The objects of $\mathcal{E}$ are the sequences $(M_i, m_i)_{i\in\mathbb{Z}}$ with $M_i \in \operatorname{mod} A$ for all i, $M_i = 0$ for almost all i, and $m_i \in \operatorname{Hom}_A(M_i \otimes_A DA, M_{i-1})$ for all i satisfying $m_{i-1} \circ (m_i \otimes_A DA) = 0$, and for each $(M_i, m_i)_i, (N_i, n_i)_i \in \mathcal{E}$ the morphism space $\mathcal{E}((M_i, m_i)_i, (N_i, n_i)_i)$ is defined as the set of all $(f_i)_{i\in\mathbb{Z}} \in \prod_{i\in\mathbb{Z}} \operatorname{Hom}_A(M_i, N_i)$ such that $n_i \circ (f_i \otimes_A DA) = f_{i-1} \circ m_i$ for all $i \in \mathbb{Z}$.
Then the following is well-known.

Lemma 2.4. *Let* A *be an algebra. Then the category* $\operatorname{mod} \hat{A}$ *of finite-dimensional* (*right*) $\hat{A}$*-modules is equivalent to the category* $\mathcal{E}_A$.

We identify $\operatorname{mod} \hat{A}$ with $\mathcal{E}_A$ by this lemma.

Using this description of $\operatorname{mod} \hat{A}$ the canonical embedding $\operatorname{mod} A \to \operatorname{mod} \hat{A}$ is defined by $M \mapsto (M_i, m_i)_i$ with $M_0 = M, M_i = 0$ for all $i \neq 0$ and $m_i = 0$ for all $i \in \mathbb{Z}$. Denote by $\eta : \operatorname{mod} A \to \underline{\operatorname{mod}}\, \hat{A}$ the composite of this embedding and the canonical functor $\operatorname{mod} \hat{A} \to \underline{\operatorname{mod}}\, \hat{A}$. Then by [8, Lemma 2.3] η is a full embedding. The following is easy to verify.

Lemma 2.5. *Let* $A = kQ$ *for some quiver* Q. *Then* $Q_0' := \{\eta(A(\text{-}, x)) | x \in Q_0\}$ *forms a section of the component of* ${}_{\mathrm{s}}\Gamma_{\hat{A}}$ *containing* Q_0'. *Therefore if we put* Q' *to be the full subquiver of* ${}_{\mathrm{s}}\Gamma_{\hat{A}}$ *defined by* Q_0', *then the component of* ${}_{\mathrm{s}}\Gamma_{\hat{A}}$ *containing* Q' *is identified with* $\mathbb{Z}Q'$.

Let A be an algebra and $\phi \in \operatorname{Aut}(A)$. Then ϕ induces an equivalence ${}^{\phi}(\text{-}) : \operatorname{mod} A \to \operatorname{mod} A$ defined by ${}^{\phi}M := M \circ \phi^{-1} : A \to \operatorname{mod} k$ for all $M \in \operatorname{mod} A$. Similarly $\hat{\phi} \in \operatorname{Aut}(\hat{A})$ induces an equivalence ${}^{\hat{\phi}}(\text{-}) : \operatorname{mod} \hat{A} \to \operatorname{mod} \hat{A}$. With this notation we have the following, which is used to investigate the action of ϕ on components of the stable Auslander-Reiten quiver of $\hat{A}$.

Lemma 2.6. *Let* A *be an algebra and* $\phi \in \operatorname{Aut}(A)$. *Then we have* ${}^{\hat{\phi}}(\eta(X)) \cong \eta({}^{\phi}X)$ *for all* $X \in \operatorname{mod} A$.

Proof. This is straitforward. □

3. Derived equivalences

For a group G acting on a class S we say that a subclass E of S is *G-stable* if $gx \in E$ for all $g \in G$ and $x \in E$.

Let A be an algebra and $\phi \in \mathrm{Aut}(A)$. Then ϕ acts on $\mathrm{mod}\, A$ as ${}^{\phi}(\text{-})$. In particular for $A(\text{-}, x)$ with $x \in A$, we have ${}^{\phi}(A(\text{-}, x)) = A(\phi^{-1}(\text{-}), x) \cong A(\text{-}, \phi x)$, and the last isomorphism is given by ϕ itself. Therefore the subset $\{A(\text{-}, x) \mid x \in A\}$ of $\mathrm{pro}\, A$ is not $\langle {}^{\phi}(\text{-})\rangle$-stable in a strict sense. This makes it difficult to give explicitly a complete set of representatives of isoclasses of indecomposable objects of $\mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, A)$ which is $\langle \mathcal{H}^{\mathrm{b}}({}^{\langle\phi\rangle}(\text{-}))\rangle$-stable. To avoid this difficulty we used in [2] the formal additive hull $\oplus A$ of A defined below instead of $\mathrm{pro}\, A$.

Definition. Let A be a spectroid. Then by $\oplus A$ we denote the following category ([7, 2.1 Example 8]). Objects are finite sequences $(x_1, \dots, x_n)$ of objects of A; morphisms are defined by $(\oplus A)((x_1, \dots, x_n), (y_1, \dots, y_m)) := \{(\mu_{ji})_{i,j} | \mu_{ji} \in A(x_i, y_j), \text{for all } i, j\}$ for all objects $(x_1, \dots, x_n), (y_1, \dots, y_m)$; and the composition is given by the matrix multiplication. We regard that A is contained in $\oplus A$ by the embedding $(f \colon x \to y) \mapsto ((f) \colon (x) \to (y))$ for all f in A.

Remark. Let A and ϕ be as above.

(1) Define a functor $\eta_A \colon \oplus A \to \mathrm{pro}\, A$ by $(x_1, \dots, x_n) \mapsto A(-, x_1) \oplus \cdots \oplus A(-, x_n)$ and $(\mu_{ji})_{i,j} \mapsto (A(-, \mu_{ji}))_{i,j}$. Then η_A is an equivalence, called the *Yoneda* equivalence.
(2) Let $F \colon A \to B$ be a functor of spectroids. Then F naturally induces functors $\oplus F \colon \oplus A \to \oplus B$ and $\tilde{F} := \mathcal{H}^{\mathrm{b}}(\oplus F) \colon \mathcal{H}^{\mathrm{b}}(\oplus A) \to \mathcal{H}^{\mathrm{b}}(\oplus B)$, which are equivalences if F is an isomorphism. Namely, $\oplus F$ is defined by $(x_1, \dots, x_n) \mapsto (Fx_1, \dots, Fx_n)$ and $(\mu_{ji}) \mapsto (F\mu_{ji})$ for all objects $(x_1, \dots, x_n)$ and all morphisms (μ_{ji}) in $\oplus A$; and $\tilde{F}$ is defined by $\oplus F$ componentwise.
(3) The automorphism ϕ acts on $\mathcal{H}^{\mathrm{b}}(\oplus A)$ as $\tilde{\phi}$, and ${}^{\phi}\mathcal{H}^{\mathrm{b}}(\eta_A)(X^{\bullet}) \cong \mathcal{H}^{\mathrm{b}}(\eta_A)(\tilde{\phi}(X^{\bullet}))$ for all $X^{\bullet} \in \mathcal{H}^{\mathrm{b}}(\oplus A)$.

We cite the following from [2, Proposition 5.1] which follows from Keller [11] (Cf. Rickard [13], [1, Proposition 1.1]).

Proposition 3.1. *Let A and B be spectroids. Then the following are equivalent:*

(1) *There is a triangle equivalence $\mathcal{D}(\mathrm{Mod}\, B) \to \mathcal{D}(\mathrm{Mod}\, A)$; and*
(2) *There is a full subcategory E of $\mathcal{H}^{\mathrm{b}}(\oplus A)$ such that*
   (a) *For any $T, U \in E$ and any $n \neq 0$, $\mathcal{H}^{\mathrm{b}}(\oplus A)(T, U[n]) = 0$;*
   (b) *A is contained in the smallest full triangulated subcategory of $\mathcal{H}^{\mathrm{b}}(\oplus A)$ containing E that is closed under direct summands and isomorphisms; and*
   (c) *E is isomorphic to B.*

Definition. We say that spectroids A and B are *derived equivalent* if one of the equivalent conditions above holds. In (2) the triple (A, E, B) is called a *tilting triple* and $E \subseteq \mathcal{H}^{\mathrm{b}}(\oplus A)$ is called a *tilting spectroid* for A.

The following statement which we proved in [2, Proposition 5.4.3] is very useful.

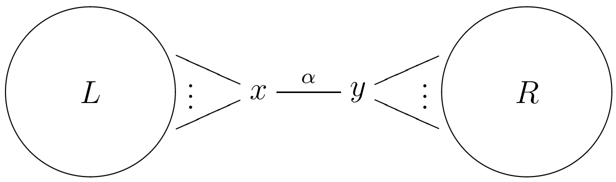
Proposition 3.2. *Let (A, E, B) be a tilting triple of locally bounded spectroids, $\eta\colon E \xrightarrow{\sim} B$ an isomorphism and $\phi \in \mathrm{Aut}(A)$. Assume that E is $\langle\tilde{\phi}\rangle$-stable. Then $T^n_\phi(A)$ is derived equivalent to $T^n_\psi(B)$ for any integer $n \geq 1$, where ψ is an automorphism of B defined by the commutativity of the diagram*

$$\begin{tikzcd} E \arrow[r, "\eta"] \arrow[d, "\tilde{\phi}"'] & B \arrow[d, "\psi"] \\ E \arrow[r, "\eta"'] & B \end{tikzcd}.$$

4. A special orientation of a tree

Let T be a tree with the set of vertices T_0. Then we denote by $\mathrm{Aut}(T)$ the group of all automorphisms of T. For each $f \in \mathrm{Aut}(T)$ we set $\mathrm{Fix}(f) := \{x \in T_0 | f(x) = x\}$. Consider the set $\mathrm{Aut}_0(T) := \{f \in \mathrm{Aut}(T) | \mathrm{Fix}(f) \neq \emptyset\}$. Then it is well known that the set $\bigcap_{f\in\mathrm{Aut}_0(T)} \mathrm{Fix}(f)$ is non-empty. Therefore $\mathrm{Aut}_0(T)$ forms a subgroup of $\mathrm{Aut}(T)$; and $\mathrm{Aut}_0(T) = \mathrm{Aut}(T)$ holds if and only if T *has a fixed vertex*, i.e., $\bigcap_{f\in\mathrm{Aut}(T)} \mathrm{Fix}(f) \neq \emptyset$ (an element of this set is called a *fixed vertex* of T). Further it is well known that if $\mathrm{Aut}_0(T) \neq \mathrm{Aut}(T)$, then $\mathrm{Aut}(T)/\mathrm{Aut}_0(T)$ has order 2 and T has a unique edge $\alpha : x$—y such that $f(\alpha) = \alpha$ for any $f \in \mathrm{Aut}(T)$, and T has the following form

(4.1)



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where L and R are subtrees of T isomorphic to each other. Here both vertices x and y are fixed by all $f \in \mathrm{Aut}_0(T)$, and $g(x) = y$ and $g(y) = x$ for all $g \in \mathrm{Aut}(T) \setminus \mathrm{Aut}_0(T)$.

Note that if a quiver Q is obtained from T by giving an orientation ρ, then each automorphism of Q is considered as an automorphism of T preserving the orientation ρ, i.e., the group $\mathrm{Aut}(Q)$ of all automorphisms of Q is a subgroup of $\mathrm{Aut}(T)$. By the remark above we further have $\mathrm{Aut}(Q) \subseteq \mathrm{Aut}_0(T)$. We call the orientation ρ *admissible* if $\mathrm{Aut}(Q) = \mathrm{Aut}_0(T)$.

Lemma 4.1. *Let T be a tree. Then there is an admissible orientation ρ of T such that the quiver $Q := (T, \rho)$ has a unique source.*

Proof. Take $x \in \bigcap_{f\in\mathrm{Aut}_0(T)} \mathrm{Fix}(f)$. Define an orientation ρ as follows: for each edge $a : y$—z in T, make a to be an arrow $y \to z$ if and only if $d(x, y) < d(x, z)$, where $d(x, t)$ is the distance between x and t for each $t \in T_0$.

(1) x is a unique source of Q. Indeed, since $d(x, x) = 0$ and $d(x, y) = 1$ for all neighbours y of x, x is a source. If $y \in T_0$ is not equal to x, then there is a unique shortest linear subtree combining x and y of length (= the number of edges) $d(x, y) > 0$. Let z be the neighbour of y in this subtree. Then $d(x, z) = d(x, y) - 1 < d(x, y)$. Thus Q has an arrow $z \to y$. Hence y is not a source.

(2) ρ is admissible. Indeed, let f be in $\mathrm{Aut}_0(T)$, and $y \to z$ an arrow in Q. Then $f(x) = x$ and $d(x, y) < d(x, z)$. Since f preserves the distance we have $d(x, f(y)) = d(f(x), f(y)) < d(f(x), f(z)) = d(x, f(z))$. Thus we have an arrow $f(y) \to f(z)$ in Q. Hence $f \in \mathrm{Aut}(Q)$. $\square$

5. Reduction to hereditary tree algebras

For an algebra A recall from [8, III.1.4] that the dimension vector $\underline{\dim}\, X^{\bullet}$ of an $X^{\bullet} = (X^i, d^i)_{i\in\mathbb{Z}} \in \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ is defined by

$$\underline{\dim}\, X^{\bullet} = \sum_{i\in\mathbb{Z}} (-1)^i \,\underline{\dim}\, X^i, \tag{5.1}$$

where $\underline{\dim}\, X^i$ is the usual dimension vector of the A-module X_i for all i.

We denote by $\bigvee_{i\in I} \mathcal{X}_i$ the disjoint union of a family $(\mathcal{X}_i)_{i\in I}$ of translation quivers, thus $\mathcal{X}_i$ forms a connected component in $\bigvee_{i\in I} \mathcal{X}_i$ for each $i \in I$.

In the next lemma we collect fundamental facts on the Auslander-Reiten quiver of the derived category of a piecewise hereditary algebra.

Lemma 5.1. *Let A be a piecewise hereditary algebra having a type Q and let $n = \mathrm{rank}\, K_0(A)$. Then*

1. *The Auslander-Reiten quiver $\Gamma_{\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)}$ of $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ has the form*
$$\Gamma_{\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)} = \bigvee_{i\in\mathbb{Z}} (\mathcal{X}_i \vee \mathcal{R}_i),$$
where $\mathcal{X}_i \cong \mathbb{Z}Q$ and R_i is isomorphic to the union of all regular components of the Auslander-Reiten quiver of kQ ($\mathcal{R}_i = \emptyset$ if Q is Dynkin, and is a union of quasi-serial components otherwise) for all $i \in \mathbb{Z}$, and the shift maps $\mathcal{X}_i$ to $\mathcal{X}_{i+1}$ and $\mathcal{R}_i$ to R_{i+1}; and further $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)(\mathcal{X}_i \vee \mathcal{R}_i, \mathcal{X}_j \vee \mathcal{R}_j) = 0$ if $i > j$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)(\mathcal{R}_i, \mathcal{X}_i) = 0$ for all i.
2. *The map $\underline{\dim} : \mathcal{X}_i \to \mathbb{Z}^n$ defined by $X^{\bullet} \mapsto \underline{\dim}\, X^{\bullet}$ is injective for each $i \in \mathbb{Z}$.*
3. *If $F : \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A) \to \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, A)$ is an equivalence of triangulated categories, then there is a unique $j \in \mathbb{Z}$ such that $F(\mathcal{X}_i) \subseteq \mathcal{X}_{i+j}$ and $F(\mathcal{R}_i) \subseteq \mathcal{R}_{i+j}$ for all $i \in \mathbb{Z}$. We call this j the* jump *of F.*

Proof. (1) Using a derived equivalence between A and kQ the assertion is reduced to the corresponding statement on kQ, which follows from [8, Proposition I.5.5].

(2) By [8, Proposition III.1.5] the assertion is also reduced to the corresponding one on kQ, which is easy to verify using (5.1).

(3) Since $\mathcal{X}_0$ is not isomorphic to any component of R_i for any $i \in \mathbb{Z}$, there is some $j \in \mathbb{Z}$ such that $F(\mathcal{X}_0) \subseteq \mathcal{X}_j$. Then since F commutes with the shift, we have $F(\mathcal{X}_i) \subseteq \mathcal{X}_{i+j}$ for all $i \in \mathbb{Z}$. Further since for each $i \in \mathbb{Z}$ there exist nonzero morphisms $X^{\bullet} \to Y^{\bullet} \to Z^{\bullet}$ with $X^{\bullet} \in \mathcal{X}_i$, $Y^{\bullet} \in \mathcal{R}_i$ and $Z^{\bullet} \in \mathcal{X}_{i+1}$, we have nonzero morphisms $FX^{\bullet} \to FY^{\bullet} \to FZ^{\bullet}$ with $FX^{\bullet} \in \mathcal{X}_{i+j}$ and $FZ^{\bullet} \in X_{i+j+1}$, which shows that $FY^{\bullet} \in \mathcal{R}_{i+j}$ by (1). Hence $F(\mathcal{R}_i) \subseteq \mathcal{R}_{i+j}$. The uniqueness of j is obvious. $\square$

Lemma 5.2. *Let A be an algebra that is derived equivalent to an algebra kQ defined by a tree Q oriented by an admissible orientation. Then there is a tilting triple (A, E, kQ) such that E is $\langle\tilde{\phi}\rangle$-stable for all $\phi \in \mathrm{Aut}(A)$.*

Proof. Put $B := kQ$. Since A has finite global dimension, we have identifications $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,A) = \mathcal{H}^{\mathrm{b}}(\mathrm{pro}\,A) = \mathcal{H}^{\mathrm{b}}(\oplus A)$ in an obvious way. We keep the notation of the previous lemma. Since A is derived equivalent to B, we have an equivalence

$$F : \mathcal{H}^{\mathrm{b}}(\oplus B) \to \mathcal{H}^{\mathrm{b}}(\oplus A)$$

of triangulated categories. Let E be the full subcategory of $\mathcal{H}^{\mathrm{b}}(A)$ with $\{F(x) \mid x \in B\}$ the set of objects. Then (A, E, B) is a tilting triple. Since the objects of B are contained in a component of the form $\mathbb{Z}Q$, we may assume that $E \subseteq \mathcal{X}_0$. Let Q_E be the full subquiver of $\mathcal{X}_0$ formed by the objects of E. Then Q_E is a section of $\mathcal{X}_0$ because the full subquiver Q_B of $\Gamma_{\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,B)}$ formed by the objects of B is a section of the component of $\Gamma_{\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,B)}$ containing it. Of course $Q_E \cong Q_B \cong Q$, and by these isomorphisms we identify Q with Q_E. Then $\mathcal{X}_0 = \mathbb{Z}Q$. Let ϕ be an automorphism of A. We show that E is $\langle\tilde{\phi}\rangle$-stable.

Now since A has only finitely many objects and the automorphism ϕ of A induces a permutation of objects of A, there is some integer $m \geq 1$ such that $\phi^m(a) = a$ for all $a \in A$. Thus we have

$$\tilde{\phi}^m(a) = a \quad \text{for all } a \in A. \tag{5.2}$$

This implies that $\tilde{\phi}^m(X) = X$ for all $X \in \oplus A$. Therefore by (5.1) we have $\underline{\dim}\,\tilde{\phi}^m(X^\bullet) = \underline{\dim}\,X^\bullet$ for all $X^\bullet \in \mathcal{H}^{\mathrm{b}}(\mathrm{pro}\,A) = \mathcal{H}^{\mathrm{b}}(\oplus A)$. Again by (5.2) the jump of $\tilde{\phi}$ must be zero, i.e., we have $\tilde{\phi}(\mathcal{X}_0) = \mathcal{X}_0$. Then $\tilde{\phi}$ induces an automorphism ρ of $\mathcal{X}_0 = \mathbb{Z}Q$. Hence for each $X^\bullet \in \mathcal{X}_0$ we have $\rho^m(X^\bullet) \in \mathcal{X}_0$ and $\underline{\dim}\,\rho^m(X^\bullet) = \underline{\dim}\,X^\bullet$. Therefore by Lemma 5.1(2) we obtain

$$\rho^m(X^\bullet) = X^\bullet \text{ for all } X^\bullet \in \mathcal{X}_0. \tag{5.3}$$

By Riedtmann [15, Theorem 4.2], we have an exact sequence

$$1 \longrightarrow \langle\tau\rangle \longrightarrow \mathrm{Aut}(\mathbb{Z}Q) \xrightarrow{p} \mathrm{Aut}(\overline{Q}) \longrightarrow 1, \tag{5.4}$$

where p is the canonical morphism. Put $\psi := p(\rho) \in \mathrm{Aut}(\overline{Q})$. We show the following.

$$\psi \in \mathrm{Aut}_0(\overline{Q}) \tag{5.5}$$

Suppose that $\psi \notin \mathrm{Aut}_0(\overline{Q})$. Then $\overline{Q}$ has the form (4.1). We show that ρ^m does not have any fixed vertex, which contradicts (5.3). If m is odd, then since ψ exchanges L and R in (4.1), so does $\psi^m = p(\rho^m)$, which shows that ρ^m cannot have fixed vertex. If m is even, then there is some $0 \neq t \in \mathbb{Z}$ such that $\rho^2(i, z) = (i + t, \psi^2(z))$ for all $(i, z) \in \mathbb{Z}Q$. Then also in this case ρ^m cannot have fixed vertex, and (5.5) is proved. Since $\mathrm{Aut}_0(\overline{Q}) = \mathrm{Aut}(Q)$, (5.5) enables us to define $\mathbb{Z}\psi \in \mathrm{Aut}(\mathbb{Z}Q)$ by $\mathbb{Z}\psi(i, z) = (i, \psi(z))$ for all $(i, z) \in \mathbb{Z}Q$. Then since $p(\rho) = \psi = p(\mathbb{Z}\psi)$, we have $\rho = (\mathbb{Z}\psi)\tau^l$ for some $l \in \mathbb{Z}$ by (5.4). But by (5.3) we have $l = 0$, thus $\rho = \mathbb{Z}\psi$. As a consequence $Q = Q_E$ is $\langle\rho\rangle$-stable, which means that E is $\langle\tilde{\phi}\rangle$-stable. $\square$

The following proposition says that a twisted multifold trivial extension of an algebra derived equivalent to a hereditary tree algebra is derived equivalent to a twisted multifold trivial extension of a hereditary tree algebra.

Proposition 5.3. *Let A be an algebra that is derived equivalent to an algebra kQ defined by a tree Q oriented by an admissible orientation. Then for any $\phi \in \mathrm{Aut}(A)$ and any integer $n \geq 1$ there is some $\psi \in \mathrm{Aut}(kQ)$ such that $T^n_\phi(A)$ is derived equivalent to $T^n_\psi(kQ)$.*

Proof. This is immediate by the lemma above and Proposition 3.2. $\square$

6. Invariants under stable equivalences

Let A be a locally bounded spectroid. We denote by $\mathcal{N}_A$ the Nakayama functor $\mathrm{mod}\, A \to \mathrm{mod}\, A$, which is known to be an equivalence when A is selfinjective. It is well-known that $\mathcal{N}_A \cong \Omega_A^{-2}\tau_A$ on $\underline{\mathrm{mod}}\, A$. Further when $A = \hat{B}$ for some algebra B, $\mathcal{N}_{\hat{B}} \cong {}^{\nu_B}(\text{-})$ on $\mathrm{mod}\, \hat{B}$.

The next lemma is a special case of [5, 3.5]. We give an alternative proof of it in this case using [8, Theorem 4.9].

Lemma 6.1. *Let H be a hereditary algebra. Then $\hat{H}$ is locally support-finite.*

Proof. By [8, Theorem 4.9] there exists an equivalence $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, H) \to \underline{\mathrm{mod}}\, \hat{H}$ such that the restriction to $\mathrm{mod}\, H$ is the identity. By this equivalence we identify $\Gamma_{\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, H)}$ and ${}_{\mathrm{s}}\Gamma_{\hat{H}}$. We use the notation in Lemma 5.1. We may assume that Γ_H is contained in $\mathcal{X}_0 \vee \mathcal{R}_0 \vee \mathcal{X}_1$. Then, in particular, the $\Omega_{\hat{H}}$-shifts of Γ_H cover the whole ${}_{\mathrm{s}}\Gamma_{\hat{H}}$, i.e.,

$$\bigcup_{i\in\mathbb{Z}} \Omega^i_{\hat{H}}((\Gamma_H)_0) = ({}_{\mathrm{s}}\Gamma_{\hat{H}})_0 \tag{6.1}$$

because the shifts of Γ_H cover the whole $\Gamma_{\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, H)}$. By the definition of the canonical embedding $\mathrm{mod}\, H \to \underline{\mathrm{mod}}\, \hat{H}$ the supports of modules in Γ_H as $\hat{H}$-modules are contained in $H^{[0]}$. Since the supports of the projective covers of modules supported in $H^{[0]}$ are contained in $H^{[0]} \cup H^{[-1]}$ by definition of repetitions, the $\Omega_{\hat{H}}$-shifts of modules in Γ_H are supported in $H^{[0]} \cup H^{[-1]}$. Now by (6.1) for any module X in ${}_{\mathrm{s}}\Gamma_{\hat{H}}$ there is some $m \in \mathbb{Z}$ such that $\mathcal{N}^m_{\hat{H}}(X) = \Omega^{-2m}_{\hat{H}}\tau^m_{\hat{H}}(X)$ belongs to $\Gamma_H \cup \Omega_{\hat{H}}(\Gamma_H)$ and is supported in $H^{[0]} \cup H^{[-1]}$. Hence X is supported in $H^{[-m]} \cup H^{[-1-m]}$ because $\mathcal{N}_{\hat{H}} = {}^{\nu_H}(\text{-})$. Accordingly, $\hat{H}$ is locally support-finite. $\square$

Lemma 6.2. *Let A and B be algebras derived equivalent to kQ and kR for some oriented tree Q and some quiver R, respectively. Assume that a twisted m-fold trivial extension of A is stably equivalent to a twisted n-fold trivial extension of B for some integers $m, n \geq 1$. Then we have $\overline{Q} \cong \overline{R}$ and $m = n$.*

Proof. We may assume that the orientation of Q is admissible. Put $H := kQ$. Let $\phi_A \in \mathrm{Aut}(A)$ and $\phi_B \in \mathrm{Aut}(B)$ and put $\Lambda = T^m_{\phi_A}(A)$ and $\Pi = T^n_{\phi_B}(B)$. Assume that Λ is stably equivalent to Π. We show that $\overline{Q} \cong \overline{R}$ and $m = n$. By Proposition 5.3 there is a $\psi \in \mathrm{Aut}(H)$ such that Λ is derived equivalent to $\Lambda' := T^m_\psi(H)$. By Lemma

6.1 $\hat{H}$ is locally support-finite. Then by [4, Proposition 2.5] and [6, Theorem 3.6] the canonical Galois covering functor $\hat{H} \to \Lambda'$ with group $\langle \hat{\psi}\nu_H^m \rangle$ gives an isomorphism ${}_{\mathrm{s}}\Gamma_{\hat{H}}/\langle {}^{\hat{\psi}\nu_H^m}(\text{-}) \rangle \cong {}_{\mathrm{s}}\Gamma_{\Lambda'}$. Since the jump of ${}^{\hat{\psi}\nu_H^m}(\text{-})$ is equal to $2m$, we get

$$ {}_{\mathrm{s}}\Gamma_{\Lambda} \cong {}_{\mathrm{s}}\Gamma_{\Lambda'} \cong \bigvee_{i=0}^{2m-1} (\mathcal{X}_i \vee \mathcal{R}_i), $$

where $\mathcal{X}_i \cong \mathbb{Z}Q$ and $\mathcal{R}_i$ is empty or a union of quasi-serial components for each i. Similarly by [6, Theorem 3.6(b)] the canonical Galois covering functor $\hat{B} \to \Pi$ with group $\langle \hat{\phi_B}\nu_B^n \rangle$ yields an isomorphism of the quotient

$$ {}_{\mathrm{s}}\Gamma_{\hat{B}}/\langle {}^{\hat{\phi_B}\nu_B^n}(\text{-}) \rangle \cong \bigvee_{i=0}^{2n-1} (\mathcal{X}'_i \vee \mathcal{R}'_i), $$

onto the union of some components of ${}_{\mathrm{s}}\Gamma_{\Pi}$, where $\mathcal{X}'_i \cong \mathbb{Z}R$ and $\mathcal{R}'_i$ is empty or a union of quasi-serial components for each i. But since Λ and Π are stably equivalent, we have ${}_{\mathrm{s}}\Gamma_{\Pi} \cong {}_{\mathrm{s}}\Gamma_{\Lambda} \cong {}_{\mathrm{s}}\Gamma_{\Lambda'}$, which implies that $\bigvee_{i=0}^{2n-1}(\mathcal{X}'_i \vee \mathcal{R}'_i)$ is isomorphic to the union of some components of $\bigvee_{i=0}^{2m-1}(\mathcal{X}_i \vee \mathcal{R}_i)$. Therefore $\mathbb{Z}R \cong \mathcal{X}'_0 \cong \mathcal{X}_i \cong \mathbb{Z}Q$ for some i. Then by [8, I.5.7] we have $\overline{Q} = \overline{R}$. Thus R is also an oriented tree. Applying the argument above for Q to R we see that ${}_{\mathrm{s}}\Gamma_{\Pi} \cong \bigvee_{i=0}^{2n-1}(\mathcal{X}'_i \vee \mathcal{R}'_i)$. Hence we have

$$ \bigvee_{i=0}^{2m-1} (\mathcal{X}_i \vee \mathcal{R}_i) \cong \bigvee_{i=0}^{2n-1} (\mathcal{X}'_i \vee \mathcal{R}'_i). $$

Counting the number of components isomorphic to $\mathbb{Z}Q$ we get $m = n$. □

7. Hereditary tree algebras

Let Q be an oriented tree and put $A := kQ$. Note in this case that if there is an arrow $\alpha : x \to y$ in Q, then $A(x, y) = k\alpha$. If $\phi \in \mathrm{Aut}(A)$, then ϕ induces an isomorphism $\operatorname{rad} A(x, y)/\operatorname{rad}^2 A(x, y) \to \operatorname{rad} A(\phi(x), \phi(y))/\operatorname{rad}^2 A(\phi(x), \phi(y))$. This shows that there is a unique arrow $\phi(x) \to \phi(y)$ in Q, which we denote by $\pi(\phi)(\alpha)$. Since $A(\phi(x), \phi(y)) = k\pi(\phi)(\alpha)$, we have $\phi(\alpha) = \phi_\alpha \pi(\phi)(\alpha)$ for a unique $\phi_\alpha \in k^\times := k \backslash \{0\}$. By setting $\pi(\phi)(x) := \phi(x)$ for all $x \in Q_0$, we can define a $\pi(\phi) \in \mathrm{Aut}(Q)$. In this way we obtain a homomorphism

$$ \pi : \mathrm{Aut}(A) \to \mathrm{Aut}(Q) $$

of groups, which is called the canonical map. Moreover $\phi \in \operatorname{Ker} \pi$ if and only if $\phi(x) = x$ for all $x \in Q_0$, and then the correspondence $\phi \mapsto (\phi_\alpha)_{\alpha \in Q_1}$ provides us an identification

$$ \operatorname{Ker} \pi = (k^\times)^{Q_1}. $$

For a morphism $\beta : Q \to R$ of quivers, we denote by $k\beta$ the induced algebra homomorphism $kQ \to kR$. Note that if $Q \xrightarrow{\beta} R \xrightarrow{\gamma} S$ are morphisms of quivers, then we have $k(\gamma\beta) = (k\gamma)(k\beta)$.

Proposition 7.1. *Let Q be a tree oriented by an admissible orientation, $\phi, \psi \in \mathrm{Aut}(kQ)$ and n an integer ≥ 1. Assume that $T^n_\phi(kQ)$ is stably equivalent to $T^n_\psi(kQ)$. Then $\pi(\phi)$ is conjugate to $\pi(\psi)$ in $\mathrm{Aut}(\overline{Q})$.*

Proof. Put $A := kQ$, $\Lambda := T^n_\phi(A)$ and $\Pi := T^n_\psi(A)$. Let $F : \underline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Pi$ be an equivalence (of triangulated categories). Then F commutes with both the Auslander-Reiten translations and the loop functors, i.e., we have

$$\tau_\Pi FX \cong F\tau_\Lambda X;\ \text{and}$$
$$\Omega_\Pi FX \cong F\Omega_\Lambda X$$

for all $X \in \underline{\mathrm{mod}}\,\Lambda$. Since $\mathcal{N}_B \cong \Omega_B^{-2}\tau_B$ on $\underline{\mathrm{mod}}\,B$ for any algebra B, we have

$$\mathcal{N}_\Pi FX \cong F\mathcal{N}_\Lambda X, \tag{7.1}$$

for all $X \in \underline{\mathrm{mod}}\,\Lambda$. Now let $\lambda_\Lambda : \mathrm{mod}\,\hat{A} \to \mathrm{mod}\,\Lambda$ and $\lambda_\Pi : \mathrm{mod}\,\hat{A} \to \mathrm{mod}\,\Pi$ be pushdown functors. Then since they are exact and preserve projectivity and almost split sequences, they commute with both the Auslander-Reiten translations and the loop functors. Hence we have

$$\mathcal{N}_\Lambda\lambda_\Lambda X \cong \lambda_\Lambda\mathcal{N}_{\hat{A}} X, \tag{7.2}$$

$$\mathcal{N}_\Pi\lambda_\Pi X \cong \lambda_\Pi\mathcal{N}_{\hat{A}} X, \tag{7.3}$$

for all $X \in \underline{\mathrm{mod}}\,\hat{A}$. Since $\hat{A}$ is locally support-finite by Lemma 6.1, we have ${}_{\mathrm{s}}\Gamma_\Lambda \cong {}_{\mathrm{s}}\Gamma_{\hat{A}}/\langle {}^{\hat{\phi}\nu^n_{\hat{A}}}(\text{-})\rangle = {}_{\mathrm{s}}\Gamma_{\hat{A}}/\langle {}^{\hat{\phi}}(\text{-}) \circ \mathcal{N}^n_{\hat{A}}\rangle$ and ${}_{\mathrm{s}}\Gamma_\Pi \cong {}_{\mathrm{s}}\Gamma_{\hat{A}}/\langle {}^{\hat{\psi}}(\text{-}) \circ \mathcal{N}^n_{\hat{A}}\rangle$. Thus

$${}_{\mathrm{s}}\Gamma_\Lambda = \bigvee_{i=0}^{2n-1} (\mathcal{X}_i \vee \mathcal{R}_i),\ \text{and}$$
$${}_{\mathrm{s}}\Gamma_\Pi = \bigvee_{i=0}^{2n-1} (\mathcal{X}'_i \vee \mathcal{R}'_i)$$

with $\mathcal{X}_i \cong \mathcal{X}'_i \cong \mathbb{Z}Q$ and $\mathcal{R}_i \cong \mathcal{R}'_i \cong \bigvee(\text{regular components of } A)$ for all i. Define Q_Λ (resp. Q_Π) to be the full subquiver of ${}_{\mathrm{s}}\Gamma_\Lambda$ (resp. ${}_{\mathrm{s}}\Gamma_\Pi$) with the set of vertices $\{\lambda_\Lambda\eta(A(\text{-},x)) | x \in Q_0\}$ (resp. $\{\lambda_\Pi\eta(A(\text{-},x)) | x \in Q_0\}$). Then by Lemma 2.5 we can identify, say, $\mathcal{X}_0 = \mathbb{Z}Q_\Lambda$ and $\mathcal{X}'_0 = \mathbb{Z}Q_\Pi$. Since $\Omega_\Pi : \underline{\mathrm{mod}}\,\Pi \to \underline{\mathrm{mod}}\,\Pi$ is an equivalence of triangulated categories, we can replace F by $\Omega^a_\Pi F$, if necessary, for some $a \in \mathbb{Z}$ to have $F(\mathcal{X}_0) = \mathcal{X}'_0$. Then F induces an isomorphism $\mathbb{Z}Q_\Lambda \to \mathbb{Z}Q_\Pi$. Since $\hat{\phi}$ (resp. $\hat{\psi}$) commutes with ν_A by Lemma 2.2, we see that ${}^{\hat{\phi}}(\text{-})$ (resp. ${}^{\hat{\psi}}(\text{-})$) commutes with ${}^{\hat{\phi}}(\text{-}) \circ \mathcal{N}^n_{\hat{A}}$ (resp. ${}^{\hat{\psi}}(\text{-}) \circ \mathcal{N}^n_{\hat{A}}$). Therefore we can define $\phi' \in \mathrm{Aut}({}_{\mathrm{s}}\Gamma_\Lambda)$ and $\psi' \in \mathrm{Aut}({}_{\mathrm{s}}\Gamma_\Pi)$ by the commutative diagrams

$$\begin{array}{ccc} {}_{\mathrm{s}}\Gamma_{\hat{A}} & \xrightarrow{{}^{\hat{\phi}}(\text{-})} & {}_{\mathrm{s}}\Gamma_{\hat{A}} \\ {\scriptstyle\lambda_\Lambda}\downarrow & & \downarrow{\scriptstyle\lambda_\Lambda} \\ {}_{\mathrm{s}}\Gamma_\Lambda & \xrightarrow[\phi']{} & {}_{\mathrm{s}}\Gamma_\Lambda \end{array}, \qquad \begin{array}{ccc} {}_{\mathrm{s}}\Gamma_{\hat{A}} & \xrightarrow{{}^{\hat{\psi}}(\text{-})} & {}_{\mathrm{s}}\Gamma_{\hat{A}} \\ {\scriptstyle\lambda_\Pi}\downarrow & & \downarrow{\scriptstyle\lambda_\Pi} \\ {}_{\mathrm{s}}\Gamma_\Pi & \xrightarrow[\psi']{} & {}_{\mathrm{s}}\Gamma_\Pi \end{array} \tag{7.4}$$

of translation quivers. Define an isomorphism $f_\Lambda : Q \to Q_\Lambda$ (resp. $f_\Pi : Q \to Q_\Pi$) of quivers by $f_\Lambda(x) := \lambda_\Lambda\eta(A(\text{-}, x))$ (resp. $f_\Pi(x) := \lambda_\Pi\eta(A(\text{-}, x))$) for all $x \in Q_0$. Then by Lemma 2.6 and (7.4) we have commutative diagrams:

$$\begin{tikzcd} Q \arrow[r, "\pi(\phi)"] \arrow[d, "f_\Lambda"', "\wr"] & Q \arrow[d, "\wr"', "f_\Lambda"] \\ Q_\Lambda \arrow[r, "\phi'|_{Q_\Lambda}"'] & Q_\Lambda \end{tikzcd}, \qquad \begin{tikzcd} Q \arrow[r, "\pi(\psi)"] \arrow[d, "f_\Pi"', "\wr"] & Q \arrow[d, "\wr"', "f_\Pi"] \\ Q_\Pi \arrow[r, "\psi'|_{Q_\Pi}"'] & Q_\Pi \end{tikzcd}.$$

These yield commutative diagrams:

$$\begin{tikzcd} \mathbb{Z}Q \arrow[r, "\mathbb{Z}\pi(\phi)"] \arrow[d, "\mathbb{Z}f_\Lambda"', "\wr"] & \mathbb{Z}Q \arrow[d, "\wr"', "\mathbb{Z}f_\Lambda"] \\ \mathbb{Z}Q_\Lambda \arrow[r, "\phi'"'] & \mathbb{Z}Q_\Lambda \end{tikzcd}, \qquad \begin{tikzcd} \mathbb{Z}Q \arrow[r, "\mathbb{Z}\pi(\psi)"] \arrow[d, "\mathbb{Z}f_\Pi"', "\wr"] & \mathbb{Z}Q \arrow[d, "\wr"', "\mathbb{Z}f_\Pi"] \\ \mathbb{Z}Q_\Pi \arrow[r, "\psi'"'] & \mathbb{Z}Q_\Pi \end{tikzcd}. \tag{7.5}$$

Next we show the commutativity of the following diagram.

$$\begin{tikzcd} \mathbb{Z}Q_\Lambda \arrow[r, "\phi'"] \arrow[d, "F"'] & \mathbb{Z}Q_\Lambda \arrow[d, "F"] \\ \mathbb{Z}Q_\Pi \arrow[r, "\psi'"'] & \mathbb{Z}Q_\Pi \end{tikzcd} \tag{7.6}$$

Since F, ϕ', ψ' are quiver morphisms, and since $\mathcal{X}_0 = \mathbb{Z}Q_\Lambda$ has no double arrows, it is enough to show that $F\phi' x = \psi' F x$ for all vertices x of $\mathcal{X}_0$. Let x be a vertex of $\mathcal{X}_0$. Then there exist vertices y and z of ${}_{\mathrm{s}}\Gamma_{\hat{A}}$ such that $\lambda_\Lambda(y) = x$ and $\lambda_\Pi(z) = Fx$. Since $\lambda_\Lambda({}^{\hat{\phi}}(\mathcal{N}_{\hat{A}}^n y)) = \lambda_\Lambda(y) = x$ and $\lambda_\Pi({}^{\hat{\psi}}(\mathcal{N}_{\hat{A}}^n z)) = \lambda_\Pi(z) = Fx$, we have

$$F\lambda_\Lambda({}^{\hat{\phi}}(\mathcal{N}_{\hat{A}}^n y)) = \lambda_\Pi({}^{\hat{\psi}}(\mathcal{N}_{\hat{A}}^n z)).$$

Now by (7.2), (7.1) and (7.4) we have

$$\begin{aligned} F\lambda_\Lambda({}^{\hat{\phi}}(\mathcal{N}_{\hat{A}}^n y)) &= F\lambda_\Lambda \mathcal{N}_{\hat{A}}^n({}^{\hat{\phi}}y) \\ &= F\mathcal{N}_\Lambda^n \lambda_\Lambda({}^{\hat{\phi}}y) \\ &= \mathcal{N}_\Pi^n F\lambda_\Lambda({}^{\hat{\phi}}y) \\ &= \mathcal{N}_\Pi^n F\phi'\lambda_\Lambda y \\ &= \mathcal{N}_\Pi^n F\phi' x. \end{aligned}$$

Further by (7.3) and (7.4), we have

$$\begin{aligned} \lambda_\Pi({}^{\hat{\psi}}(\mathcal{N}_{\hat{A}}^n z)) &= \lambda_\Pi \mathcal{N}_{\hat{A}}^n({}^{\hat{\psi}}z) \\ &= \mathcal{N}_\Pi^n \lambda_\Pi({}^{\hat{\psi}}z) \\ &= \mathcal{N}_\Pi^n \psi' \lambda_\Pi z \\ &= \mathcal{N}_\Pi^n \psi' F x. \end{aligned}$$

It follows from these equalities that $\mathcal{N}_\Pi^n F\phi' x = \mathcal{N}_\Pi^n \psi' F x$ for all $x \in \mathcal{X}_0$. This verifies the commutativity of (7.6) because $\mathcal{N}_\Pi$ is an equivalence.

Define an isomorphism $\alpha : \mathbb{Z}Q \to \mathbb{Z}Q$ by putting $\alpha := (\mathbb{Z}f_\Pi)^{-1} F \mathbb{Z} f_\Lambda$. Then by (7.5) and (7.6) we have a commutativity of the inner central square of the following diagram

$$\begin{tikzcd}
\overline{Q} \arrow[rrr, "\pi(\phi)"] \arrow[ddd, "p(\alpha)"', "\wr"] & & & \overline{Q} \arrow[ddd, "p(\alpha)", "\wr"'] \\
& \mathbb{Z}Q \arrow[ul, "\mathrm{pr}"'] \arrow[r, "\mathbb{Z}\pi(\phi)"] \arrow[d, "\alpha"', "\wr"] & \mathbb{Z}Q \arrow[ur, "\mathrm{pr}"] \arrow[d, "\alpha", "\wr"'] & \\
& \mathbb{Z}Q \arrow[dl, "\mathrm{pr}"] \arrow[r, "\mathbb{Z}\pi(\psi)"'] & \mathbb{Z}Q \arrow[dr, "\mathrm{pr}"'] & \\
\overline{Q} \arrow[rrr, "\pi(\psi)"'] & & & \overline{Q}
\end{tikzcd},$$

where $\mathrm{pr} : \mathbb{Z}Q \to \overline{Q}$ and $p : \mathrm{Aut}(\mathbb{Z}Q) \to \mathrm{Aut}(\overline{Q})$ are the canonical morphisms. Since the four trapezoids are commutative and since pr is surjective both on vertices and on arrows, the outer square of the diagram is also commutative. Hence $\pi(\phi)$ is conjugate to $\pi(\psi)$ in $\mathrm{Aut}(\overline{Q})$. $\square$

Proposition 7.2. *Let Q be an oriented tree with a unique source and n an integer ≥ 1. Then we have $T_\phi^n(kQ) \cong T_{k\pi(\phi)}^n(kQ)$ for all $\phi \in \mathrm{Aut}(kQ)$. Therefore $T_\phi^n(kQ)$ is determined by $\pi(\phi)$.*

Proof. Put $A := kQ$ and $\psi := \pi(\phi) \in \mathrm{Aut}(Q)$. Let $\Phi : kQ \to A$ be the identity, which is a display-functor. Then we have a display-functor $\hat{\Phi} : k\hat{Q} \to \hat{A}$ where $\hat{I} := \mathrm{Ker}\, \hat{\Phi}$ is generated by full commutativity relations and some zero relations by Lemma 2.1. We put $\overline{\beta} := \hat{\Phi}(\beta)$ for all morphisms β in $k\hat{Q}$. Since $\pi(\phi) = \pi(k\psi)$, there is some $\lambda \in \mathrm{Ker}\, \pi = (k^\times)^{Q_1}$ such that $\phi = \lambda \cdot k\psi$. Put $g := \widehat{(k\psi)}\nu_A^n$. Then $\hat{\phi}\nu_A^n = \hat{\lambda} g$ by Lemma 2.3 and we have to show that $\hat{A}/\langle \hat{\lambda} g \rangle \cong \hat{A}/\langle g \rangle$. To this end it is enough to construct an automorphism μ of $\hat{A}$ such that the diagram

$$\begin{tikzcd}
\hat{A} \arrow[r, "\mu"] \arrow[d, "g"'] & \hat{A} \arrow[d, "\hat{\lambda} g"] \\
\hat{A} \arrow[r, "\mu"'] & \hat{A}
\end{tikzcd} \tag{7.7}$$

is commutative.

First we construct an automorphism ρ of $k\hat{Q}$, and then we verify that $\rho(\hat{I}) \subseteq \hat{I}$, which makes it possible for ρ to induce an endomorphism μ of $\hat{A}$. Here μ turns out to be an automorphism because $\hat{I}$ is locally finite-dimensional. Finally we check that this μ makes the diagram (7.7) commutative.

Now set $\rho(x) := x$ for all vertices x in $\hat{Q}$ and $\rho(\gamma) := \rho_\gamma \cdot \gamma$ for all arrows γ in $\hat{Q}$, where $\rho_\gamma \in k^\times$ are defined as follows:

(i) In the case where $\gamma = \alpha^{[i]}$ with $\alpha \in Q_1$ and $i \in \mathbb{Z}$, define ρ_γ inductively using the rules

$$\begin{cases} \rho_{\alpha^{[i]}} := 1 & \text{for } i \in \{0, 1, \dots, n-1\}; \\ \rho_{\alpha^{[j+n]}} := \rho_{(\psi^{-1}\alpha)^{[j]}} \cdot \lambda_\alpha & \text{for } i \geq n; \text{ and} \\ \rho_{\alpha^{[j-n]}} := \rho_{(\psi\alpha)^{[j]}} \cdot \lambda_{\psi\alpha}^{-1} & \text{for } i \leq -1. \end{cases}$$

Then for all $\alpha \in Q_1$ and $i \in \mathbb{Z}$, we have

$$\rho_{(\psi\alpha)^{[i+n]}} = \rho_{\alpha^{[i]}} \cdot \lambda_{\psi\alpha}. \tag{7.8}$$

(ii) In the case where $\gamma = \beta^{*[i]}$ with $\beta = \alpha_m \dots \alpha_1$ a maximal path in Q and $i \in \mathbb{Z}$, define

$$\rho_{\beta^{*[i]}} := (\rho_{\alpha_m^{[i]}} \dots \rho_{\alpha_1^{[i]}})^{-1}. \tag{7.9}$$

Next we show that $\rho(\hat{I}) \subseteq \hat{I}$. For zero relations $\alpha_m \dots \alpha_1 \in \hat{I}$, we have

$$\begin{aligned} \overline{\rho(\alpha_m \dots \alpha_1)} &= \overline{\rho(\alpha_m) \dots \rho(\alpha_1)} \\ &= \rho_{\alpha_m} \dots \rho_{\alpha_1} \cdot \overline{\alpha_n \dots \alpha_1} \\ &= 0. \end{aligned}$$

Hence it is enough to check that $\overline{\rho(\mu_1)} = \overline{\rho(\mu_2)}$ for each pair of parallel paths μ_1, μ_2 from a vertex $x^{[i]}$ to a vertex $y^{[i+1]}$ for some $x, y \in Q_0$ and some $i \in \mathbb{Z}$ with $\overline{\mu_1} \neq 0$, $\overline{\mu_2} \neq 0$. For a path $\alpha_m \dots \alpha_1$ in $\hat{Q}$ (resp. in Q) we set $\rho_{\alpha_m \dots \alpha_1} := \rho_{\alpha_m} \dots \rho_{\alpha_1}$ (resp. $\lambda_{\alpha_m \dots \alpha_1} := \lambda_{\alpha_m} \dots \lambda_{\alpha_1}$). Now μ_1 and μ_2 have the form

$$\begin{cases} \mu_1 = \varepsilon^{[i+1]} \alpha^{*[i]} \gamma^{[i]} \\ \mu_2 = \zeta^{[i+1]} \beta^{*[i]} \delta^{[i]} \end{cases}$$

for some paths $\gamma, \delta, \varepsilon, \zeta$ in Q and maximal paths α, β in Q as in the following diagram ($a \overset{\sigma}{\rightsquigarrow} b$ denotes that σ is a path from a to b):

$$\begin{tikzcd}
& & x^{[i]} \arrow[dl, squiggly, "\delta^{[i]}"'] \arrow[dr, squiggly, "\gamma^{[i]}"] & \\
b^{[i]} \arrow[r, squiggly, "\beta^{[i]}"] & w^{[i]} \arrow[dl, "\beta^{*[i]}"'] & a^{[i]} \arrow[r, squiggly, "\alpha^{[i]}"] & z^{[i]} \arrow[dl, "\alpha^{*[i]}"] \\
b^{[i+1]} \arrow[dr, squiggly, "\zeta^{[i+1]}"'] & & a^{[i+1]} \arrow[dl, squiggly, "\varepsilon^{[i+1]}"] & \\
& y^{[i+1]} & &
\end{tikzcd}$$

By definition of ρ we have

$$\begin{aligned} \overline{\rho(\mu_1)} &= \overline{\rho(\varepsilon^{[i+1]}) \rho(\alpha^{*[i]}) \rho(\gamma^{[i]})} \\ &= \rho_{\varepsilon^{[i+1]}} {\rho_{\alpha^{[i]}}}^{-1} \rho_{\gamma^{[i]}} \cdot \overline{\mu_1}, \end{aligned}$$

and

$$\begin{aligned}\overline{\rho(\mu_2)} &= \rho_{\zeta^{[i+1]}}{\rho_{\beta^{[i]}}}^{-1}\rho_{\delta^{[i]}} \cdot \overline{\mu_2} \\ &= \rho_{\zeta^{[i+1]}}{\rho_{\beta^{[i]}}}^{-1}\rho_{\delta^{[i]}} \cdot \overline{\mu_1}.\end{aligned}$$

Hence it is enough to show that

$$\rho_{\varepsilon^{[i+1]}}{\rho_{\alpha^{[i]}}}^{-1}\rho_{\gamma^{[i]}} = \rho_{\zeta^{[i+1]}}{\rho_{\beta^{[i]}}}^{-1}\rho_{\delta^{[i]}}. \tag{7.10}$$

Since $0 \neq \overline{\mu_1} \in \hat{A}(x^{[i]}, y^{[i+1]}) \cong DA(y,x)$, we have $A(y,x) \neq 0$, which means that there is a path $\theta\colon y \rightsquigarrow x$ in Q. Therefore we have the following paths in Q:

$$\begin{tikzcd}
a \arrow[rrr, squiggly, "\alpha"] \arrow[dr, squiggly, "\varepsilon"] & & & z \\
& y \arrow[r, squiggly, "\theta"] & x \arrow[ur, squiggly, "\gamma"] \arrow[dr, squiggly, "\delta"] & \\
b \arrow[ur, squiggly, "\zeta"] \arrow[rrr, squiggly, "\beta"'] & & & w
\end{tikzcd}$$

Hence $\alpha = \gamma\theta\varepsilon$ and $\beta = \delta\theta\zeta$ because Q is an oriented tree. Then

$$\begin{aligned}\rho_{\varepsilon^{[i+1]}}{\rho_{\alpha^{[i]}}}^{-1}\rho_{\gamma^{[i]}} &= \rho_{\varepsilon^{[i+1]}}\left(\rho_{\gamma^{[i]}}\rho_{\theta^{[i]}}\rho_{\varepsilon^{[i]}}\right)^{-1}\rho_{\gamma^{[i]}} \\ &= \rho_{\varepsilon^{[i+1]}}{\rho_{\theta^{[i]}}}^{-1}{\rho_{\varepsilon^{[i]}}}^{-1},\end{aligned}$$

and

$$\rho_{\zeta^{[i+1]}}{\rho_{\beta^{[i]}}}^{-1}\rho_{\delta^{[i]}} = \rho_{\zeta^{[i+1]}}{\rho_{\theta^{[i]}}}^{-1}{\rho_{\zeta^{[i]}}}^{-1}.$$

Note that both a and b are sources in Q. Hence by assumption, we have $a = b$, thus $\varepsilon = \zeta$. Therefore the equality (7.10) holds. Hence the automorphism ρ of $k\hat{Q}$ induces an automorphism μ of $\hat{A}$. It remains to verify the commutativity of the diagram (7.7). This is obvious on objects . Therefore it is enough to check the commutativity on arrows in $\hat{Q}$, namely to show the following two facts:

$$\mu(g(\overline{\alpha^{[i]}})) = (\hat{\lambda}g)(\mu(\overline{\alpha^{[i]}})) \quad \text{for all } \alpha \in Q_1 \text{ and } i \in \mathbb{Z}\text{; and} \tag{7.11}$$

$$\mu(g(\overline{\beta^{*[i]}})) = (\hat{\lambda}g)(\mu(\overline{\beta^{*[i]}})) \quad \text{for all maximal path } \beta \text{ in } Q \text{ and } i \in \mathbb{Z}. \tag{7.12}$$

The statement (7.11) follows from (7.8). Indeed, using $\overline{(\psi\alpha)^{[i+n]}} = (\psi\alpha)^{[i+n]}$ in $\hat{A}$, we have

$$\begin{aligned}\mu(g(\overline{\alpha^{[i]}})) &= \overline{\rho((\psi\alpha)^{[i+n]})} \\ &= \rho_{(\psi\alpha)^{[i+n]}} \cdot \overline{(\psi\alpha)^{[i+n]}} \\ &= \rho_{\alpha^{[i]}}\lambda_{\psi\alpha} \cdot (\psi\alpha)^{[i+n]} \text{ in } \hat{A}\end{aligned}$$

by (7.8) and

$$\begin{aligned}(\hat{\lambda}g)(\mu(\overline{\alpha^{[i]}})) &= (\hat{\lambda}g)(\rho_{\alpha^{[i]}} \cdot \overline{\alpha^{[i]}}) \\ &= \rho_{\alpha^{[i]}}\hat{\lambda}((\psi\alpha)^{[i+n]}) \\ &= \rho_{\alpha^{[i]}}\lambda_{\psi\alpha} \cdot (\psi\alpha)^{[i+n]} \text{ in } \hat{A}.\end{aligned}$$

Thus (7.11) holds.

For (7.12) put $\beta := \alpha_m \dots \alpha_1 \colon\ y \rightsquigarrow x$. Then

$$\begin{aligned}g(\overline{\beta^{*[i]}}) &= (k\psi)\check{}(\beta^{*[i+n]}) \\ &= (\beta^* \circ (k\psi)^{-1})^{[i+n]} \\ &= ((\psi\beta)^*)^{[i+n]}.\end{aligned}$$

The last equality follows from $(\beta^* \circ (k\psi)^{-1})(\psi\beta) = 1$. Thus

$$\begin{aligned}\mu(g(\overline{\beta^{*[i]}})) &= \overline{\rho((\psi\beta)^{*[i+n]})} \\ &= \rho_{(\psi\beta)^{*[i+n]}} \cdot \overline{(\psi\beta)^{*[i+n]}} \\ &= (\rho_{(\psi\alpha_m)^{[i+n]}} \dots \rho_{(\psi\alpha_1)^{[i+n]}})^{-1} \cdot (\psi\beta)^{*[i+n]} \\ &= (\rho_{\alpha_m^{[i]}} \dots \rho_{\alpha_1^{[i]}}\lambda_{\psi\alpha_m} \dots \lambda_{\psi\alpha_1})^{-1} \cdot (\psi\beta)^{*[i+n]} \\ &= \rho_{\beta^{*[i]}}\lambda_{\psi\beta}^{-1} \cdot (\psi\beta)^{*[i+n]}.\end{aligned}$$

On the other hand,

$$\begin{aligned}g(\mu(\overline{\beta^{*[i]}})) &= g(\rho_{\beta^{*[i]}} \cdot \overline{\beta^{*[i]}}) \\ &= \rho_{\beta^{*[i]}} \cdot g(\overline{\beta^{*[i]}}) \\ &= \rho_{\beta^{*[i]}}(\psi\beta)^{*[i+n]},\end{aligned}$$

and

$$\begin{aligned}(\hat{\lambda}g)(\mu(\overline{\beta^{*[i]}})) &= \rho_{\beta^{*[i]}} \cdot \hat{\lambda}((\psi\beta)^{*[i+n]}) \\ &= \rho_{\beta^{*[i]}} \cdot ((\psi\beta)^* \circ \lambda^{-1})^{[i+n]} \\ &= \rho_{\beta^{*[i]}}\lambda_{\psi\beta}^{-1} \cdot (\psi\beta)^{*[i+n]}.\end{aligned}$$

The last equality follows from $((\psi\beta)^* \circ \lambda^{-1})(\lambda_{\psi\beta} \cdot \psi\beta) = 1 = (\lambda_{\psi\beta}^{-1} \cdot (\psi\beta)^*)(\lambda_{\psi\beta} \cdot \psi\beta)$. Thus (7.12) holds. As a consequence, the diagram (7.7) is commutative. □

Remark. (1) The proposition above is still valid even if we replace the word source by the word sink. In this case we can use the formula $\rho_{\beta^{*[i]}} := (\rho_{\alpha_m^{[i+1]}} \dots \rho_{\alpha_1^{[i+1]}})^{-1}$. instead of (7.9).

(2) The proposition above can be generalized by a similar proof as follows: Let (Q, I) be a commutative directed quiver such that two nonzero paths in (Q, I) starting from distinct tails of some maximal paths do not have the same head. Then $T_\phi^n(kQ/I) \cong T_{k\pi(\phi)}^n(kQ/I)$ for all $\phi \in \mathrm{Aut}(kQ/I)$.

Corollary 7.3. *Let Q be an oriented tree with a unique source, $\phi, \psi \in \mathrm{Aut}(kQ)$ and n an integer ≥ 1. If $\pi(\phi)$ is conjugate to $\pi(\psi)$ in $\mathrm{Aut}(Q)$, then we have $T^n_\phi(kQ) \cong T^n_\psi(kQ)$.*

Proof. By assumption there is a $\beta \in \mathrm{Aut}(Q)$ such that $\beta\pi(\phi) = \pi(\psi)\beta$. This implies that $(k\beta)\hat{}(k\pi(\phi))\hat{} = (k\pi(\psi))\hat{}(k\beta)\hat{}$. Put $A := kQ$. Since $(k\beta)\hat{}$ commutes with ν_A by Lemma 2.2, we have a commutative diagram

$$\begin{tikzcd} \hat{A} \arrow[r, "(k\pi(\phi))\hat{}\ \nu_A^n"] \arrow[d, "(k\beta)\hat{}"'] & \hat{A} \arrow[d, "(k\beta)\hat{}"] \\ \hat{A} \arrow[r, "(k\pi(\psi))\hat{}\ \nu_A^n"] & \hat{A}. \end{tikzcd}$$

This shows that $T^n_{k\pi(\phi)}(A) \cong T^n_{k\pi(\psi)}(A)$, from which the conclusion follows by Proposition 7.2. $\square$

Theorem 7.4. *Let Q be a tree with a fixed vertex oriented by an admissible orientation with a unique source. Then two twisted multifold trivial extensions of kQ are isomorphic if they are stably equivalent.*

Proof. In this case we have $\mathrm{Aut}(\overline{Q}) = \mathrm{Aut}_0(\overline{Q}) = \mathrm{Aut}(Q)$. Hence this is a direct consequence of Lemma 6.2, Proposition 7.1 and Corollary 7.3. $\square$

8. Proof of main theorem

It is enough to show that if Λ and Π are stably equivalent, then they are derived equivalent. By assumption A and B are algebras derived equivalent to kQ and kR for some oriented tree Q and some quiver R, respectively; and Λ is a twisted m-fold trivial extension of A and Π is a twisted n-fold trivial extension of B for some integers $m, n \geq 1$. By Lemma 4.1 there is an admissible orientation ρ of $\overline{Q}$ such that the quiver $Q' := (\overline{Q}, \rho)$ has a unique source. Since kQ and kQ' are derived equivalent, we may assume from the beginning that Q is a tree oriented by an admissible orientation and with a unique source. Now assume that Λ and Π are stably equivalent. Then by Lemma 6.2 we have $\overline{Q} = \overline{R}$ and $m = n$. Therefore kQ and kR are derived equivalent, and hence we may assume that $Q = R$, and that both A and B are derived equivalent to kQ. By Proposition 5.3 Λ is derived equivalent to $T^n_\phi(kQ)$ and Π is derived equivalent to $T^n_\psi(kQ)$ for some $\phi, \psi \in \mathrm{Aut}(kQ)$. Since Λ and Π are stably equivalent, $T^n_\phi(kQ)$ and $T^n_\psi(kQ)$ are stably equivalent. By Proposition 7.1 there is a $\rho \in \mathrm{Aut}(\overline{Q})$ such that

$$\rho\pi(\phi) = \pi(\psi)\rho. \tag{8.1}$$

Case 1. $\rho \in \mathrm{Aut}_0(\overline{Q})$**.** In this case $\rho \in \mathrm{Aut}_0(\overline{Q}) = \mathrm{Aut}(Q)$. Thus by (8.1) $\pi(\phi)$ is conjugate to $\pi(\psi)$ in $\mathrm{Aut}(Q)$. Then by Corollary 7.3 we have $T^n_\phi(kQ) \cong T^n_\psi(kQ)$. As a consequence, Λ and Π are derived equivalent.

Case 2. $\rho \notin \mathrm{Aut}_0(\overline{Q})$**.** In this case $\overline{Q}$ does not have a fixed vertex, and ρ fixes a unique edge, say $\alpha : x$—y. Further $\rho(x) = y$ and $\rho(y) = x$, and $\overline{Q}$ has the form (4.1). The quiver Q was obtained from $\overline{Q}$ by letting x or y a unique source as in the proof of Lemma 4.1. We may assume that x is a unique source of Q. Let Q' be a quiver obtained from $\overline{Q}$ by letting y a unique source, and put $A' := kQ'$. Then (8.1) yields a commutative diagram

$$\begin{tikzcd} Q \arrow[r, "\pi(\phi)"] \arrow[d, "\rho"'] & Q \arrow[d, "\rho"] \\ Q' \arrow[r, "\pi(\psi)"'] & Q' \end{tikzcd}$$

of quivers, which gives the following commutative diagram by Lemmas 2.3 and 2.2.

$$\begin{tikzcd} \hat{A} \arrow[r, "(k\pi(\phi))\tilde{}\ \nu_A^n"] \arrow[d, "(k\rho)\tilde{}"'] & \hat{A} \arrow[d, "(k\rho)\tilde{}"] \\ \hat{A}' \arrow[r, "(k\pi(\psi))\tilde{}\ \nu_A^n"'] & \hat{A}' \end{tikzcd}$$

Hence we have

$$T^n_{k\pi(\phi)}(A) \cong T^n_{k\pi(\psi)}(A'). \tag{8.2}$$

Next we show the following.

Claim. *$T^n_{k\pi(\psi)}(A')$ and $T^n_{k\pi(\psi)}(A)$ are derived equivalent.*

Clearly

$$T := \bigoplus_{z \in L_0 \cup \{x\}} \tau_A^{-1} A(\text{-}, z) \oplus \bigoplus_{t \in R_0 \cup \{y\}} A(\text{-}, t)$$

is a classical tilting module with $\mathrm{End}_A(T) \cong A'$. Let E be the tilting spectroid in $\mathcal{H}^{\mathrm{b}}(\oplus A)$ corresponding to T, i.e., E is the full subcategory of $\mathcal{H}^{\mathrm{b}}(\oplus A)$ with the set of objects $\{H(z) | z \in L_0 \cup \{x\}\} \cup R_0 \cup \{y\}$, where $H(z) \in \mathcal{H}^{\mathrm{b}}(\oplus A)$ corresponds to a minimal projective resolution of $\tau_A^{-1} A(\text{-}, z)$ under the Yoneda equivalence $\mathcal{H}^{\mathrm{b}}(\oplus A) \to \mathcal{H}^{\mathrm{b}}(\mathrm{pro}\, A)$ (section 3). Then the triple (A, E, A') is a tilting triple and there exists an isomorphism $\Phi : E \to A'$ sending $H(z)$ to z and t to t for all $z \in L_0 \cup \{x\}$ and $t \in R_0 \cup \{y\}$. It is easy to see that $(k\pi(\psi))\tilde{}\,(H(z)) = H(\psi z)$ for all $z \in L_0$, that $(k\pi(\psi))\tilde{}\,(t) = \psi t$ for all $t \in R_0$ and that $(k\pi(\psi))\tilde{}$ fixes both $H(x)$ and y. Thus E is $\langle (k\pi(\psi))\tilde{}\, \rangle$-stable. Then by Proposition 3.2 $T^n_{k\pi(\psi)}(A)$ is derived equivalent to $T^n_{\psi'}(A')$, where $\psi' \in \mathrm{Aut}(A')$ is defined by the commutativity of the diagram

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "(k\pi(\psi))\tilde{}"'] & A' \arrow[d, "\psi'"] \\ E \arrow[r, "\Phi"'] & A' \end{tikzcd} .$$

In addition, it follows from the commutativity of this diagram and the definition of Φ that $\pi(\psi') = \pi(\psi)$. This implies $T^n_{\psi'}(A') \cong T^n_{k\pi(\psi)}(A')$ by Proposition 7.2. Thus the claim follows.

This claim together with (8.2) implies that $T^n_{k\pi(\phi)}(A)$ and $T^n_{k\pi(\psi)}(A)$ are derived equivalent. Hence $T^n_{\phi}(A)$ and $T^n_{\psi}(A)$ are derived equivalent again by Proposition 7.2. As a consequence Λ and Π are derived equivalent. □

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Department of Mathematics, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558, Japan

E-mail address: `asashiba@sci.osaka-cu.ac.jp`