# Spectrum Estimates On Hilbert Bundles With Applications To Vector Bundles Over Riemannian Manifolds *

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**Abstract**

By using the Poincaré type inequalities introduced in [28], estimate of essential spectrum and eigenvalues is studied for self-adjoint operators on Hilbert bundles. As applications to vector bundles over a (noncompact) complete Riemannian manifold, some estimates of eigenvalues and the space-dimension of $L^2$-harmonic sections are obtained. In particular, some well-known results in the Hodge theory for compact Riemannian manifolds are extended.

**Keywords:** Essential spectrum, eigenvalue, Hilbert bundle, vector bundle, Hodge Laplacian.
**AMS Subjclass:** 58G25, 58G30.

## 1 Introduction

Recall that the famous Hodge's decomposition theorem provides a representation of the de Rham cohomology by the space of harmonic forms over a compact Riemannian manifold. Recently, some efforts have been made by Bueler [8] (for the heat weighted Laplacian), Bueler-Prokhorenkov [9] (for some Gaussian type weighted Hodge Laplacians on topologically tame manifolds), and Ahmed-Stroock [1] (for a class of weighted Hodge Laplacians with ultracontractive semigroups), to establish such a decomposition theorem on the space of (weighted) $L^2$-forms over noncompact Riemannian manifolds. A key step to do this is to show that 0 does not lie in the essential spectrum of the corresponding weighted Hodge Laplacian (cf. Theorem 5.10 and Corollary 5.11 in [8]). This is one motivation for us to study the spectrum for general weighted Hodge Laplacians.

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On the other hand, the essential spectrum of linear operators on a $L^2$-space of Hilbert valued functions are studied by the authors in [13, 28] using Poincaré type inequalities (see (2.3) below). To study spectrum estimates on differential forms (or more generally, vector bundles) over Riemannian manifolds, we first extend some related results obtained in [13, 28] to Hilbert bundles. This is the main task in section 2.

Next, in section 3 we study spectrum estimates on vector bundles over a complete Riemannian manifold. We first present a semigroup comparison theorem (see Theorem 3.1 below) which is essentially classical. By combining this theorem with results obtained in section 2, one may pass problems on vector bundles to the corresponding ones on the base manifold. Therefore, we are able to estimate eigenvalues on vector bundles following the line of [29] where the same thing was done on the base manifold, see Theorem 3.4 and Corollary 3.5 below. Moreover, we obtain some estimates for the dimension of the space of $L^2$-harmonic sections, see Theorem 3.3 and Corollarys 3.6 and 3.7. In particular, we extend (and improve in some sense) Gromov's theorem on the first Betti number (and a more general result by Berard, Besson and Gallot [5]) to the present general framework with an explicit "$\eta$", see Corollary 3.7 and the paragraph before it for details.

Finally, we apply results obtained in section 3 to weighted Hodge Laplacians on differential forms. In particular, we extend the above mentioned Gromov's theorem with an explicit "$\eta$" depending only on the dimension of the base manifold, see Corollary 4.2 for details. Moreover, we obtain a Hodge type decomposition theorem (see Theorem 4.3 below). As has been shown by Bueler and Prokhorenkov in [9] that, there are examples for noncompact Riemannian manifolds such that this theorem provides the Hodge's representation of the de Rham cohomology (cf. (4.5) below).

# 2 Spectrum Estimates On Hilbert Bundles

Let $(E, \mathcal{F}, \mu)$ be a complete measure space, and $H := \{(H_x, \langle\cdot,\cdot\rangle_x) : x \in E\}$ a family of separable real Hilbert spaces (i.e., a Hilbert bundle over $E$). Assume that there is a (possibly finite) sequence $\{e_j\} \subset \prod_{x\in E} H_x$ such that for $\mu$-a.e. $x \in E$, $\{e_j(x)\}$ is an orthonormal basis in $H_x$. Set

$$\mathcal{M} = \left\{ f \in \prod_{x\in E} H_x : \langle f, \mathrm{e}_j\rangle \text{ is } \mathcal{F}\text{-measurable for all } j \right\},$$

where $\langle f, \mathrm{e}_j\rangle(x) := \langle f(x), e_j(x)\rangle_x$. We call $\mathcal{M}$ the space of $\mathcal{F}$-measurable sections of $H$.

For $p \ge 1$, let $L^p_H(\mu) = \{f \in \mathcal{M} : |f| \in L^p(\mu)\}$, where $L^p(\mu)$ denotes the $L^p$-space of real valued functions. As in usual, we denote $\mu(u) = \int_E u\,\mathrm{d}\mu$ for $u \in L^1(\mu)$ and regard $f = g$ in $L^p_H(\mu)$ provided $f = g$ $\mu$-a.e. Then $L^2_H(\mu)$ is a Hilbert space with the inner product $\langle f, g\rangle_{L^2_H(\mu)} := \mu(\langle f, g\rangle)$. We refer to [24] for more knowledge on Hilbert bundles.

Recall that $A \subset L^p_H(\mu)$ is said to be $L^p$-uniformly integrable if

$$\lim_{r\to\infty}\sup\{\mu(|f|^p 1_{\{|f|>r\}}) : f\in A\} = 0.$$

A linear operator $P$ on $L^p_H(\mu)$ is called $L^p$-uniformly integrable if so is $\{Pf : \mu(|f|^p) \le 1\}$. Moreover, denote by $\sigma(P)$ and $\sigma_{\rm ess}(P)$, respectively, the spectrum and the essential spectrum of a linear operator $P$.

Let $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ be a positive definite symmetric closed from on $L^2_H(\mu)$, and let $\tilde{P}_t$ and $(\tilde{L}, \mathcal{D}(\tilde{L}))$ denote, respectively, the associated contraction semigroup and its generator. It is well-known that $\tilde{L}$ is self-adjoint on $L^2_H(\mu)$, and (c.f. [16] or [18])

$$\tilde{\mathcal{E}}(f,g) = -\mu(\langle f, \tilde{L}g\rangle), \quad f\in \mathcal{D}(\tilde{\mathcal{E}}),\, g\in\mathcal{D}(\tilde{L}), \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{P}_t f = \tilde{L}\tilde{P}_t f = \tilde{P}_t\tilde{L}f, \quad t\ge 0,\, f\in\mathcal{D}(\tilde{L}). \tag{2.2}$$

The first aim in this section is to study the essential spectrum of $\tilde{L}$ by using the following Poincaré type inequality:

$$\mu(|f|^2) \le r\tilde{\mathcal{E}}(f,f) + \beta(r)\mu(|f|)^2, \quad f\in\mathcal{D}(\tilde{\mathcal{E}}),\, r>r_0, \tag{2.3}$$

where $r_0\ge 0$ is a constant and $\beta$ is a positive function defined on $(r_0,\infty)$. We may assume that $\beta$ in (2.3) is decreasing since the inequality remains true with $\beta$ replaced by $\bar\beta(r) := \inf\{\beta(s) : s\in(r_0,r]\}$ for $r>r_0$. As have been shown in [28] (also [13]) for $H_x=\mathbb{R}$ that (2.3) extends known Poincaré-Sobolev type inequalities. The relationship between $\sigma_{\rm ess}(\tilde{L})$ and (2.3) has already been worked out recently by the authors in [13] for the case that $H_x$ is independent of $x$, see also [28] for diffusions on manifolds. A key step of the study is the following lemma, which extends Lemma 3.1 in [13] and hence an earlier result due to Wu [30, 31].

**Lemma 2.1.** *Assume that $\mu$ is a probability measure and $p\ge 1$ is fixed. Let $P$ be a bounded linear operator on $L^p_H(\mu)$ with kernel $p(x,y)$, i.e., for $\mu$-a.e. $x,y\in E$, $p(x,y) : H_y\to H_x$ is a bounded linear operator such that*

$$Pf(x) = \int_E p(x,y)f(y)\mu(\mathrm{d}y), \quad f\in L^p_H(\mu).$$

*Suppose that for $\mu$-a.e. $x\in E$, we have*

$$\sum_j\left(\int_E |p(x,y)^* e_j(x)|_y\mu(\mathrm{d}y)\right)^2 < \infty, \tag{2.4}$$

*where $p(x,y)^*$ is the adjoint operator of $p(x,y)$. If $\{Pf : \|f\|_\infty\le 1\}$ is $L^p$-uniformly integrable, then for any $L^p$-uniformly integrable set $A\subset L^p_H(\mu)$, $P(A) := \{Pf : f\in A\}$ is relatively compact in $L^p_H(\mu)$.*

*Proof.* We shall use the following Bourbaki theorem (see page 112 in [7]): *a bounded set in the dual space $B'$ of a separable Banach space $B$ is compact and metrisable w.r.t. the weak topology $\sigma(B', B)$.* If $P(A)$ is not relatively compact in $L^p_H(\mu)$, then there exist $\varepsilon > 0$ and a sequence $\{f_n\}_{n=1}^\infty \subset A$ such that $\|Pf_n - Pf_m\|_{L^p_H(\mu)} \geq \varepsilon$, $n \neq m$. Since $P$ is bounded and $A$ is $L^p$-uniformly integrable, we may take $K > 0$ such that

$$\|Pf_{n,K} - Pf_{m,K}\|_{L^p_H(\mu)} \geq \varepsilon/2, \qquad n \neq m, \tag{2.5}$$

where $f_{n,K} = f_n 1_{\{|f_n| \leq K\}}$. We fix a version of $f_n$ for each $n$, and let $\mathcal{F}'$ be the $\mu$-completion of the $\sigma$-field $\sigma(\{\langle f_{n,K}, \mathrm{e}_j\rangle : n, j \geq 1\})$ which is $\mu$-separable. Let

$$\mathcal{M}' = \left\{ f \in \prod_{x\in E} H_x : \langle f, \mathrm{e}_j\rangle \text{ is } \mathcal{F}'\text{-measurable for all } j \geq 1 \right\} \subset \mathcal{M}.$$

Let $L^p_H(\mu)'$ be defined as $L^p_H(\mu)$ for $\mathcal{F}'$ and $\mathcal{M}'$ in place of $\mathcal{F}$ and $\mathcal{M}$ respectively, which is separable since $\mathcal{F}'$ is $\mu$-separable. For $f \in L^1_H(\mu)$ let $\mu(f|\mathcal{F}') := \sum_j \mu(\langle f, e_j\rangle|\mathcal{F}')e_j$, where $\mu(\cdot|\mathcal{F}')$ is the conditional expectation w.r.t. $\mu$ under $\mathcal{F}'$. By Bourbaki theorem with $B = L^1_H(\mu)'$, there exists $f \in L^\infty_H(\mu)'$ such that $f_{n_i,K} \to f$ weakly for some $n_i \uparrow \infty$, i.e., for any $g \in L^1_H(\mu)'$, one has $\mu(\langle f_{n_i,K}, g\rangle) \to \mu(\langle f, g\rangle)$. Noting that for any $g \in L^1_H(\mu)$ and any $f' \in L^\infty_H(\mu)'$, one has $\mu(\langle f', g\rangle) = \mu(\langle \mu(g|\mathcal{F}'), f'\rangle)$, we obtain $\mu(\langle f_{n_i,K}, g\rangle) \to \mu(\langle f, g\rangle)$ for all $g \in L^1_H(\mu)$. Then for $\mu$-a.e. $x$,

$$\begin{aligned}
|Pf(x) - Pf_{n_i,K}(x)|_x^2 &= \sum_j \langle P_{n_i,K} f(x) - Pf(x), e_j(x)\rangle_x^2 \\
&= \sum_j \left( \int_E \langle f_{n_i,K}(y) - f(y), p(x,y)^* e_j(x)\rangle_y \mu(\mathrm{d}y) \right)^2 \\
&\leq \big(K + \|f\|_\infty\big)^2 \sum_j \left( \int_E |p(x,y)^* e_j(x)|_y \mu(\mathrm{d}y) \right)^2.
\end{aligned}$$

By (2.4) and the dominated convergence theorem, we obtain, for $\mu$-a.e. $x$,

$$\lim_{n_i\to\infty} |Pf(x) - Pf_{n_i,K}(x)|_x^2 = \sum_j \lim_{n_i\to\infty} \mu(\langle f_{n_i,K} - f, p(x,\cdot)^* e_j(x)\rangle)^2 = 0.$$

Since $\{|Pf_{n_i,K}|^p : i \geq 1\}$ is uniformly integrable and $\mu$ is a probability measure, $Pf_{n_i,K} \to Pf$ in $L^p_H(\mu)$. This is a contradiction to (2.5). $\square$

Directly following an argument in [13], we obtain the following result. We present below a complete proof of this result for completeness.

**Theorem 2.2.** *Assume that $\mu$ is a probability measure. If $\sigma_{\mathrm{ess}}(-\tilde{L}) \subset [r_0^{-1}, \infty)$ for some $r_0 \geq 0$, then* (2.3) *holds for some $\beta \in C(r_0, \infty)$. Conversely, if $\tilde{P}_t$ has kernel satisfying* (2.4) *for each $t > 0$, then* (2.3) *implies $\sigma_{\mathrm{ess}}(-\tilde{L}) \subset [r_0^{-1}, \infty)$.*

*Proof.* For any $r > r_0$, let $r_1 = \frac{1}{2}(r + r_0)$. If $\sigma_{\mathrm{ess}}(-L) \subset [r_0^{-1}, \infty)$, then $\sigma_{\mathrm{ess}}(-L) \cap [0, r_1^{-1}) = \emptyset$. Let $\lambda_0 \le \lambda_1 \cdots \le \lambda_{n_r}$ be all the eigenvalues of $-\tilde{L}$ (including multiplicity) satisfying $\lambda_i \le r_1^{-1}$, and $\{f_0, \cdots, f_{n_r}\}$ the corresponding normalized eigenfunctions. Let $\mathcal{H}_r = \mathrm{spn}\{f_0, \cdots, f_{n_r}\}$, we have $\mathcal{H}_r = Q_r(L^2_H(\mu))$, where $Q_r = \int_0^{r_1^{-1}} \mathrm{d}E_\lambda$ and $\{E_\lambda : \lambda \ge 0\}$ is the resolution of the identity of $-\tilde{L}$. For any $f \in \mathcal{D}(\tilde{L})$, let $g = Q_r(f)$ and $h = f - g$. We have $g = \sum_{i=0}^{n_r} \mu(\langle f, f_i\rangle) f_i \in \mathcal{D}(\tilde{L})$ and hence $h \in \mathcal{D}(\tilde{L})$. Observing that $h = \int_{r_1^{-1}}^{\infty} \mathrm{d}\, E_\lambda f$ and hence $\mu(|h|^2) = \int_{r_1^{-1}}^{\infty} \mathrm{d}\langle f, E_\lambda f\rangle_{L^2_H(\mu)}$, we obtain

$$\mu(|h|^2) \le r_1 \int_0^\infty \lambda \mathrm{d}\langle f, E_\lambda f\rangle_{L^2_H(\mu)} = -r_1 \mu(\langle f, \tilde{L} f\rangle) = r_1 \tilde{\mathcal{E}}(f, f). \tag{2.6}$$

Next, since $\mathcal{H}_r$ is a finite-dimensional space, $\beta_1(r) := \sup_{0 \ne g \in \mathcal{H}_r} \frac{\mu(|g|^2)}{\mu(|g|)^2} \in [1, \infty)$. We have

$$\mu(|g|^2) \le \beta_1(r) \mu(|g|)^2. \tag{2.7}$$

Let $\varepsilon_r = \frac{r - r_1}{2r\beta_1(r)(1+n_r)^2}$, and let $c_r > 0$ be the smallest constant such that $\mu(|f_i|^2 1_{\{|f_i| \ge c_r\}}) \le \varepsilon_r,\ i = 0, \cdots, n_r$. Then

$$\begin{aligned}
\mu(|g|) &\le \sum_{i=0}^{n_r} \mu(|\langle f_i, f\rangle|) \le \sum_{i=0}^{n_r} \big( c_r \mu(|f|) + \mu(|\langle f, f_i\rangle| 1_{\{|f_i| \ge c_r\}}) \big) \\
&\le (1 + n_r) c_r \mu(|f|) + (1 + n_r)\sqrt{\varepsilon_r \mu(|f|^2)}.
\end{aligned}$$

Therefore, (2.7) implies

$$\mu(|g|^2) \le 2\beta_1(r)(1 + n_r)^2 \varepsilon_r \mu(|f|^2) + 2\beta_1(r)(1 + n_r)^2 c_r^2 \mu(|f|)^2.$$

Combining this with (2.6), we obtain

$$\mu(|f|^2) \le r_1 \tilde{\mathcal{E}}(f, f) + 2\beta_1(r)(1 + n_r)^2 c_r^2 \mu(|f|)^2 + 2\beta_1(r)(1 + n_r)^2 \varepsilon_r \mu(|f|^2).$$

This proves (2.3) with $\beta \in C(r_0, \infty)$ such that

$$\beta(r) \ge \frac{2\beta_1(r)(1 + n_r)^2 c_r^2 r}{r_1} = \frac{4\beta_1(r)(1 + n_r)^2 c_r^2 r}{r_0 + r}.$$

Next, assume that $\tilde{P}_t$ has kernel and (2.3) holds. Assume that there exists $\lambda \in [0, r_0^{-1}) \cap \sigma_{\mathrm{ess}}(-\tilde{L})$. Let $r_1 > r_0$ be such that $r_1 \lambda < 1$ and take

$$\varepsilon = \min\left\{ \frac{1 - r_1 \lambda}{2\sqrt{2}\, r_1},\ \frac{\mathrm{e}^{-2\lambda t}(1 - \lambda r_1)}{20 t \beta(r_1)} \right\}$$

which is positive. By Weyl's criterion (see [21], Theorem VII.12 and comments on page 264), there exists a sequence $\{f_n\} \subset \mathcal{D}(\tilde{L})$ such that $\|f_n\|_{L^2_H(\mu)} = 1$, $\mu(\langle f_n, f_m\rangle) = 0$ for $n \neq m$, and

$$\|(\lambda + \tilde{L})f_n\|_{L^2_H(\mu)} \leq \varepsilon, \qquad n \geq 1. \tag{2.8}$$

For any $m, n \geq 1$, we have $h(s) := \|\tilde{P}_s(f_n - f_m) - \mathrm{e}^{-\lambda s}(f_n - f_m)\|^2_{L^2_H(\mu)} \leq 8, s \geq 0$. By (2.8),

$$\begin{aligned} h'(s) &= 2\langle(\tilde{P}_s - \mathrm{e}^{-\lambda s})(f_n - f_m), (\tilde{P}_s\tilde{L} + \lambda\mathrm{e}^{-\lambda s})(f_n - f_m)\rangle_{L^2_H(\mu)} \\ &= -2\lambda h(s) + 2\langle(\tilde{P}_s - \mathrm{e}^{-\lambda s})(f_n - f_m), \tilde{P}_s(\lambda + \tilde{L})(f_n - f_m)\rangle_{L^2_H(\mu)} \\ &\leq 4\varepsilon\sqrt{h(s)} < 16\varepsilon. \end{aligned}$$

Then

$$\begin{aligned} \|\tilde{P}_t(f_n - f_m)\|_{L^1_H(\mu)} &\geq \mathrm{e}^{-\lambda t}\|f_n - f_m\|_{L^1_H(\mu)} - \|\tilde{P}_t(f_n - f_m) - \mathrm{e}^{-\lambda t}(f_n - f_m)\|_{L^2_H(\mu)} \\ &\geq \mathrm{e}^{-\lambda t}\|f_n - f_m\|_{L^1_H(\mu)} - \sqrt{16\varepsilon t}. \end{aligned} \tag{2.9}$$

Next, by (2.3) and (2.8), for $n \neq m$ we have

$$\begin{aligned} 2 = \|f_n - f_m\|^2_{L^2_H(\mu)} &\leq r_1\mu(\langle f_n - f_m, \tilde{L}(f_m - f_n)\rangle) + \beta(r_1)\|f_n - f_m\|^2_{L^1_H(\mu)} \\ &= 2r_1\lambda + r_1\mu(\langle f_n - f_m, (\tilde{L} + \lambda)(f_m - f_n)\rangle) + \beta(r_1)\|f_n - f_m\|^2_{L^1_H(\mu)} \\ &\leq 2r_1\lambda + 2\sqrt{2}r_1\varepsilon + \beta(r_1)\|f_n - f_m\|^2_{L^1_H(\mu)}. \end{aligned}$$

This implies

$$\|f_n - f_m\|_{L^1_H(\mu)} \geq \sqrt{\frac{2(1 - r_1\lambda) - 2\sqrt{2}r_1\varepsilon}{\beta(r_1)}} \geq \sqrt{\frac{1 - r_1\lambda}{\beta(r_1)}}, \quad n \neq m. \tag{2.10}$$

Combining (2.9) with (2.10), we obtain

$$\|\tilde{P}_t(f_n - f_m)\|_{L^1_H(\mu)} \geq \mathrm{e}^{-\lambda t}\sqrt{\frac{1 - r_1\lambda}{\beta(r_1)}} - \sqrt{16\varepsilon t} > 0 \tag{2.11}$$

for any $n \neq m$. Since $\{f_n : n \geq 1\}$ is $L^1$-uniformly integrable, by Lemma 2.1 there exists $n_i \uparrow \infty$ such that $\|\tilde{P}_t(f_{n_i} - f_{n_{i+1}})\|_{L^1_H(\mu)} \to 0$, which is a contradiction to (2.11). $\square$

Next, we turn to study the discrete eigenvalues estimation for $\tilde{L}$. Let $\bar{\lambda} = \inf \sigma_{\text{ess}}(-\tilde{L})$, and set $\inf \emptyset = \infty$. Assume that $\sigma(-\tilde{L}) \cap [0, \bar{\lambda}) \neq \emptyset$, where $\sigma(-\tilde{L})$ denotes the spectrum of $-\tilde{L}$. We list all eigenvalues of $-\tilde{L}$ (including multiplicity) in $[0, \bar{\lambda})$ as follows: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ We will follow the line of [29] where the eigenvalues estimation was studied on real-valued function spaces.

**Lemma 2.3.** *Let $\bar{\lambda}$ and $\{\lambda_n\}$ be as in above. Assume that $\tilde{P}_t$ has kernel $\tilde{p}_t(x, y)$ and there exists $l \in \mathbb{N}$ such that $\dim H_x = l$, $\mu$-a.e. $x$. Let $\{f_i\}$ be the family of normalized eigenfunctions for $\{\lambda_i\}$. We have*

$$l \int_E \|\tilde{p}_{t/2}(\cdot, y)\|_{op}^2 \mu(\mathrm{d}y) \geq \mathrm{e}^{-\lambda_n t} \sum_{i=1}^n |f_i|^2, \tag{2.12}$$

*for any $n \leq \#\{i : \lambda_i < \bar{\lambda}\}$.*

*Proof.* Let $x \in E$ be such that $l = \dim H_x$ and $\{e_j(x)\}$ is an orthonormal basis in $H_x$. For any $1 \leq j \leq l$, let

$$g_j(y) = \sum_{i=1}^n \langle f_i(x), e_j(x)\rangle_x f_i(y), \qquad y \in E.$$

We have

$$\begin{aligned}\mathrm{e}^{-\lambda_n t/2} \sum_{i=1}^n |f_i|^2(x) &\leq \sum_{j=1}^l \langle \tilde{P}_{t/2} g_j, e_j\rangle(x) = \sum_{j=1}^l \int_E \langle \tilde{p}_{t/2}(x, y) g_j(y), e_j(x)\rangle_x \mu(\mathrm{d}y) \\ &\leq \left( l \int_E \|\tilde{p}_{t/2}(x, y)\|_{op}^2 \mu(\mathrm{d}y) \right)^{1/2} \left( \sum_{j=1}^l \|g_j\|_{L_H^2(\mu)}^2 \right)^{1/2}.\end{aligned}$$

The proof is completed by noting that

$$\sum_{j=1}^l \|g_j\|_{L_H^2(\mu)}^2 = \sum_{j=1}^l \sum_{i=1}^n \langle f_i, e_j\rangle^2(x) = \sum_{i=1}^n |f_i|^2(x).$$

□

**Theorem 2.4.** *In the situation of Lemma 2.3 and assume that $\mu$ is a probability measure. If there exits $t > 0$ such that*

$$C(t) := \int_{E \times E} \|\tilde{p}_{t/2}(x, y)\|_{op}^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y) < \infty,$$

*then $\sigma_{\text{ess}}(\tilde{L}) = \emptyset$ and hence $\bar{\lambda} = \infty$, equivalently, (2.3) holds for $r_0 = 0$ and some $\beta \in C(0, \infty)$ by Theorem 2.2. Moreover,*

$$\lambda_n \geq \frac{1}{t} \log \frac{n}{lC(t)}, \qquad n \geq 1.$$

*Consequently, $\#\{i : \lambda_i \leq \lambda\} \leq lC(t)\mathrm{e}^{\lambda t}$ for each $\lambda \geq 0$.*

*Proof.* If $C(t) < \infty$, then $\tilde{P}_{t/2}$ is $L^2$-uniformly integrable. By Lemma 2.1, $\tilde{P}_{t/2}$ is compact and hence $\sigma_{\text{ess}}(\tilde{L}) = \emptyset$. Next, by (2.12) we obtain $lC(t) \geq n\text{e}^{-\lambda_n t}$. $\square$

Obviously, to apply Theorems 2.2 and 2.4, one has to study the existence of the heat kernel for $\tilde{P}_t$ and then to estimate it. A convenient way to do so is to compare $\tilde{P}_t$ with a semigroup on $L^2(\mu)$. This trick has been widely used in spectrum geometry, especially, in the study of spectrum on differential forms over compact manifolds, see for instance [3, 4] and references therein. The next result is implied by Theorem 16 in [6] (see also [23] for related results), and the final one is a result on the existence of heat kernels for operators on $L^2_H(\mu)$.

**Theorem 2.5.** *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric closed from on $L^2(\mu)$ which is bounded from below. Assume that the associated semigroup $P_t$ is positivity preserving. Then the following two statements are equivalent:*
*(1) $|\tilde{P}_t f| \leq P_t|f|$ for all $t > 0$ and $f \in L^2_H(\mu)$.*
*(2) If $f \in \mathcal{D}(\tilde{\mathcal{E}})$ one has $|f| \in \mathcal{D}(\mathcal{E})$, and $\mathcal{E}(|f|, |g|) \leq |\tilde{\mathcal{E}}(f, g)|$ for all $f, g$ in a core of $\tilde{L}$ with $\langle f, g\rangle_{L^2_H(\mu)} = \mu(|f| \cdot |g|)$.*

**Proposition 2.6.** *Assume that $\mu$ is a probability measure. Let $\tilde{P}$ be a bounded linear operator on $L^2_H(\mu)$, and $p$ a nonnegative measurable function on $E \times E$ such that*

$$Pu = \int p(\cdot, y)u(y)\mu(\text{d}y)$$

*provides a bounded linear operator $P$ on $L^2(\mu)$. If $l := \#\{e_j\} < \infty$ and $|\tilde{P}f| \leq P|f|$ for any $f \in L^2_H(\mu)$, then $\tilde{P}$ has kernel $\tilde{p}$ satisfying $\|\tilde{p}(x, y)\|_{op} \leq lp(x, y)$.*

*If in addition that $E$ is a metric space and for each $x \in E$, $\tilde{p}(x, \cdot)$ and $p(x, \cdot)$ are continuous and locally bounded on the support of $\mu$, i.e., for any $y$ in the support of $\mu$ and any unit $\omega \in H_y$ there exists $e \in \mathcal{M}$ with $e(y) = \omega$ such that in a neighborhood of $y$, one has $|e| = 1$ and $\tilde{p}(x, \cdot)e(\cdot), p(x, \cdot)$ are bounded and continuous. Then $\|\tilde{p}(x, y)\|_{op} \leq p(x, y)$.*

*Proof.* For any $i, j \in \mathbb{N}$ with $i, j \leq l$, let $\mu^{ij}$ be a set function on $\mathcal{F} \times \mathcal{F}$ defined by

$$\mu^{ij}(A) = \int_{A_1} \langle e_i(x), \tilde{P}(1_{A(x)}e_j)(x)\rangle_x \mu(\text{d}x), \qquad A \in \mathcal{F} \times \mathcal{F},$$

where $A_1 = \{x \in E :$ there exists $y \in E$ such that $(x, y) \in A\}$ and $A(x) = \{y \in E : (x, y) \in A\}$. Since $\tilde{P}$ is bounded, it is easy to check that $\mu^{ij}$ is a signed measure. Moreover, by Jordan's decomposition theorem and that $|\tilde{P}f| \leq P|f|$ for any $f \in L^2(\mu)$, we have $|\mu^{ij}| := (\mu^{ij})^+ + (\mu^{ij})^- \leq p\mu \times \mu$. Then $\mu^{ij}$ is absolutely continuous w.r.t. $\mu \times \mu$ with density $p^{ij}$ satisfying $|p^{ij}| \leq p$. Define $\tilde{p}(x, y) : H_y \to H_x$ by

$$\tilde{p}(x, y)\omega = \sum_{i,j=1}^{l} p^{ij}(x, y)\langle \omega, e_j(y)\rangle_y e_i(x), \quad \omega \in H_y.$$

It is easy to check that $\|\tilde p(x,y)\|_{op} \le lp(x,y)$ for any $x,y\in E$ and $\tilde p$ is a kernel of $\tilde P$. Indeed, for any $f,g\in L^2_H(\mu)$, we have

$$\begin{aligned}
\langle g, \tilde P f\rangle_{L^2_H(\mu)} &= \sum_{i,j=1}^{l} \int_E \langle \tilde P(\langle f, e_j\rangle e_j), \langle g, e_i\rangle e_i\rangle \mathrm{d}\mu \\
&= \sum_{i,j=1}^{l} \int_{E\times E} \langle f, e_j\rangle(y)\langle g, e_i\rangle(x)\mu^{ij}(\mathrm{d}x\mathrm{d}y) \\
&= \sum_{i,j=1}^{l} \int_{E\times E} p^{ij}(x,y)\langle f, e_j\rangle(y)\langle g, e_i\rangle(x)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\
&= \left\langle g, \int_E \tilde p(\cdot,y) f(y)\mu(\mathrm{d}y)\right\rangle_{L^2_H(\mu)}.
\end{aligned}$$

Next, let the additional conditions hold. For $x\in E$, $y$ in the support of $\mu$ and any $\omega\in H_y$ with $|\omega|_y=1$, let $e\in\mathcal{M}$ be such that $e(y)=\omega$, $|e|=1$ and $\tilde p(x,\cdot)e(\cdot)$ and $p(x,\cdot)$ are bounded and continuous in a neighborhood $N_y$ of $y$. Let $\{f_n\}$ be a sequence of nonnegative continuous functions with supports contained in $N_y$ such that $f_n\mu\to\delta_y$ weakly as $n\to\infty$. We have

$$\left|\int_E [\tilde p(x,z)e(z)]f_n(z)\mu(\mathrm{d}z)\right|_x = \left|\tilde P(f_n e)(x)\right|_x \le Pf_n(x) = \int_E p(x,z)f_n(z)\mu(\mathrm{d}z).$$

By letting $n\to\infty$, we obtain $|\tilde p(x,y)\omega|_x \le p(x,y)$. Therefore $\|\tilde p\|_{op}\le p$ since we may take $\tilde p(x,\cdot)=0$ outside of the support of $\mu$. □

# 3 Vector Bundles Over Riemannian Manifolds

Let $M$ be a $d$-dimensional connected complete Riemannian manifold, and $\Omega$ a $l$-dimensional Riemannian vector bundle over $M$. Let $\mathcal{M}$ and $\Gamma(\Omega)$ denote, respectively, the Borel-measurable and smooth sections of $\Omega$. Moreover, let $\Gamma_0(\Omega)$ consist of all elements in $\Gamma(\Omega)$ with compact support. Let $\{X_i\}$ be a locally normal frame, and $\nabla_{X_i}$ the usual covariant derivative along $X_i$. Then the horizontal Laplacian reads $\square=\sum_{i=1}^d \nabla^2_{X_i}$ which is naturally defined on $\Gamma(\Omega)$. Let $\mu(\mathrm{d}x)=\mathrm{e}^{V(x)}\mathrm{d}x$ for some $V\in C^2(M)$, where $\mathrm{d}x$ denotes the Riemannian volume element. We note that for each $o\in M$, there exists $\{e_j\}_{j=1}^l\subset\mathcal{M}$ such that $\{e_j(x)\}$ is an orthonormal basis in $\Omega_x$ for each $x\in E$, and $e_j$ is smooth outside of the cut-locus of $o$ for each $j$.

Consider the operator

$$\tilde L = \square + \nabla_{\nabla V} - R,$$

where $R$ is a symmetric measurable endomorphism of $\Omega$ such that $(\tilde L, \Gamma_0(\Omega))$ is essentially self-adjoint on $L^2_\Omega(\mu)$ (note that in this case $H_x=\Omega_x$), and $\tilde L$ is negative definite, i.e., $\mu(\langle f, \tilde{L} f\rangle) \le 0$ for all $f \in \Gamma_0(\Omega)$. Let $(\tilde{L}, \mathcal{D}(\tilde{L}))$ be the unique self-adjoint extension of $(\tilde{L}, \Gamma_0(\Omega))$ which is negative definite too. Let

$$\underline{R}(x) = \inf\{\langle R\omega, \omega\rangle_x : \omega \in \Omega_x, |\omega| = 1\}, \quad x \in M.$$

We assume that $\underline{R} \in L^1_{loc}(\mathrm{d}x)$ and there exists $C \ge 0$ such that $\mu(|\nabla u|^2) + \mu(\underline{R}u^2) \ge -C\mu(u^2)$ for all $u \in C_0^\infty(M)$. Then the following form is closable and let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ denote its closure (see e.g. Corollary VI.1.28 in [16]):

$$\mathcal{E}(u, v) := \mu(\langle \nabla u, \nabla v\rangle) + \mu(\underline{R}uv), \qquad u, v \in C_0^\infty(M).$$

Be careful that the boundedness from below of $(\mathcal{E}, C_0^\infty(M))$ does not imply that of $\underline{R}$, see e.g. Remark 2.4 in [19]. Let $(L, \mathcal{D}(L))$ be the smallest closed extension (i.e., the Friedrichs extension) of $(\Delta + \nabla V - \underline{R}, C_0^\infty(M))$ (which is self-adjoint by Theorem VI.2.6 in [16]), and $P_t^{\underline{R}}$ the corresponding strongly continuous semigroup.

**Theorem 3.1.** *Let $\{x_t : t \ge 0\}$ be the diffusion process generated by $\Delta + \nabla V$ on $M$. We have*

$$|\tilde{P}_t f| \le P_t^{\underline{R}}|f| = \mathbb{E}^x\left\{|f|(x_t)\exp\left[-\int_0^t \underline{R}(x_s)\mathrm{d}s\right]\right\}, \quad f \in L^2_\Omega(\mu), \tag{3.1}$$

*where the second formula holds provided the right-hand side is well-defined. Consequently, $\tilde{P}_t$ and $P_t^{\underline{R}}$ have smooth kernels.*

*Proof.* We need only to prove the first inequality in (3.1) since the second equality is known as Feyman-Kac formula. By Theorem 2.5, it suffices to show that for any $f, g \in \Gamma_0(\Omega)$ with $\langle f, g\rangle_{L^2_H(\mu)} = \mu(|f| \cdot |g|\rangle)$, one has

$$|\tilde{\mathcal{E}}(f, g)| \ge \mathcal{E}(|f|, |g|). \tag{3.2}$$

Since $\langle f, g\rangle \le |f| \cdot |g|$, we have $\langle f, g\rangle = |f| \cdot |g|$ and hence $f = \frac{|f|g}{|g|}$ on $\{|g| > 0\}$. By Kato's inequality, we have $|\nabla|g|| \le |\nabla g|$ for all $g \in \Gamma_0(\Omega)$, cf. Lemma VI.31 in [4] and its proof. Moreover, since any order derivatives of $g$ are zero on $\{|g| = 0\}$, we obtain

$$\begin{aligned}
\tilde{\mathcal{E}}(f, g) &= -\mu(\langle f, \tilde{L}g\rangle) \ge -\mu(\langle f, (\Box + \nabla_{\nabla V})g\rangle) + \mu(\underline{R}|f| \cdot |g|) \\
&= \mu\Big(1_{\{|g|>0\}}\Big\langle \nabla\frac{|f|g}{|g|}, \nabla g\Big\rangle\Big) + \mu(\underline{R}|f| \cdot |g|) \\
&= \mu\Big(1_{\{|g|>0\}}\frac{|f|}{|g|}|\nabla g|^2\Big) + \frac{1}{2}\mu\big(1_{\{|g|>0\}}\nabla_{\nabla(|f|/|g|)}\langle g, g\rangle\big) + \mu(\underline{R}|f| \cdot |g|) \\
&\ge \mu\Big(1_{\{|g|>0\}}\frac{|f| \cdot |\nabla|g||^2}{|g|}\Big) + \mu\Big(|g|\Big\langle \nabla\frac{|f|}{|g|}, \nabla|g|\Big\rangle\Big) + \mu(\underline{R}|f| \cdot |g|) \\
&= \mu(\langle \nabla|f|, \nabla|g|\rangle) + \mu(\underline{R}|f| \cdot |g|) = \mathcal{E}(|f|, |g|).
\end{aligned}$$

Therefore (3.2) holds.

Finally, since $\cap_{n=1}^\infty \mathcal{D}(\tilde{L}^n) \subset \Gamma(\Omega)$ and $\cap_{n=1}^\infty \mathcal{D}\big((\Delta+\nabla V-\underline{R})^n\big) \subset C^\infty(M)$, by the argument in the proof of Theorem 5.2.1 in [10], we conclude that $\tilde{P}_t$ and $P_t^{\underline{R}}$ have smooth kernels. □

From now on, we let $P_t^0$ denote the diffusion semigroup generated by $\Delta+\nabla V$, and $p_t^0(x,y)$ its kernel w.r.t. $\mu$ which is positive since $M$ is connected. Denote by $b(\tilde{L})$ the dimension of the $\tilde{L}$-harmonic space $\mathbf{H}(\tilde{L}) := \{f \in \mathcal{D}(\tilde{L}) : \tilde{L}f = 0\}$. It is well-known by Hodge's theory, when $M$ is compact and $\tilde{L}$ is the minus Hodge Laplacian on differential $p$-forms, that $b(\tilde{L})$ coincides with the $p$-th Betti number. Moreover, let $\bar\lambda$ and $\{\lambda_i\}$ be as in the last section.

A direct application of Theorem 3.1 is to establish the inequality (2.3) for $\tilde{\mathcal{E}}$ from a known inequality for $\mathcal{E}$. We refer to [28] for criteria of such inequalities for the Dirichlet form associated by $P_t^0$.

**Corollary 3.2.** *Assume that $\underline{R} \geq -c$ for some $c \geq 0$. if there exist $r_1 \in [0, c^{-1})$ and a positive function $\beta_1$ defined on $(r_1,\infty)$ such that*

$$\mu(u^2) \leq r\mu(|\nabla u|^2) + \beta_1(r)\mu(|u|)^2, \quad r > r_1,\, u \in C_0^\infty(M), \tag{3.3}$$

*then* (2.3) *holds for $r_0 = r_1/(1-cr_1)$ and*

$$\beta(r) = \inf\Big\{\frac{\beta_1(s)}{1-cs} : \frac{r_1}{1-cr_1} < s < r \wedge \frac{1}{c}\Big\}, \quad r > r_0.$$

*Proof.* For any $f \in \Gamma_0(\Omega)$, by Theorem 3.1 we have

$$\mu(|\tilde{P}_t f|^2) \leq e^{2ct}\mu((P_t^0|f|)^2), \quad t \geq 0.$$

Therefore, by (2.1) and (2.2),

$$\begin{aligned}\tilde{\mathcal{E}}(f,f) &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mu(|\tilde{P}_t f|^2)|_{t=0}\\ &\geq -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[e^{2ct}\mu((P_t^0|f|)^2)\big]\big|_{t=0} = \mu(|\nabla|f||^2) - c\mu(|f|^2).\end{aligned}$$

The proof is completed by combining this with (3.3). □

Now we study the number $b(\tilde{L})$ by using Theorem 3.1.

**Theorem 3.3.** (1) *We have $b(\tilde{L}) = 0$ provided either* $\inf\sigma(\underline{R}-\Delta-\nabla V) > 0$ *or*

$$\underline{\lim}_{t\to\infty}\int_M \mathbb{E}^x \exp\Big[-2\int_0^t \underline{R}(x_s)\mathrm{d}s\Big]\mu(\mathrm{d}x) = 0. \tag{3.4}$$

(2) *Assume that $\mu$ is a probability measure. If $\underline{R} \geq 0$, then $b(\tilde{L}) \leq l$. Moreover, for any $f \in \mathbf{H}(\tilde{L})$, $|f|$ is constant.*
(3) *Assume that $\underline{R}$ is bounded from below. We have $\bar\lambda \geq \inf\sigma_{\mathrm{ess}}(-(\Delta+\nabla V)+\underline{R})$. Consequently, let $\rho(x)$ be the Riemannian distance between $x$ and a fixed point $o \in M$, if $\delta := \underline{\lim}_{\rho\to\infty}\underline{R} + \inf\sigma_{\mathrm{ess}}(-\Delta-\nabla V) > 0$, then $\bar\lambda \geq \delta > 0$ and hence $b(\tilde{L}) < \infty$.*

*Proof.* (1) Let $f \in \mathbf{H}(\tilde L)$. If $\inf \sigma(-\Delta - \nabla V + \underline{R}) > 0$, then $|f| = |\tilde P_t f| \le P_t^{\underline{R}} |f| \to 0$ as $t \to \infty$, hence $f = 0$. Next, Theorem 3.1 implies

$$\mu(|f|)^2 = \mu(|\tilde P_t f|)^2 \le \mu(|f|^2) \int \mathbb{E}^x \exp\Big[ -2 \int_0^t \underline{R}(x_s)\Big] \mu(\mathrm{d}x).$$

Then $f = 0$ provided (3.4) holds.

(2) For fixed $n \le b(\tilde L)$, let $\{f_1, \cdots, f_n\} \subset \mathbf{H}(\tilde L)$ be an orthonormal family. If $\underline{R} \ge 0$, by Proposition 2.6 and Theorem 3.1 we have $\|\tilde p_t(x,y)\|_{op} \le p_t^0(x,y)$. Then by Lemma 2.3, for any compact set $B \subset M$ there holds

$$\sum_{i=1}^n \int_B |f_i|^2 \mathrm{d}\mu \le l \int_B p_t^0(x,x) \mu(\mathrm{d}x), \quad t > 0. \tag{3.5}$$

We now intend to show that $p_t^0(x,x) \downarrow 1$ as $t \uparrow \infty$ for all $x \in M$. Observing that

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t^0(x,x) = -\int_M |\nabla p_{t/2}^0(x,\cdot)|^2(y) \mu(\mathrm{d}y) \le 0,$$

then $p_t^0(x,x)$ is decreasing in $t$. Next, noting that the Dirichlet form for $\Delta + \nabla V$ is irreducible since $M$ is connected, we have $\|P_t^0 u - \mu(u)\|_{L^2(\mu)} \to 0$ as $t \to \infty$ for any $u \in L^2(\mu)$ (see e.g. the Appendix in [2]). For fix $x \in M$, letting $u(y) = p_1^0(x,y)$, we obtain

$$\|P_t^0 u - \mu(u)\|_{L^2(\mu)}^2 = \mu\big((p_{t+1}^0(\cdot,x) - 1)^2\big) = p_{2(t+1)}^0(x,x) - 1.$$

Therefore $p_t(x,x) \to 1$ as $t \to \infty$. Now by first letting $t \to \infty$ and then $B \to M$, we obtain from (3.5) that $n \le l$, hence $b(\tilde L) \le l$ since $n$ is arbitrary. Moreover, for $f \in \mathbf{H}(\tilde L)$, we have

$$|f|(x)^2 = |P_t f|(x)^2 \le p_{2t}^0(x,x)\mu(|f|^2).$$

By letting $t \to \infty$ we obtain $|f|(x)^2 \le \mu(|f|^2)$ and hence $|f|^2 = \mu(|f|^2)$, $\mu$-a.e.

(3) We note that by Donnely-Li's decomposition principle [11], we have $\sigma_{\mathrm{ess}}(\tilde L) = \sigma_{\mathrm{ess}}(\tilde L|_{B^c})$ for any compact domain $B$, where $\tilde L|_{B^c}$ denotes the operator $\tilde L$ on $B^c$ with Dirichlet boundary conditions. Although their proof in [11] is only given for the Laplacian for functions, but it works also for our present case. Indeed, let $\{f_n\}_{n=1}^\infty \subset \mathcal{D}(\tilde L)$ be such that $\mu(|f_n|^2) = 1$ and $\tilde{\mathcal{E}}(f_n, f_n) \le c_1$ for some $c_1 > 0$ and all $n \ge 1$, we have $\sup_n \mu(|\nabla f_n|^2) < \infty$ since $\underline{R}$ is bounded from below. Moreover, since $V$ is locally bounded, we obtain $\sup_n \int_B |\nabla f_n|^2 \mathrm{d}x < \infty$. Therefore $\{1_B f_n\}$ is relatively compact on $L_H^2(\mu)$ (note that $\mathrm{supp} f_n \subset B$ and $V$ is bounded on $B$). Hence Donnely-Li's argument applies.

By the above decomposition principle, we have

$$\inf \sigma_{\mathrm{ess}}(-(\Delta + \nabla V) + \underline{R}) = \lim_{n\to\infty} \inf \sigma([-(\Delta + \nabla V) + \underline{R}]|_{B_n^c})$$

and the same formula holds for $\tilde L$ in place of $-(\Delta + \nabla V) + \underline{R}$. Then the proof is completed by (3.2) and noting that $\inf \sigma([-(\Delta + \nabla V) + \underline{R})]|_{B_n^c}) \ge \inf_{B_n^c} \underline{R} + \inf \sigma(-(\Delta + \nabla V)|_{B_n^c})$. $\square$

We remark that half part of Theorem 3.3 (1), i.e. $\inf\sigma(\underline{R}-\Delta-\nabla V)>0$ implies $b(\tilde L)=0$, is already known by Elworthy and Rosenberg [12] for differential forms. Moreover, as is well-known in Hodge's theory, Theorem 3.3 (2) is optimal in the sense that there exist examples such that $b(\tilde L)=l$ and $\underline{R}\geq 0$, for instance, the Betti numbers on torus (see e.g. [14]). Finally, Donnley-Li's decomposition principle has become an efficient tool for estimating $\inf\sigma_{\rm ess}(-\Delta-\nabla V)$, we refer to [17] and references therein for details. Also, one may estimate this quantity by using functional inequalities according to Theorem 2.2. In particular, if $\mu$ is a probability measure and $P_t^0$ is uniformly integrable in $L^2(\mu)$, then by Lemma 2.1 (see also [13]) $\inf\sigma_{\rm ess}(-\Delta-\nabla V)=\infty$.

We now study the estimates of $\lambda_n$, and then use the basic estimate (3.6) below to obtain more estimates of $b(\tilde L)$.

**Theorem 3.4.** *Assume that $\mu$ is a probability measure and $\underline{R}\geq -c$ for some $c\in\mathbb{R}$. If $p_t^0(x,x)$ is integrable w.r.t. $\mu$ for some $t>0$, then $\bar\lambda=\infty$ and*

$$\lambda_n\geq\sup_{t>0}\frac{1}{t}\log\frac{n\mathrm{e}^{-ct}}{l\int_M p_t^0(x,x)\mu(\mathrm{d}x)},\quad n\geq 1. \tag{3.6}$$

*In the case that $p_t^0(x,x)$ is not integrable, let $\delta_{s,t}=\mu(\{x:p_t^0(x,x)>s\})$. If there exist some positive $\beta$ defined on $(0,\infty)$ such that*

$$\mu(u^2)\leq r\mu(|\nabla u|^2)+\beta(r)\mu(|u|)^2,\quad r>0,\ u\in C_0^\infty(M). \tag{3.7}$$

*Then $\bar\lambda=\infty$ and*

$$\begin{aligned}\lambda_n\geq\sup\bigg\{\Big(\frac{1}{t}\log\frac{n\varepsilon\mathrm{e}^{-ct}}{sl}\Big)\wedge\sup_{r>0}\frac{1}{r}\big[1-cr-2\beta(r)(\varepsilon+\delta_{s,t})\big]:\\ \varepsilon\in(0,1),s,t>0\bigg\}.\end{aligned} \tag{3.8}$$

*Especially, if (3.7) holds for $\beta(r)=\exp[\alpha(1+r^{-1/\delta})]$ for some $\alpha>0$ and $\delta>1$, one has $\lambda_n\geq\lambda([\log n-\theta]^+)^\delta$ for some $\lambda,\theta>0$ and all $n\geq 1$.*

*Proof.* The first assertion follows from Theorem 2.4 by noting that in the present case we have $\|\tilde p_{t/2}\|_{op}\leq p_{t/2}^0(x,y)\mathrm{e}^{ct/2}$ according to Proposition 2.6 and Theorem 3.1. Next, if (3.7) holds then $\bar\lambda=\infty$ according to Theorem 2.2 and Theorem 3.3 (3). If (3.7) holds for $\beta(r)=\exp[\alpha(1+r^{-1/\delta})]$ for some $\alpha>0$ and $\delta>1$, by Corollary 5.12 in [28] (in which $V\in C^\infty(M)$ is assumed, but the argument there works also for $V\in C^2(M)$), we have $p_t^0(x,x)\leq\exp[\lambda(1+t^{-1/(\delta-1)})]$ for some $\lambda>0$ and all $t>0$. Then the last assertion follows from (3.6).

It reminds to prove (3.8). The proof is essentially taken from [29]. Let $A_{s,t}=\{x:p_t^0(x,x)\leq s\}$. By (2.12) we obtain

$$\mathrm{e}^{-\lambda_n t}\sum_{i=1}^n 1_{A_{s,t}}|f_i|^2 \le l\mathrm{e}^{ct} p_t^0(\cdot,\cdot)1_{A_{s,t}} \le l\mathrm{e}^{ct}s.$$

This implies

$$\lambda_n \ge \frac{1}{t}\log\left[\frac{1}{ls\mathrm{e}^{ct}}\sum_{i=1}^n \mu(1_{A_{s,t}}|f_i|^2)\right] \ge \frac{1}{t}\log\frac{n\varepsilon}{ls\mathrm{e}^{ct}} \tag{3.9}$$

provided $\mu(1_{A_{s,t}}|f_i|^2) \ge \varepsilon$ for all $1 \le i \le n$. On the other hand, if there exists $i$ such that $\mu(1_{A_{s,t}}|f_i|^2) < \varepsilon$, we have

$$\mu(|f_i|)^2 = \big(\mu(1_{A_{s,t}}|f_i|) + \mu(1_{A_{s,t}^c}|f_i|)\big)^2 \le 2(\varepsilon + \delta_{s,t}).$$

Combining this with (3.7), we obtain

$$\begin{aligned} 1 &= \mu(|f_i|^2) \le r\mu(|\nabla|f_i||^2) + \beta(r)\mu(|f_i|)^2 \le r\tilde{\mathcal{E}}(f_i,f_i) + cr + 2\beta(r)(\varepsilon+\delta_{s,t})\\ &= r\lambda_i + cr + 2\beta(r)(\varepsilon+\delta_{s,t}), \quad r>0. \end{aligned}$$

Therefore

$$\lambda_n \ge \lambda_i \ge \sup_{r>0}\frac{1}{r}\big[1 - cr - 2\beta(r)(\varepsilon+\delta_{s,t})\big].$$

Combining this with (3.9) we obtain (3.8). $\square$

To estimate $p_t^0(x,x)$, we assume

$$(\mathrm{Ric} - \mathrm{Hess}_V)(X,X) \ge -K|X|^2, \quad X \in TM, \tag{3.10}$$

for some $K \ge 0$. By the dimension-free Harnack inequality obtained in [26], we have

$$[P_t^0 u(x)]^2 \le P_t^0 u^2(y)\exp\left[\frac{K\rho(x,y)^2}{1-\mathrm{e}^{-2Kt}}\right], \qquad f \in L^2(\mu),\ t>0, \tag{3.11}$$

where $\rho(x,y)$ denotes the Riemannian distance between $x$ and $y$. Then for any $u$ with $\mu(u^2)=1$, we have

$$1 \ge [P_{t/2}^0 u(x)]^2 \mu(B_o(r))\exp\left[-\frac{K(\rho(x)+r)^2}{1-\mathrm{e}^{-Kt}}\right], \quad r>0,\, t>0, \tag{3.12}$$

where $o \in M$ is a fixed point, $\rho(x) := \rho(o,x)$ and $B_o(r)$ is the geodesic ball with center $o$ and radius $r$. Taking $u(y) = p_{t/2}^0(x,y)\big/\sqrt{p_t^0(x,x)}$, we obtain

$$p_t^0(x,x) \le \frac{1}{\mu(B_o(r))}\exp\left[\frac{K(\rho(x)+r)^2}{1-\mathrm{e}^{-Kt}}\right], \qquad r>0, t>0, x\in M. \tag{3.13}$$

Therefore, there exists $c_1, c_2 > 0$ such that $A_{s,1} \supset \{x : \rho(x) \le c_1\sqrt{\log s} - c_2\}$ for all $s > 1$.

**Corollary 3.5.** *Assume that $\mu$ is a probability measure and* (3.10) *holds. If $\underline{R} \geq -c$ and* (3.7) *holds, then*

$$\lambda_n \geq \sup_{\varepsilon\in(0,1),s>1} \left\{ \Big(\log \frac{n\varepsilon \mathrm{e}^{-c}}{sl}\Big) \wedge \sup_{r>0} \frac{1}{r}\Big[1 - cr - 2\beta(r)\big(\varepsilon + \mu\big(\rho > c_1\sqrt{\log s} - c_2\big)\big)\Big]\right\}$$

*for some $c_1, c_2 > 0$ and all $n \geq 1$. Consequently, if in addition* (3.7) *holds for $\beta(r) = \exp[\alpha(1 + r^{-1/\delta})]$ for some $\alpha, \delta > 0$, then $\lambda_n \geq \lambda([\log n - \theta]^+)^\delta$ for some $\lambda, \theta > 0$ and all $n \geq 1$.*

*Proof.* By Theorem 3.4, it suffices to check the second assertion for $\delta \in (0, 1]$. By Corollary 6.3 in [28], there exists $\alpha_1, \alpha_2 > 0$ such that $\mu\big(\rho \geq c_1\sqrt{\log s} - c_2\big) \leq \alpha_1 \exp\big[-\alpha_2(\log s)^{1/(2-\delta)}\big]$ for all $s > 1$. Then the proof is completed by some simple calculations. $\square$

We come back to estimate $b(\tilde{L})$ by using (3.6).

**Corollary 3.6.** *Assume that $\mu$ is a probability measure and $\underline{R} \geq -c$ for some $c \in \mathbb{R}$. We have*

$$b(\tilde{L}) \leq \inf_{t>0} \mathrm{e}^{ct} l \int_M p_t^0(x, x)\mu(\mathrm{d}x). \tag{3.14}$$

*If* (3.10) *holds, then*

$$b(\tilde{L}) \leq \inf_{t>0,r>0} \frac{l\mathrm{e}^{ct}}{\mu(B_o(r))} \int_M \exp\left[\frac{K(\rho(x) + r)^2}{1 - \mathrm{e}^{-Kt}}\right]\mu(\mathrm{d}x). \tag{3.15}$$

*In particular, if $M$ is compact with diameter $D$, we have*

$$b(\tilde{L}) \leq l \inf_{t>0} \exp\left[\frac{KD^2}{1 - \mathrm{e}^{-Kt}} + ct\right]. \tag{3.16}$$

*Proof.* Assume that the right-hand side of (3.14) is finite, by Theorem 3.3 one has $\bar{\lambda} = \infty$ and hence $b(\tilde{L}) < \infty$. Let $n = b(\tilde{L})$, we have $\lambda_n = 0$. Then (3.14) follows from (3.6). Moreover (3.15) follows from (3.14) and (3.13). If $M$ is compact, by (3.11) we obtain $p_t^0(x, x) \leq \exp\big[\frac{KD^2}{1-\mathrm{e}^{-Kt}}\big]$, therefore we obtain (3.16). $\square$

We note that (3.16) does not imply the famous Gromov type result: *there exists $\eta > 0$ such that if $D^2 \inf \underline{R} > -\eta$ then $b(\tilde{L}) \leq l$.* This result was first proved by Gromov for the first Betti number for compact $M$ with some $\eta$ depending only on $d$, using geometric method (cf. [15]). A complete proof of this result for $V = 0$, with some $\eta$ depending only on $l, KD^2$ and $d$, was given by Berard, Besson and Gallot [5] by using a heat kernel comparison deduced from isoperimetric inequalities, We present below a different proof by using the a gradient estimate obtained in [25], with an explicit $\eta$ depending only on $l$ and $KD^2$. As a consequence of our next result, we obtain the Gromov's theorem in the next section with an explicit $\eta$ depending on $d$, see Corollary 4.2.

**Corollary 3.7.** *Assume that $M$ is compact with diameter $D$. Let $K \geq 0$ be such that* (3.10) *holds. Put*

$$\begin{aligned}\eta &:= \sup_{t>0} \frac{1}{1+t} \log \frac{l+1}{l(1+\frac{1}{4t}\exp[KD^2(\frac{1}{8}+\frac{1}{1-\mathrm{e}^{-KD^2}})])} \\ &\geq \frac{1}{1+l\exp[KD^2(1/8+1/(1-\mathrm{e}^{-KD^2})]} \log \frac{l+1}{l+1/4}.\end{aligned}$$

*If $D^2 \inf \underline{R} > -\eta$, then $b(\tilde{L}) \leq l$.*

*Proof.* Let $x \in M$ be fixed. For any $y \in M$, let $u(y) = p^0_{D^2}(x,y)$. We have $p^0_{D^2(1+t)}(y,x) = P^0_{D^2t}u(y)$. From the proof of Corollary 3.5 we see that $\|u\|_\infty \leq \exp\big[\frac{KD^2}{1-\mathrm{e}^{-KD^2}}\big]$. By Theorem 4.4 in [25] with $\lambda = 0$, we obtain

$$\begin{aligned}|\nabla p^0_{D^2(t+1)}(\cdot,x)|(y) &\leq \frac{1}{4D^2t}\exp\left[\frac{KD^2}{1-\mathrm{e}^{-KD^2}}\right]\int_0^D \exp[Kr^2/8]\mathrm{d}r \\ &\leq \frac{1}{4Dt}\exp\left[KD^2\Big(\frac{1}{8}+\frac{1}{1-\mathrm{e}^{-KD^2}}\Big)\right].\end{aligned}$$

Since $\int_M p^0_{D^2(t+1)}(y,x)\mu(\mathrm{d}y) = 1$, there exists $y \in M$ such that $p^0_{D^2(t+1)}(y,x) \leq 1$. We obtain

$$p^0_{D^2(t+1)}(x,x) \leq 1 + \frac{1}{4t}\exp\left[KD^2\Big(\frac{1}{8}+\frac{1}{1-\mathrm{e}^{-KD^2}}\Big)\right].$$

If $D^2 \inf \underline{R} > -\eta$, then there exist $t_0 > 0$ and $\varepsilon > 0$ such that

$$\begin{aligned}D^2 \inf \underline{R} &\geq \varepsilon - \frac{1}{t_0+1}\log\frac{l+1}{l(1+\frac{1}{4t_0}\exp[KD^2(\frac{1}{8}+\frac{1}{1-\mathrm{e}^{-KD^2}}])} \\ &\geq \varepsilon - \frac{1}{t_0+1}\log\frac{l+1}{l\int_M p_{D^2(1+t_0)}(x,x)\mu(\mathrm{d}x)}.\end{aligned}$$

By this and (3.15) we obtain

$$b(\tilde{L}) \leq l\mathrm{e}^{-\varepsilon(1+t_0)}\frac{l+1}{l} < l+1.$$

Therefore $b(\tilde{L}) \leq l$ since $b(\tilde{L}) \in \mathbb{Z}_+$. Finally, taking $t = l\exp[KD^2/8 + KD^2/(1 - \mathrm{e}^{KD^2})]$, we obtain the desired lower bound of $\eta$. □

Finally, we present below three simple examples to illustrate our results obtained in this section.

**Example 3.1.** Let $M$ be noncompact and $\mu$ is a probability measure. If there exists $\alpha > 0$ such that $\overline{\lim}_{\rho\to\infty}(\Delta + \nabla V)\rho \leq -\alpha$, where the limit is taken outside of the cut-locus of $o$, and $\underline{\lim}_{\rho\to\infty}\underline{R} > -\alpha^2/4$, then $0 \notin \sigma_{\rm ess}(\tilde{L})$ and hence $b(\tilde{L}) < \infty$. Indeed (see e.g. (2.8) in [27]), we have $\lim_{n\to\infty} \inf \sigma(-(\Delta + \nabla V)|_{B_o(n)^c}) \geq \alpha^2/4$. By Donnely-Li's decomposition principle [11], we have $\inf \sigma_{\rm ess}(-(\Delta + \nabla V)) \geq \alpha^2/4$. Therefore, by Theorem 3.3 (3), $0 \notin \sigma_{\rm ess}(\tilde{L})$ provided $\underline{\lim}_{\rho\to\infty}\underline{R} > -\alpha^2/4$.

**Example 3.2.** Let $M$ be noncompact with Ricci curvature bounded from below, and $\delta > 1$ a constant. Let $V = -\alpha\tilde{\rho}$ for some $\alpha > 0$ with $\tilde{\rho} \in C^\infty(M)$ such that $\rho^\delta - \tilde{\rho}$ is bounded and $\mu$ is a probability measure, where $\rho$ is as in above. The existence of $\tilde{\rho}$ is guaranteed by a classical approximation theorem and the volume comparison theorem. By Corollary 2.5 in [28], (3.7) holds with $\beta(r) = \exp[c_1(1 + r^{-\delta/[2(\delta-1)]})]$ for some $c_1 > 0$. Therefore by Theorem 3.4, if $\underline{R}$ is bounded from below, then there exist $\lambda, \theta > 0$ such that

$$\lambda_n \geq \lambda([\log n - \theta]^+)^{2(\delta-1)/\delta}, \qquad n \geq 1, \tag{3.17}$$

provided $\delta > 2$. If in addition (3.10) holds, then (3.17) holds for all $\delta > 1$.

**Example 3.3.** Let $M$ be compact. We have $p_t^0(x, x) \leq \alpha(1+t^{-d/2})$ for some $\alpha > 0$. By Theorem 3.4 we obtain $\lambda_n \geq \lambda[n^{2/d} - \theta]$ for some $\lambda, \theta > 0$ and all $n \geq 1$. This is sharp in the order of $n$ according to the known asymptotic estimates obtained in [20] for $V = 0$.

# 4 Applications to Weighted Hodge Laplacians

In this section, we apply results obtained in section 3 to weighted Hodge Laplacians on differential forms. Throughout this section, we fix $p \in [0, d] \cap \mathbb{Z}_+$, and let $\Omega = \Lambda^p$ be the bundle of $p$-forms. Then $l = \frac{d!}{p!(d-p)!}$. Let $\Delta_\mu^p = {\rm d}_\mu^*{\rm d} + {\rm d}{\rm d}_\mu^*$, where ${\rm d}_\mu^*$ is the adjoint of the exterior derivative d on $L^2_{\Lambda^p}(\mu)$. We have (see [8]) ${\rm d}_\mu^* = \delta - {\rm i}_{\nabla V}$ and hence

$$\Delta_\mu^p = \Delta^p - {\rm L}_{\nabla V} = -\Box - {\rm L}_{\nabla V} + \mathcal{R}, \tag{4.1}$$

where $\Delta^p := \delta{\rm d} + {\rm d}\delta$ and $\mathcal{R}$ denote, respectively, the usual Hodge Laplacian and the curvature operator on $p$-forms, $\Box$ is the horizontal (negative) Laplacian as in section 3, and ${\rm L}_X = {\rm d}{\rm i}_X + {\rm i}_X{\rm d}$ is the Lie differentiation in the direction $X$. The formula (4.1) was obtained in [8] for the case that $M$ is oriented, and it remains true in general by passing to the oriented double cover of $M$. Moreover, $(\Delta_\mu^p, \Gamma_0(\Lambda^p))$ is essentially self-adjoint, see e.g. page 692 in [8].

Let $\{E_j\}_{j=1}^d$ be a locally normal frame with dual $\{\omega_j\} \subset \Lambda^1$. We have, for any differential form $\phi$,

$$\mathrm{i}_{\nabla V}\mathrm{d}\phi = \sum_{i,j=1}^{d} \mathrm{i}_{\langle \nabla V, E_i\rangle E_i}(\omega_j \wedge \nabla_{E_j}\phi) = \nabla_{\nabla V}\phi - \sum_{i,j=1}^{d}\langle \nabla V, E_i\rangle \omega_j \wedge (\mathrm{i}_{E_i}\nabla_{E_j}\phi),$$

$$\mathrm{d}\mathrm{i}_{\nabla V}\phi = \sum_{i,j=1}^{d} \mathrm{Hess}_V(E_i, E_j)\omega_i \wedge (\mathrm{i}_{E_j}\phi) + \sum_{i,j=1}^{d}\langle \nabla V, E_j\rangle \omega_i \wedge (\mathrm{i}_{E_j}\nabla_{E_i}\phi).$$

Therefore

$$\mathrm{L}_{\nabla V} = \sum_{i,j=1}^{d} \mathrm{Hess}_V(E_i, E_j)\omega_i \wedge \mathrm{i}_{E_j} + \nabla_{\nabla V} := \mathrm{Hess}_V + \nabla_{\nabla V}.$$

Combining this with (4.1), we obtain

$$\Delta_\mu^p = -\Box - \nabla_{\nabla V} + \mathcal{R} - \mathrm{Hess}_V. \tag{4.2}$$

**Theorem 4.1.** *All results in section 3 hold true for $\Omega = \Lambda^p, \tilde{L} = -\Delta_\mu^p, R = \mathcal{R} - \mathrm{Hess}_V$ and $l = \frac{d!}{p!(d-p)!}$.*

Recall that the Gromov's theorem mentioned in the last section says that for compact $M, \Omega = \Lambda^1$ and $V = 0$, there exists a positive $\eta$ depending only on $d$ such that $D^2 \inf \underline{R} \geq -\eta$ implies $b(\tilde{L}) \leq d$. The following result extends the above Gromov's theorem with an explicit $\eta$ depending only on $d$. We remark that when $M$ is compact, $b(-\Delta_\mu^p)$ is equal to the $p$-th Betti number, see Theorem 5.2 in [8] (note that [8] treated the case that $V \in C^\infty(M)$, but arguments given there work also for $V \in C^2(M)$). Hence Corollary 3.7 and Corollary 4.2 below improve the corresponding results on Betti numbers mentioned right before Corollary 3.7, in the sense that (in particular when $\mathcal{R}$ is not constant) one may choose a nice $V$ such that $\mathcal{R} - \mathrm{Hess}_V$ has a lower bound bigger than that of $\mathcal{R}$.

**Corollary 4.2.** *For the case where $\Omega = \Lambda^1$, we have $R = \mathrm{Ric} - \mathrm{Hess}_V$ and $l = d$. If*

$$D^2 \inf \underline{R} \geq -\frac{\log[(d+1)/(d+1/4)]}{1 + d(1+d^{-1})^{d+9/8}},$$

*then $b(-\Delta_\mu^1) \leq d$.*

*Proof.* Let $\eta$ be defined in Corollary 3.7, we have $\eta < \log \frac{d+1}{d}$. If $-KD^2 := D^2 \inf \underline{R} \geq -\eta$ then $KD^2 < \log \frac{d+1}{d}$ and hence

$$\eta > \frac{\log[(d+1)/(d+1/4)]}{1 + d\exp[(\log \frac{d+1}{d})(\frac{1}{8} + 1 + d)]} = \frac{\log[(d+1)/(d+1/4)]}{1 + d(1+d^{-1})^{d+9/8}} := \eta'.$$

Therefore, if $D^2 \inf \underline{R} \geq -\eta'$ then $D^2 \inf \underline{R} > -\eta$ and hence $b(-\Delta_\mu^1) \leq d$ by Corollary 3.6. $\square$

We remark that in the situation of Corollary 4.2, for any $n > d$ there exist examples with big enough $KD^2$ such that $b(\tilde{L}) \geq n$. Moreover, the above Gromov's theorem does not hold with $D^2$ replacing by the volume of $M$. See e.g. pages 138–139 in [4] for details.

Let $((\mathrm{d}_\mu^*)^p, \mathcal{D}((\mathrm{d}_\mu^*)^p)))$ and $(\mathrm{d}^p, \mathcal{D}(\mathrm{d}^p))$ denote, respectively, the operators $\mathrm{d}_\mu^*$ and d on $L^2_{\Lambda^p}(\mu)$ with domains. Let $C_\mu^{\infty,p} := \cap_{n=1}^\infty \mathcal{D}((\Delta_\mu^p)^n)$. By Theorem 5.3 in [8], $C_\mu^{\infty,p} \subset \Gamma(\Lambda^p)$. Let $\mathrm{d}|_{C_\mu^{\infty,p}}$ (resp. $\mathrm{d}_\mu^*|_{C_\mu^{\infty,p}}$) denote the restriction of $d$ (resp. $d_\mu^*$) on $C_\mu^{\infty,p}$.

**Theorem 4.3.** *Let $R = \mathcal{R} - \mathrm{Hess}_V$. If either $\inf \sigma(-\Delta - \nabla V + \underline{R}) > 0$ or $\underline{\lim}_{\rho\to\infty} \underline{R} > \sup \sigma_{\mathrm{ess}}(\Delta + \nabla V)$, then $\mathrm{im}\,\mathrm{d}^{p-1}$ is closed and*

$$\begin{aligned}
&L^2_{\Lambda^p}(\mu) = \mathrm{im}(\mathrm{d}_\mu^*)^{p+1}|_{\mathcal{D}((\mathrm{d}_\mu^*)^{p+1})} \oplus \mathrm{im}\,\mathrm{d}^{p-1}|_{\mathcal{D}(\mathrm{d}^{p-1})} \oplus \ker\Delta_\mu^p|_{\mathcal{D}(\Delta_\mu^p)},\\
&C_\mu^{\infty,p} = \mathrm{im}\,\mathrm{d}_\mu^*|_{C_\mu^{\infty,p+1}} \oplus \mathrm{im}\,\mathrm{d}|_{C_\mu^{\infty,p-1}} \oplus \ker\Delta_\mu^p|_{\mathcal{D}(\Delta_\mu^p)},\\
&\ker\Delta_\mu^p|_{\mathcal{D}(\Delta_\mu^p)} \cong \frac{\ker \mathrm{d}^p|_{\mathcal{D}(\mathrm{d}^p)}}{\mathrm{im}\,\mathrm{d}^{p-1}|_{\mathcal{D}(\mathrm{d}^{p-1})}} \cong \frac{\ker \mathrm{d}|_{C_\mu^{\infty,p}}}{\mathrm{im}\,\mathrm{d}|_{C_\mu^{\infty,p-1}}}.
\end{aligned} \tag{4.3}$$

*Proof.* By Theorem 3.1 and Theorem 3.3 (3), each condition in Theorem 4.3 implies $0 \notin \sigma_{\mathrm{ess}}(\Delta_\mu^p)$. Then the proof is completed by some classical results (see e.g. Theorem 5.10, Corollary 5.11 in [8], and Corollary 10 in [9]). □

To compare Theorem 4.3 with the corresponding result obtained by Ahmed and Stroock [1], we present the following corollary.

**Corollary 4.4.** *Assume that $\mu$ is a probability measure and $\underline{R}$ is bounded from below. If there exists positive $U \in C^2(M)$ such that $U + V$ is bounded, $\{U \leq N\}$ is compact for each $N > 0$, $|\nabla U| \to \infty$ as $U \to \infty$, and*

$$\overline{\lim}_{U\to\infty} \frac{\Delta U}{|\nabla U|^2} < 1. \tag{4.4}$$

*Then $\sigma_{\mathrm{ess}}(\Delta + \nabla V) = \emptyset$ and hence* (4.3) *holds.*

*Proof.* We have

$$(\Delta - \nabla U)\mathrm{e}^{\varepsilon U} = \varepsilon[\Delta U - |\nabla U|^2 + \varepsilon|\nabla U|^2]\mathrm{e}^{\varepsilon U}, \quad \varepsilon > 0.$$

If (4.4) holds, then there exists $\varepsilon \in (0,1)$ such that $\Delta U - (1-\varepsilon)|\nabla U|^2 \to -\infty$ as $U \to \infty$. By the proof of Theorem 1.2 in [26] (or the paragraph after it), we have

$$\begin{aligned}
&\inf\left\{\frac{\int_M |\nabla u|^2 \mathrm{e}^{-U}\mathrm{d}x}{\int_M u^2\mathrm{e}^{-U}\mathrm{d}x} : 0 \neq u \in C_0^\infty(M), u = 0 \text{ on } \{U \leq n\}\right\}\\
&\geq \varepsilon \inf_{U \geq n}[(1-\varepsilon)|\nabla U|^2 - \Delta U] \to \infty
\end{aligned}$$

as $n \to \infty$. Since $U + V$ is bounded, we have $\inf \sigma(-(\Delta + \nabla V)|_{\{U>n\}}) \to \infty$ as $n \to \infty$. Therefore $\sigma_{\mathrm{ess}}(\Delta + \nabla V) = \emptyset$ by Donnely-Li's decomposition principle. The proof is completed by Theorem 4.3. □

Under several conditions, Theorem 5.1 in [1] provides the following decomposition: *for any* $\phi \in \Gamma(\Lambda^p) \cap L_{\Lambda^p}(\mu)$ *with* $\mathrm{d}_\mu^*\phi \in L^2_{\Lambda^{p-1}}(\mu)$, *there exists a unique* $\eta \in \ker\Delta_\mu^p|_{\mathcal{D}(\Delta_\mu^p)}$ *such that* $\phi - \eta = \mathrm{d}\xi$ *for some* $\xi \in \Gamma(\Lambda^{p-1}) \cap L^2_{\Lambda^{p-1}}(\mu)$. This decomposition is implied by (4.3) (e.g. the first formula). Their conditions (e.g. (1.1), part of (2.8) and (4.4) in [1]) imply that $\overline{\lim}_{U\to\infty} \frac{\Delta U}{|\nabla U|^2} \le 0$, hence Corollary 4.4 applies. Note that their conditions are strong enough to imply the ultracontractivity of $\tilde{P}_t$, so all $L^2$- harmonic forms there are bounded. We refer to [22] for some general criteria of ultracontractivity for semigroups on manifolds.

We remark that for noncompact "topologically tame" manifolds (see Definition 10.1 in [8]), [9] constructed an example for $V$ such that $\sigma_{\mathrm{ess}}(\Delta_\mu^p) = \emptyset$ and $\ker\Delta_\mu^p \cong H^p_{deRham}$, the $p$-th de Rham cohomology, and hence results in Theorem 4.3 hold. More precisely, let $M = M_0 \cup \big(\cup_i M_i\big)$ be a connected Riemannian manifold, where $M_0$ is a compact domain, $M_i$ is isometrically diffeomorphic to $(0,\infty)\times Q_i$ (with the product metric) with a compact manifold $Q_i$ for each $i$, and $\bar{E}_i \cap \bar{E}_j = \emptyset$. Therefore the curvature operator on forms is bounded. Let $\rho$ be the distance function from $M_0$, i.e. $\rho(x) = r$ for $x = (r,q) \in E_i = (0,\infty)\times Q_i$. Let $V \in C^\infty(M)$ such that $V = -c\rho^2$ outside a neighborhood of $M_0$ and $\mu$ is a probability measure. By Proposition 6 and Theorem 15 in [9], one has

$$\ker\Delta_\mu^p \cong \frac{\ker \mathrm{d}|_{\Gamma(\Lambda^p)}}{\mathrm{im}\, \mathrm{d}|_{\Gamma(\Lambda^{p-1})}}. \tag{4.5}$$

Moreover, Theorem 2 in [9] says that $\sigma_{\mathrm{ess}}(\Delta_\mu^p) = \emptyset$. Actually, by Corollary 3.3 in [28] and the proof of Corollary 2.5 in [28], $P_t^0$ is hyperbounded and hence so is $\tilde{P}_t$ by Theorem 3.1. Consequently, by Corollary 3.3 in [28], (3.8) holds for $\beta(r) = \exp[\alpha(1+r^{-1})]$ for some $\alpha > 0$. Then it follows from Corollary 3.5 that $\lambda_n \ge \lambda[\log n - \theta]^+$ for some $\lambda, \theta > 0$ and all $n \ge 1$.

Finally we present below an example to illustrate Theorem 4.3.

**Example 4.1.** Let $M$ be noncompact with a pole $o$, and $\rho$ the Riemannian distance function from $o$. Assume that the Ricci curvature is bounded from below by $-K$ for some $K \ge 0$, and the sectional curvatures are nonpositive. Take $V \in C^2(M)$ such that $V = -c_1\rho^\delta + c_2$ outside of a neighborhood of $o$, where $c_1 > 0, \delta \ge 1$ and $c_2 \in \mathbb{R}$ such that $\mu$ is a probability measure. By Hessian comparison theorem we have $\underline{\lim}_{\rho\to\infty}\mathrm{Hess}_{-V} \ge 0$. Moreover, by Laplacian comparison theorem, $\overline{\lim}_{\rho\to\infty} L\rho \le \sqrt{K(d-1)} - c_1$. Then, by Examples 3.1 and 3.2, we have $0 \notin \sigma_{\mathrm{ess}}(\Delta_\mu^p)$ and hence (4.3) holds according to the proof of Theorem 4.3, provided any one of the following two conditions is fulfilled:

(1) $\delta > 1$ and $\mathcal{R}$ is bounded from below.

(2) $\delta = 1$ and $\underline{\lim}_{\rho\to\infty}\mathcal{R} > -\frac{1}{4}\big(c_1 - \sqrt{K(d-1)}\big)^{+2}$.

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