Classification of Restricted Lie Algebras with Tame Principal Block

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0. Introduction

According to the general structure theory of finite algebraic groups, the representation theory of cocommutative Hopf algebras over an algebraically closed field k of characteristic $p > 0$ subdivides into disciplines that differ with regard to their methods and results. The by now classical modular representation theory of finite groups is concerned with those Hopf algebras whose dual algebras are semisimple. The representation theory of the blocks of these Hopf algebras is governed by the structure of their defect groups. The theory of defect groups rests on the Mackey decomposition theorem, which ensures that the relevant representation theoretic properties are inherited by subgroups. Two other important features of classical modular representation theory are the symmetry of the underlying algebras as well as the invertibility of their Cartan matrices.

It turns out that none of these methods and results retain their validity upon passage to distribution algebras associated to infinitesimal group schemes, i.e., cocommutative Hopf algebras whose dual algebras are local. In default of a general block theory and methods of descent one is led to employ geometric methods related to schemes of unipotent and multiplicative subgroups. Roughly speaking, these tools allow the reduction of problems to algebras whose module categories are amenable to the methods from abstract representation theory. Following this philosophy we combine in this paper these techniques to classify the tame infinitesimal groups of height ≤ 1 as well as the blocks of their distribution algebras in case the underlying base field has characteristic $p \geq 3$. The distribution algebra of an infinitesimal group of height ≤ 1 is the restricted enveloping algebra $u(L)$ of its restricted Lie algebra $(L, [p])$. Thus, our goal is to determine the structure of L and $u(L)$ under the assumption that the principal block of $u(L)$ is tame.

The class of finite dimensional algebras (associative, with an identity) over an algebraically closed field k may be divided into three disjoint classes (see [7, 13]). For the class of representation-finite algebras, which have only finitely many isoclasses of indecomposable modules, the representation theory is well understood. The second class, called the tame

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algebras, consists of representation-infinite algebras for which the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one-parameter families. The third class is formed by the wild algebras whose representation theory comprises the representation theories of all finite dimensional algebras over k. Accordingly, we may realistically hope to classify the indecomposable finite dimensional modules only for the representation-finite and tame algebras. The representation theory of arbitrary tame algebras is still only emerging. However, special biserial algebras, which play a prominent rôle in this paper, form a distinguished class of tame algebras whose representation theory is rather well understood. Important examples of such algebras are provided by blocks of group algebras with cyclic or dihedral defect groups (see [9, 14, 44]) as well as the algebras appearing in the Gel'fand-Ponomarev classification of the singular Harish-Chandra modules over the Lorentz group [34].

Our paper is organized as follows. In view of the results of [25] the Lie algebra L is an extension of $s\ell(2)$ by a representation-finite center. Basic properties of their enveloping algebras are summarized in section 1. The important special case of a one-dimensional p-unipotent center is addressed in section 2. We show that there are three isomorphism types of such extensions, the trivial (split) extension, a nilpotent type and a semisimple type. By combining filtrations of principal indecomposables by Verma modules with Galois covering techniques we establish in section 4 the wildness of the extensions of nilpotent type. The verification the special biseriality of the dense orbit of extensions of semisimple type necessitates detailed information on certain extensions of the Kronecker algebra. At this point the symmetry of restricted enveloping algebras with tame principal block enters in a crucial way: it enables us to lift zero relations from radical cube zero algebras to arbitrary extensions of the Kronecker algebra. With this result in hand, the special biseriality of extensions of semisimple type is a consequence of the decomposablity of certain Loewy factors of principal indecomposable modules. Here module varieties again enter the stage.

For the determination of the structure of the principal block of $u(L)$ one may assume that the toral radical of L is trivial. In section 7 we consider this case and show in Theorem 7.1 that the Morita eqivalence class of a tame principal block is entirely determined by the dimension of the p-unipotent radical of L. It turns out that all tame blocks are special biserial with exactly two simple modules and singular Cartan matrix. Moreover, $u(L)$ is tame whenever its principal block is. As a consequence subalgebras of tame restricted Lie algebras may be wild, a fact which markedly contrasts with the modular representation theory of finite groups. Theorem 7.4 presents the classification of the restricted Lie algebras with tame principal block.

The theory of defects suggests that the principal block of a group algebra is the "most complicated" block. In fact, the representation type of the principal block is "inherited" by all other blocks. While such a relationship remains in effect for cocommutative Hopf algebras with representation-finite principal block [24], the results of section 8 imply that defect theory does in general not provide the right paradigm. Using reduced enveloping algebras, we show in Theorem 8.10 that, for Lie algebras with non-trivial toral radical, the restricted enveloping algebra $u(L)$ may have wild blocks even if its principal block is tame.

The final section 9 provides an application by computing the Krull-Gabriel dimension of restricted enveloping algebras.

1. Principal Indecomposables and the Gabriel Quiver

Throughout this paper we will be working over an algebraically closed field k of positive characteristic $p \geq 3$. Unless mentioned otherwise, a k-vector space is assumed to be finite-dimensional. Modules over a k-algebra Λ are always understood to be left modules. With regard to restricted Lie algebras, we shall use [52] as a general reference for undefined terminology. For basic notions from abstract representation theory the reader is referred to [2, 3, 14].

Given a k-algebra Λ, we let $\mathrm{ind}_\Lambda(d)$ be the set of isoclasses of d-dimensional indecomposable Λ-modules. We say that Λ is *tame*, if it is not representation-finite and if for every $d > 0$ there exist $(\Lambda, k[X])$-bimodules $Q_1, \ldots, Q_{m(d)}$ that are finitely generated free right $k[X]$-modules, such that all but finitely many elements of $\mathrm{ind}_\Lambda(d)$ are of the form $Q_i \otimes_{k[X]} S$ for some simple $k[X]$-module S and $i \in \{1, \ldots, m(d)\}$. It is well known that an algebra is tame if and only if its basic algebra is tame. The reader is referred to [8] and [10] for equivalent concepts of tameness.

Let $(L, [p])$ be a restricted Lie algebra with center $C(L)$ and restricted enveloping algebra $u(L)$. We denote by $\mathcal{B}_0(L)$ the *principal block* of the Hopf algebra $u(L)$, that is, the block whose idempotent is not annihilated by the co-unit of $u(L)$. If $\mathcal{B}_0(L)$ is of tame representation type, then [25, (6.4)] asserts that $L/C(L) \cong s\ell(2)$, and that $C(L)$ is the direct sum of a torus and a nil-cyclic Lie algebra. The general structure of the central extensions of $s\ell(2)$ is well understood. Since the trace form of the two-dimensional standard $s\ell(2)$-module is non-degenerate, and every derivation of $s\ell(2)$ is inner, we have

$$H^2(s\ell(2), k) \hookrightarrow H^1(s\ell(2), s\ell(2)^*) \cong H^1(s\ell(2), s\ell(2)) = (0)$$

for the Chevalley-Eilenberg cohomology groups of the ordinary Lie algebra $s\ell(2)$ with coefficients in k (cf. [18, (1.3)]). Thus, $H^2(s\ell(2), C(L))$ vanishes, and each central extension of $s\ell(2)$ splits, so that $L \cong s\ell(2) \oplus C(L)$ (see [36, (VII.3.3)]). The general theory of restrictable Lie algebras (cf. [52, (II.2.1)]) then shows that any central extension of the restricted Lie algebra $s\ell(2)$ by the center $(C(L), [p])$ has a p-map

$$(x, v)^{[p]} := (x^{[p]}, \psi(x) + v^{[p]}) \qquad \forall\, x \in s\ell(2),\ v \in C(L),$$

where $\psi : s\ell(2) \longrightarrow C(L)$ is a p-semilinear map.

For future reference we record the following important consequence of the above observations.

Proposition 1.1 *Let $(L, [p])$ be a restricted Lie algebra over k.*
(1) *If L is a central extension of $s\ell(2)$, then $u(L)$ is symmetric.*
(2) *If $\mathcal{B}_0(L)$ is tame, then $u(L)$ is symmetric.*

Proof. (1). In view of [47] it suffices to show that the adjoint representation $\mathrm{ad} : L \longrightarrow g\ell(L)$ satisfies $\mathrm{tr}(\mathrm{ad}\, x) = 0$ for every $x \in L$. This, however, follows directly from the decomposition $L = s\ell(2) \oplus C(L)$ as well as $s\ell(2) = [s\ell(2), s\ell(2)]$.

(2). By [25, (6.4)] the algebra L is a central extension of $s\ell(2)$, so that (1) applies. □

In this section we study the principal indecomposable modules and the Gabriel quiver of those central extensions of $s\ell(2)$, where the toral part of the center is trivial. Specifically, we let V be an n-dimensional nil-cyclic Lie algebra with generator v_0. Hence $V = \bigoplus_{i=0}^{n-1} kv_0^{[p]^i}$, where $v_0^{[p]^n} = 0$ and $n \geq 1$. Given a p-semilinear map $\psi : s\ell(2) \longrightarrow V$, we denote by $s\ell(2)_\psi$ the restricted Lie algebra with underlying k-space $s\ell(2) \oplus V$, whose product and p-map are given by

$$[x+v, y+w] := [x,y] \text{ and } (x+v)^{[p]} := x^{[p]} + \psi(x) + v^{[p]} \quad \forall\ x \in s\ell(2), v \in V,$$

respectively.

Given any restricted Lie algebra $(L, [p])$, we let $\mathcal{V}_L := \mathcal{V}_L(k) := \{x \in L\ ;\ x^{[p]} = 0\}$ be the *rank variety* of the trivial L-module. For an arbitrary $u(L)$-module M we recall that

$$\mathcal{V}_L(M) := \{x \in \mathcal{V}_L\ ;\ M|_{u(kx)} \text{ is not free}\} \cup \{0\}$$

is the *rank variety* of M. By general theory [27, 28, 40] $\mathcal{V}_L(M)$ is a conical affine variety whose dimension coincides with the complexity of M. In particular, M is projective if and only if $\mathcal{V}_L(M) = \{0\}$.

If $X \subset L$ is a subset of L, then X_p denotes the smallest $[p]$-invariant subspace containing X. The following result provides an easy necessary, yet, as will be seen in section 4, by no means sufficient criterion for the tameness of $\mathcal{B}_0(s\ell(2)_\psi)$.

Lemma 1.2 *Suppose that $\mathcal{B}_0(s\ell(2)_\psi)$ is tame. Then there exists $x_0 \in \mathcal{V}_{s\ell(2)}$ such that $V = (\psi(x_0))_p$.*

Proof. Given any $x \in s\ell(2)$, we either have $(\psi(x))_p = V$ or $(\psi(x))_p \subset V^{[p]}$. Assume the Lemma to be false, and consider the restricted Lie algebra $L' := s\ell(2)_\psi/V^{[p]}$. Then $L' = s\ell(2) \oplus kw_0$, with $w_0^{[p]} = 0$. Moreover, for $x \in \mathcal{V}_{s\ell(2)}$ and $w \in kw_0$, we have $(x+w)^{[p]} = x^{[p]} + w^{[p]} = 0$, so that $\mathcal{V}_{L'}$ contains the three-dimensional variety $\mathcal{V}_{s\ell(2)} \times kw_0$. On the other hand, $\mathcal{B}_0(L')$ is tame or representation-finite, so that Rickard's Theorem [43, Thm.2] implies $\dim \mathcal{V}_{L'} \leq 2$. This contradiction establishes the Lemma. $\square$

Observe that the ideal $I := u(s\ell(2)_\psi)V \subset u(s\ell(2)_\psi)$ is nilpotent, generated by the central element v_0, and that $u(s\ell(2)) \cong u(s\ell(2)_\psi)/I$. In particular, the algebras $u(s\ell(2)_\psi)$ and $u(s\ell(2))$ have the same simple modules.

It was shown in [38] that $u(s\ell(2))$ possesses p simple modules $S(0), \ldots, S(p-1)$, where $S(i)$ is the simple highest weight module of heighest weight $i \in \mathrm{GF}(p)$ and of dimension $\dim_k S(i) = i+1$. For $0 \leq i \leq \frac{p-3}{2}$ the modules $S(i)$ and $S(p-2-i)$ belong to the same block $\mathcal{B}(i)$. The simple module $S(p-1)$, the so-called *Steinberg module*, is projective. The simple $u(s\ell(2))$-modules have no self-extensions, and $\dim_k \mathrm{Ext}^1_{u(s\ell(2))}(S(i), S(p-2-i)) = 2$ for $0 \leq i \leq p-2$ (cf.[26, 42]). Our next result compares the Gabriel quivers of $u(s\ell(2))$ and $u(s\ell(2)_\psi)$.

Proposition 1.3 *Suppose that $V = (\psi(\mathcal{V}_{s\ell(2)}))_p$. Let S, T be simple $u(s\ell(2)_\psi)$-modules of dimension $\leq p-1$. Then we have $\dim_k \mathrm{Ext}^1_{u(s\ell(2)_\psi)}(S,T) = \dim_k \mathrm{Ext}^1_{u(s\ell(2))}(S,T)$.*

Proof. Let $M := Hom_k(S,T)$. The cohomology-5-term sequence associated to the spectral sequence $H^r(u(s\ell(2)), H^s(u(V),M)) \Rightarrow H^n(u(s\ell(2)_\psi),M)$ (see [41, (I.6.6)]) gives rise to a short exact sequence

$$(*)\qquad \begin{aligned}(0) &\longrightarrow \mathrm{Ext}^1_{u(s\ell(2))}(S,T) \longrightarrow \mathrm{Ext}^1_{u(s\ell(2)_\psi)}(S,T) \longrightarrow H^1(u(V),M)^{s\ell(2)_\psi}\\ &\longrightarrow \mathrm{Ext}^2_{u(s\ell(2))}(S,T).\end{aligned}$$

Since V acts trivially on M, and $s\ell(2)_\psi$ acts trivially on V, it follows that

$$(**)\qquad H^1(u(V),M)^{s\ell(2)_\psi} \cong M^{s\ell(2)_\psi} \cong \mathrm{Hom}_{u(s\ell(2)_\psi)}(S,T).$$

If $S \not\cong T$, then the latter space vanishes, and the asserted result follows.

Alternatively, we have to show that $\mathrm{Ext}^1_{u(s\ell(2)_\psi)}(S,S) = (0)$. By assumption, there exists an element $x_0 \in \mathcal{V}_{s\ell(2)}$ such that $V = (\psi(x_0))_p$. Since $x_0^{[p]^{n+1}} = 0$, the element $x_0 \in u(s\ell(2)_\psi)$ is nilpotent. We also note that $x_0^p = x_0^{[p]} = \psi(x_0)$ lies centrally in $u(s\ell(2)_\psi)$. Accordingly, the element x_0^p belongs to the Jacobson radical J of $u(s\ell(2)_\psi)$.

Now let

$$(0) \longrightarrow S \longrightarrow M \longrightarrow S \longrightarrow (0)$$

be a self-extension of $u(s\ell(2)_\psi)$-modules. The above observation implies that

$$x_0^p.M \subset JM$$

is a submodule of dimension $\dim_k x_0^p.M \leq 2\dim_k S - p < \dim_k S$. As JM has no nonzero submodules of dimension $\leq \dim_k S - 1$, we obtain $x_0^p M = (0)$. In view of $(\psi(x_0))_p = V$, this implies $V.M = (0)$, so that M is a $u(s\ell(2))$-module. Since the $u(s\ell(2))$-module S possesses no non-trivial self-extensions, the above sequence splits, whence $\mathrm{Ext}^1_{u(s\ell(2)_\psi)}(S,S) = (0)$. □

Remark. Since $S(p-1)$ is a projective $u(s\ell(2))$-module, the exact sequence $(*)$ shows that

$$\mathrm{Ext}^1_{u(s\ell(2)_\psi)}(S(p-1),T) = (0) = \mathrm{Ext}^1_{u(s\ell(2)_\psi)}(T,S(p-1))$$

for every simple $u(s\ell(2)_\psi)$-module $T \not\cong S(p-1)$. Consequently, the block $\mathcal{B}(p-1)$ belonging to $S(p-1)$ is primary. Moreover, $(*)$ and $(**)$ give rise to an isomorphism

$$\mathrm{Ext}^1_{u(s\ell(2)_\psi)}(S(p-1),S(p-1)) \cong \mathrm{Hom}_{u(s\ell(2)_\psi)}(S(p-1),S(p-1)),$$

so that this group has dimension 1. Consequently, $\mathcal{B}(p-1)$ is a Nakayama algebra.

Given a simple module S, we let $P(S)$ and $\mathcal{B}(S)$ be the projective cover and the block belonging to S, respectively. If $S = S(i)$, then we write $P(i)$ and $\mathcal{B}(i)$ instead of $P(S(i))$ and $\mathcal{B}(S(i))$, respectively.

Suppose that $(\psi(\mathcal{V}_{s\ell(2)}))_p = V$. Owing to (1.3) and [42, Thm.1] (see [26] for $p=3$) the algebra $u(s\ell(2)_\psi)$ has $\frac{p+1}{2}$ blocks, $\mathcal{B}(0),\ldots,\mathcal{B}(\frac{p-3}{2})$, and $\mathcal{B}(p-1)$. For $0 \leq i \leq \frac{p-3}{2}$ the block $\mathcal{B}(i)$ has two simple modules $S(i)$ and $S(p-2-i)$.

Corollary 1.4 *Suppose that $V = (\psi(\mathcal{V}_{s\ell(2)}))_p$. We have $I\mathcal{B}(i) \subset J^2\mathcal{B}(i)$ for $0 \leq i \leq \frac{p-3}{2}$.*

Proof. Let $r, s \in \{i, p-2-i\}$. Directly from [3, (2.4.3)] and (1.3) we obtain

$$\mathrm{Hom}_{u(s\ell(2)_\psi)}(JP(r)/(J^2+I)P(r), S(s)) \cong \mathrm{Hom}_{u(s\ell(2)_\psi)}(JP(r)/J^2P(r), S(s)).$$

Accordingly, the two semisimple $\mathcal{B}(i)$-modules $JP(r)/(J^2+I)P(r)$ and $JP(r)/J^2P(r)$ have the same multiplicities, and are therefore isomorphic. It follows that $I\mathcal{B}(i) \subset J^2\mathcal{B}(i)$. $\square$

In the following we let $\ell(M)$ denote the length of a module M.

Lemma 1.5 *Let S be a simple $u(s\ell(2)_\psi)$-module of dimension $\leq p-1$. Then the factor module $v_0^iP(S)/v_0^{i+1}P(S)$ is the projective cover of the $u(s\ell(2))$-module S for $0 \leq i \leq p^n-1$. In particular, $\dim_k P(S) = 2p^{n+1}$ and $\ell(P(S)) = 4p^n$. Moreover, each of the two simple $\mathcal{B}(S)$-modules occurs in $P(S)$ with multiplicity $2p^n$.*

Proof. Recall that $I = u(s\ell(2)_\psi)V$ is the ideal generated by v_0, and that $u(s\ell(2)_\psi)/I \cong u(s\ell(2))$. Since $P(S)/IP(S) = P(S)/v_0P(S)$ is a projective $u(s\ell(2))$-module with top S, it is the projective cover $\hat{P}(S)$ of the $u(s\ell(2))$-module S.

Left multiplication by the central element v_0^i induces a surjective, $u(s\ell(2)_\psi)$-linear map

$$P(S)/v_0P(S) \xrightarrow{\pi_i} v_0^iP(S)/v_0^{i+1}P(S)$$

for each $i \in \{0, \ldots, p^n-1\}$.

Note that $k[v_0] \subset u(s\ell(2)_\psi)$ is the restricted enveloping algebra of the p-subalgebra $V \subset s\ell(2)_\psi$. Hence $P(S)|_{k[v_0]}$ is, as a projective module for the local algebra $k[v_0]$, free. As $k[v_0] \cong k[X]/(X^{p^n})$, we have

$$P(S)|_{k[v_0]} \cong k[v_0]^r \cong k[X]/(X^{p^n})^r$$

for some $r \in \mathbb{N}$. Consequently, $\dim_k v_0^iP(S)/v_0^{i+1}P(S) = r$ for $0 \leq i \leq p^n-1$. Thus, π_i is an isomorphism for every such i. Our assertions now follow from the well-known structure of $\hat{P}(S)$ (cf. [42, Thm.1] and [26]). $\square$

Remark. Let $\mathcal{B} \subset u(s\ell(2)_\psi)$ be a block with simple modules S_1, S_2, $P := P_1 \oplus P_2$ its projective generator. By (1.5) the simple module S_i occurs $2p^n$ times as a composition factor of P_j. Thus, $\dim_k \mathrm{Hom}_{u(s\ell(2)_\psi)}(P_i, P_j) = 2p^n$, and $\dim_k \mathrm{End}_{u(s\ell(2)_\psi)}(P) = 8p^n$.

We continue by determining the Loewy layers of the principal indecomposable $u(s\ell(2)_\psi)$-modules. In view of the remark following (1.3) we only consider blocks with two simple modules.

Proposition 1.6 *Suppose that $V = (\psi(\mathcal{V}_{s\ell(2)}))_p$. Let $\mathcal{B} \subset u(s\ell(2)_\psi)$ be a block with two simple modules S_0, S_1. Then we have $J^\ell P(S_i)/J^{\ell+1}P(S_i) = S_{\ell+i} \oplus S_{\ell+i}$ (with indices taken* mod(2)*) for $1 \leq \ell \leq 2p^n-1$. Moreover, $J^{2m+1}P(S_i) = v_0^mJP(S_i)$ for $0 \leq m \leq p^n-1$, and $P(S_i)$ has Loewy length $2p^n+1$.*

Proof. We write $P_i := P(S_i)$ for ease of notation, and define

$$q := \max\{r \in \mathbb{N}_0 \ ; \ J^{2s+1}P_i \text{ is properly contained in } v_0^s P_i \ \ \forall\ s \leq r\}.$$

Then we have $v_0^q P_i \neq (0)$, and by (1.5) $v_0^q P_i/v_0^{q+1}P_i$ is a principal indecomposable $u(s\ell(2))$-module. Owing to [26, 42] this module has Loewy length 3, so that $J^{2q+3}P_i \subset v_0^{q+1}P_i$. By choice of q this implies $J^{2q+3}P_i = v_0^{q+1}P_i$. We proceed by considering the short exact sequence

$$(0) \longrightarrow J^{2q+3}P_i/J^{2q+4}P_i \longrightarrow J^{2q+2}P_i/J^{2q+4}P_i \longrightarrow J^{2q+2}P_i/J^{2q+3}P_i \longrightarrow (0).$$

Note that the right-hand term is a semisimple submodule of $v_0^q P_i/v_0^{q+1}P_i$. By (1.5) it is therefore either zero, or isomorphic to S_i. In the former case, we have $J^{2q+2}P_i = (0)$. Thus, $v_0^{q+1}P_i = J^{2q+3}P_i = (0)$, and $J^{2q+1}P_i$ is a semisimple submodule of the principal indecomposable $u(s\ell(2))$-module $v_0^q P_i$. Thus, $\ell(J^{2q+1}P_i) \leq 1$. Thanks to (1.4) we have $Jv_0^q P_i \subset J^{2q+1}P_i$, proving that $\ell(J^{2q+1}P_i) \geq 3$, a contradiction. Accordingly, the right-hand term of the above sequence is isomorphic to S_i. If the left-hand term is non-zero, then the identity $J^{2q+3}P_i = v_0^{q+1}P_i$ implies that it coincides with the top of $v_0^{q+1}P_i/v_0^{q+2}P_i \cong S_i$. An application of (1.3) now shows that the above sequence splits. This readily yields $J^{2q+3}P_i = (0)$, a contradiction.

Consequently, the first term vanishes, and we obtain $(0) = J^{2q+3}P_i = v_0^{q+1}P_i$. Since $v_0^q P_i \neq (0)$, (1.5) readily implies $q = p^n - 1$, whence $J^{2p^n+1}P_i = (0)$. Since we have seen above that $(0) \neq J^{2q+2}P_i = J^{2p^n}P_i$, it follows that the module P_i has Loewy length $2p^n + 1$.

Let $m \leq q$. By our observations above, we have

$$v_0^{m+1}P_i \subset J^{2m+2}P_i \subset J^{2m+1}P_i \subset v_0^m P_i,$$

with the right-hand inclusion being proper. Owing to (1.5) $Jv_0^m P_i$ is the unique maximal submodule of $v_0^m P_i$ containing $v_0^{m+1}P_i$. Since $Jv_0^m P_i \subset J^{2m+1}P_i$, we have equality, and (1.5) in conjunction with [26, 42] shows

$$v_0^m P_i/J^{2m+1}P_i \cong S_i \ ; \ J^{2m+1}P_i/J^{2m+2}P_i \cong S_{i+1} \oplus S_{i+1} \ ; \ J^{2m+2}P_i/v_0^{m+1}P_i \cong S_i,$$

where the indices are to be interpreted mod(2). This readily implies the asserted result. □

2. One-Dimensional Central Extensions of sℓ(2)

In this section we will consider the case where $V = kv_0$ is one-dimensional. It will turn out that the information obtained here actually suffices for the treatment of the general case.

Recall that for any p-semilinear map $\psi : s\ell(2) \longrightarrow kv_0$ we have a central extension $s\ell(2)_\psi$ with Lie bracket

$$[x + v, y + w] = [x, y] \qquad \forall\ x, y \in s\ell(2),\ v, w \in kv_0,$$

and p-map

$$(x+v)^{[p]} = x^{[p]} + \psi(x) \qquad \forall\, x \in s\ell(2),\ v \in kv_0.$$

The purpose of this section is to show that there are exactly three isomorphism types of these Lie algebras. In the sequel, we let $\mathrm{Aut}_p(L)$ be the automorphism group of the restricted Lie algebra $(L,[p])$. Given $g \in \mathrm{Aut}_p(s\ell(2))$ and $\alpha \in k^\times$, we consider

$$\hat{g}_\alpha : s\ell(2) \oplus kv_0 \longrightarrow s\ell(2) \oplus kv_0 \;\; ; \;\; x+v \mapsto g(x) + \alpha v.$$

Obviously, $\hat{g}_\alpha$ is an isomorphism of vector spaces.

Lemma 2.1 *Let $g \in \mathrm{Aut}_p(s\ell(2))$ and $\alpha \in k^\times$. Then $\hat{g}_\alpha : s\ell(2)_\psi \longrightarrow s\ell(2)_{\alpha(\psi\circ g^{-1})}$ is an isomorphism of restricted Lie algebras. Moreover, if $s\ell(2)_{\psi'} \cong s\ell(2)_\psi$, then there exist $g \in \mathrm{Aut}_p(s\ell(2))$ and $\alpha \in k^\times$ such that $\psi' = \alpha(\psi \circ g^{-1})$.*

Proof. Let $x, y \in s\ell(2)$, $v, w \in kv_0$. Then we have

$$\begin{aligned} \hat{g}_\alpha([x+v, y+w]) &= \hat{g}_\alpha([x,y]) = g([x,y]) = [g(x), g(y)] = [g(x)+\alpha v, g(y)+\alpha w] \\ &= [\hat{g}_\alpha(x), \hat{g}_\alpha(y)], \end{aligned}$$

as well as

$$\begin{aligned} \hat{g}_\alpha(x+v)^{[p]} &= (g(x)+\alpha v)^{[p]} = g(x)^{[p]} + \alpha(\psi\circ g^{-1})(g(x)) = g(x^{[p]}) + \alpha\psi(x) \\ &= \hat{g}_\alpha(x^{[p]} + \psi(x)) = \hat{g}_\alpha((x+v)^{[p]}), \end{aligned}$$

as desired.

Let $\mu : s\ell(2)_\psi \longrightarrow s\ell(2)_{\psi'}$ be an isomorphism of restricted Lie algebras. Then μ is an automorphism of the ordinary Lie algebra $L := s\ell(2) \oplus kv_0$. Since $s\ell(2) = [L, L]$ and $kv_0 = C(L)$ is the center of L, the map μ restricts to automorphisms of these subalgebras. Hence there exist $g \in \mathrm{Aut}(s\ell(2))$ and $\alpha \in k^\times$ such that $\mu = \hat{g}_\alpha$. As $s\ell(2)$ has trivial center, the map g necessarily is an automorphism of the restricted Lie algebra $s\ell(2)$. By assumption μ is compatible with the p-maps, whence

$$\begin{aligned} g(x)^{[p]} + \psi'(g(x)) &= (g(x)+\alpha v)^{[p]} = \mu(x+v)^{[p]} = \mu((x+v)^{[p]}) = \mu(x^{[p]} + \psi(x)) \\ &= g(x^{[p]}) + \alpha\psi(x) = g(x)^{[p]} + \alpha\psi(x) \quad \forall\, x \in s\ell(2),\ v \in kv_0. \end{aligned}$$

Hence $\psi'(g(x)) = \alpha\psi(x)$, so that the desired identity follows. $\square$

Now let $s\ell(2) = kf \oplus kh \oplus ke$ be the standard basis of $s\ell(2)$, i.e.,

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;\; ; \;\; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;\; ; \;\; e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We consider three particular choices of ψ that will turn out to be orbit representatives with respect to a certain group action:

$$\psi_0 := 0 \;\; ; \;\; \psi_n(e) = \psi_n(h) = 0,\ \psi_n(f) = v_0 \;\; ; \;\; \psi_s(e) = 0 = \psi_s(f),\ \psi_s(h) = v_0.$$

The corresponding one-dimensional central extensions of $s\ell(2)$ will be denoted $s\ell(2)_0$, $s\ell(2)_n$ and $s\ell(2)_s$, respectively.

Proposition 2.2 *Let $s\ell(2)_\psi$ be a four-dimensional central extension of $s\ell(2)$. Then $s\ell(2)_\psi$ is isomorphic to $s\ell(2)_0$, or $s\ell(2)_n$, or $s\ell(2)_s$.*

Proof. We let

$$(\ , \ ) : s\ell(2) \times s\ell(2) \longrightarrow k \ \ ; \ \ (x, y) := \mathrm{tr}(x \circ y)$$

be the trace form of $s\ell(2)$ on its standard module, and recall that $(\, , \,)$ is a non-degenerate SL(2)-invariant form. Here SL(2) acts on $s\ell(2)$ via the adjoint representation, that is, via conjugation. We will denote this action by $(g, x) \mapsto g \cdot x$. Accordingly, every element $g \in \mathrm{SL}(2)$ defines an automorphism of $s\ell(2)$, which we also denote by g. It is well-known that every element of $\mathrm{Aut}_p(s\ell(2))$ is of the form $x \mapsto g \cdot x$ for a suitably chosen $g \in \mathrm{SL}(2)$ (see [39, p.281ff]). In the sequel, we will interpret ψ as a p-semilinear map $s\ell(2) \longrightarrow k$. Then there exists an element $x_\psi \in s\ell(2)$ such that

$$\psi(y) = (x_\psi, y)^p \qquad \forall \ y \in s\ell(2).$$

Given $g \in \mathrm{SL}(2)$ and $\alpha \in k^\times$ we have, observing the invariance of the trace form,

$$\alpha^p(\psi \circ g^{-1})(y) = \alpha^p(x_\psi, g^{-1} \cdot y)^p = (\alpha(g \cdot x_\psi), y)^p,$$

so that $x_{\alpha^p(\psi \circ g^{-1})} = \alpha(g \cdot x_\psi)$. In view of (2.1) we see that $s\ell(2)_\psi$ is isomorphic to the extension given by an orbit representative of x_ψ relative to the action

$$(\alpha, g) \cdot x := \alpha \, g \cdot x$$

of the group $k^\times \times \mathrm{SL}(2)$. There are three orbits given by: $0, \ e, \ h$. The standard orthogonality relations of the trace form (cf. [48, (II.1.7)]) show that these representatives yield non-zero multiples of ψ_0, ψ_n and ψ_s, respectively. $\square$

Remark. Let $G := k^\times \times \mathrm{SL}(2)$. By the proof of (2.2) we have

$$s\ell(2) = G \cdot 0 \cup G \cdot h \cup G \cdot e.$$

Direct computation shows that $\dim \overline{G \cdot h} = 3$ so that $G \cdot h$ is the dense open orbit of $s\ell(2)$ relative to G. Moreover, we have $\{0\} \subset \overline{G \cdot e} \subset \overline{G \cdot h}$.

3. Verma Modules

In this section we collect a few general results on certain induced modules that will be referred to as Verma modules. As before, we consider a one-dimensional central extension $s\ell(2)_\psi$ that is defined by a non-zero p-semilinear map $\psi : s\ell(2) \longrightarrow kv_0$. Observe that $(\psi(\mathcal{V}_{s\ell(2)}))_p = kv_0$ in this case, for otherwise we would have $\mathcal{V}_{s\ell(2)} \subset \ker \psi$, which, by equality of dimensions, implies $\mathcal{V}_{s\ell(2)} = \ker \psi$. As $e + f \notin \mathcal{V}_{s\ell(2)}$, this is impossible. By (2.2) we may assume that $\psi(e) = 0$. In this situation we define p-subalgebras

$$s\ell(2)^+_\psi := ke, \quad s\ell(2)^0_\psi := k(h + \psi(h)), \quad s\ell(2)^-_\psi := kf \oplus kv_0, \quad B_\psi := s\ell(2)^0_\psi \oplus s\ell(2)^+_\psi.$$

Let

$$\begin{aligned}\mathcal{X}_\psi &:= \{\lambda \in s\ell(2)^*_\psi \ ; \ \lambda(ke \oplus kf \oplus kv_0) = (0) \text{ and } \lambda(h)^p = \lambda(h)\} \text{ and} \\ \mathcal{X} &:= \{\lambda \in s\ell(2)^* \ ; \ \lambda(ke \oplus kf) = (0) \text{ and } \lambda(h)^p = \lambda(h)\}.\end{aligned}$$

In the sequel, we will often identify the elements of $\mathcal{X}$ and $\mathcal{X}_\psi$. If λ belongs to $\mathcal{X}_\psi$ or $\mathcal{X}$, then the element λ' of $\mathcal{X}_\psi$ or $\mathcal{X}$ is given by $\lambda'(h) := p - 2 - \lambda(h)$.

Note that each element $\lambda \in \mathcal{X}_\psi$ induces a character $u(B_\psi) \longrightarrow k$, which we will also denote by λ. Let M be a $u(s\ell(2)_\psi)$-module. A nonzero element $m \in M \setminus \{0\}$ is a *highest weight vector with highest weight* $\lambda \in \mathcal{X}_\psi$ if $am = \lambda(a)m$ for every $a \in u(B_\psi)$. If, in addition, $M = u(s\ell(2)_\psi)m$, then M is called a *highest weight module with highest weight* λ. The simple module $S(i)$ is a highest weight module with highest weight given by $\lambda(h) = i$. We will occasionally write $S(\lambda)$ instead of $S(i)$. Note that $S(\lambda)$ belongs to a block with two simple modules if and only if $\lambda(h) \neq -1$.

We continue by defining universal highest weight modules, the so-called "Verma modules". Given a character $\lambda \in \mathcal{X}_\psi$, we let k_λ be the one-dimensional $u(B_\psi)$-module on which $u(B_\psi)$ acts via λ, and put

$$Z(\lambda) := u(s\ell(2)_\psi) \otimes_{u(B_\psi)} k_\lambda.$$

The corresponding construction for $u(s\ell(2))$ gives rise to the Verma module $Z_{s\ell(2)}(\lambda) := u(s\ell(2)) \otimes_{u(kh \oplus ke)} k_\lambda$ with highest weight $\lambda \in \mathcal{X}$.

For an arbitrary k-algebra Λ, we let Ω_Λ be the Heller operator. The reader is referred to [2, 3] concerning its definition and basic properties.

Proposition 3.1 *The following statements hold:*

(1) $s\ell(2)_\psi = s\ell(2)^-_\psi \oplus s\ell(2)^0_\psi \oplus s\ell(2)^+_\psi$ *is a triangular decomposition of* $s\ell(2)_\psi$ *with a torus* $s\ell(2)^0_\psi$*, and unipotent constituents* $s\ell(2)^\pm_\psi$.

(2) *We have isomorphisms* $v_0^i Z(\lambda)/v_0^{i+1} Z(\lambda) \cong Z_{s\ell(2)}(\lambda)$ *for* $0 \le i \le p-1$.

(3) *If* $\lambda(h) \neq -1$*, then the* p^2*-dimensional module* $Z(\lambda)$ *is uniserial of length* $2p$ *with simple top* $S(\lambda)$ *and simple socle* $S(\lambda')$.

(4) *If* $\lambda(h) \neq -1$*, then* $\Omega^2_{u(s\ell(2)_\psi)}(Z(\lambda)) \cong Z(\lambda)$.

Proof. (1). The first statement follows immediately from the fact that $\psi(e) = 0$ as well as $(h + \psi(h))^{[p]} = h + \psi(h)$.

(2). Directly from the definition we obtain $\dim_k Z(\lambda)/v_0 Z(\lambda) = p$. Moreover, the canonical projection $\pi_\psi : u(s\ell(2)_\psi) \longrightarrow u(s\ell(2))$ induces a surjective $u(s\ell(2)_\psi)$-linear map

$$Z(\lambda) \longrightarrow Z_{s\ell(2)}(\lambda) \quad ; \quad u \otimes \alpha \mapsto \pi_\psi(u) \otimes \alpha.$$

Consequently, this map gives rise to an isomorphism $Z(\lambda)/v_0 Z(\lambda) \cong Z_{s\ell(2)}(\lambda)$ of $u(s\ell(2)_\psi)$-modules. Left multiplication by the central element v_0^i induces a surjective homomorphism

$$Z(\lambda)/v_0 Z(\lambda) \longrightarrow v_0^i Z(\lambda)/v_0^{i+1} Z(\lambda)$$

of $u(s\ell(2)_\psi)$-modules. Since $\mathcal{V}_{s\ell(2)_\psi}(Z(\lambda)) \subset ke$ (cf. [20, (3.4)]) the module $Z(\lambda)|_{k[v_0]}$ is projective, and we have $Z(\lambda)|_{k[v_0]} \cong k[v_0]^r$ for some $r \in \mathbb{N}_0$. It follows that $\dim_k v_0^i Z(\lambda)/v_0^{i+1} Z(\lambda) = r$ for $0 \le i \le p-1$, implying that the above maps are in fact isomorphisms.

(3). Owing to (2) the module $Z(\lambda)/v_0Z(\lambda)$ has (Loewy) length 2, whence $J^2Z(\lambda) \subset v_0Z(\lambda)$. Since $\lambda(h) \neq -1$, (1.4) yields $v_0Z(\lambda) \subset J^2Z(\lambda)$, so that equality holds. Part (2) now implies that $J^{2i}Z(\lambda)/J^{2i+2}Z(\lambda)$ is a uniserial module of length 2 for $i \in \{0, \ldots, \frac{p-1}{2}\}$. Accordingly, $(J^\ell Z(\lambda))_{\ell \geq 0}$ is a composition series of $Z(\lambda)$, so that $Z(\lambda)$ is uniserial.

Since v_0 belongs to the Jacobson radical of $u(s\ell(2)_\psi)$ the canonical surjection $Z(\lambda) \longrightarrow Z_{s\ell(2)}(\lambda)$ of $u(s\ell(2)_\psi)$-modules induces an isomorphism $\mathrm{Top}(Z(\lambda)) \cong \mathrm{Top}(Z_{s\ell(2)}(\lambda))$. As the latter module is known to have top $S(\lambda)$, the assertion follows. According to (1) the uniseriality of $Z(\lambda)$ yields $\mathrm{Soc}(Z(\lambda)) = \mathrm{Soc}(v_0^{p-1}Z(\lambda)) \cong \mathrm{Soc}(Z_{s\ell(2)}(\lambda)) \cong S(\lambda')$.

(4). Suppose that $\lambda(h) \neq -1$. Thanks to [20, (3.4)] we have $\mathcal{V}_{s\ell(2)_\psi}(Z(\lambda)) \subset ke$. For dimension reasons (see (1.5)) the module $Z(\lambda)$ is not projective, so that equality holds. We may now apply [19, (2.5)] to obtain $\Omega^2_{u(s\ell(2)_\psi)}(Z(\lambda)) \cong Z(\lambda)$. $\square$

Lemma 3.2 *Suppose that $\lambda(h) \neq -1$. There are isomorphisms $J^qZ(\lambda) \cong Z(\mu)/J^{2p-q}Z(\mu)$, where $\mu = \lambda$ if q is even, and $\mu = \lambda'$ if q is odd.*

Proof. We proceed by induction on q. Suppose that $q = 1$, and let $i \in \{0, \ldots, p-2\}$ be such that $i \equiv \lambda(h) \mod(p)$. From the Cartan-Weyl formulae (cf. [52, (I.1.3)]) we see that $f^{i+1} \otimes 1 \in Z(\lambda)$ is a highest weight vector with highest weight λ'. Since its image in $Z_{s\ell(2)}(\lambda)$ generates $\mathrm{Rad}(Z_{s\ell(2)}(\lambda)) \cong S(\lambda')$, (3.1) yields $JZ(\lambda) = u(s\ell(2)_\psi)(f^{i+1} \otimes 1) + v_0Z(\lambda) = u(s\ell(2)_\psi)(f^{i+1} \otimes 1) + J^2Z(\lambda)$. Consequently, $JZ(\lambda) = u(s\ell(2)_\psi)(f^{i+1} \otimes 1)$, and the highest weight vector $f^{i+1} \otimes 1$ induces a surjection

$$Z(\lambda') \longrightarrow JZ(\lambda) \ \ ; \ \ u \otimes 1 \mapsto uf^{i+1} \otimes 1.$$

Since $\ell(JZ(\lambda)) = \ell(Z(\lambda)) - 1 = \ell(Z(\lambda')) - 1$, (3.1(3)) shows that the above map gives rise to an isomorphism $Z(\lambda')/J^{2p-1}Z(\lambda') \cong JZ(\lambda)$.

Now suppose that $q > 1$. By inductive hypothesis there exists a surjection $Z(\mu) \longrightarrow J^{q-1}Z(\lambda)$, which induces a surjection $JZ(\mu) \longrightarrow J^qZ(\lambda)$. We combine this with the case $q = 1$ to obtain a surjection $Z(\mu') \longrightarrow J^qZ(\lambda)$. Owing to (3.1(3)) we have $\ell(J^qZ(\lambda)) = 2p - q = \ell(Z(\mu')/J^{2p-q}Z(\mu'))$, so that there is an isomorphism $Z(\mu')/J^{2p-q}Z(\mu') \cong J^qZ(\lambda)$. $\square$

4. Wildness of the Algebra $\mathbf{u(s\ell(2)_n)}$

We continue by considering the case where $\psi = \psi_n$, i.e., $\psi(e) = 0 = \psi(h)$ and $\psi(f) = v_0$. In this situation, the triangular decomposition is induced by a $\mathbb{Z}$-grading on $s\ell(2)_\psi$, whose homogeneous components ke, kh, kf and kv_0 have degrees 1, 0, -1 and $-p$, respectively.

Lemma 4.1 *There exists a short exact sequence*

$$(0) \longrightarrow Z(\lambda') \longrightarrow P(\lambda) \longrightarrow Z(\lambda) \longrightarrow (0).$$

Proof. Note that our $\mathbb{Z}$-grading is compatible with the p-map, i.e., $(s\ell(2)_\psi)_i^{[p]} \subset (s\ell(2)_\psi)_{pi}$. Consequently, the grading induces a $\mathbb{Z}$-grading on the restricted enveloping algebra $u(s\ell(2)_\psi)$. The PBW-Theorem implies that this grading satisfies the hypotheses of [37, (2.1)]. Hence [37, (4.5)] applies, and the projective $u(s\ell(2)_\psi)$-module $P(\lambda)$ affords a "Z-filtration", i.e., a filtration with factors being Verma modules. According to (1.1) the algebra $u(s\ell(2)_\psi)$ is symmetric. Hence (3.1(3)) yields $\mathrm{Soc}(P(\lambda)) = S(\lambda) = \mathrm{Soc}(Z(\lambda'))$, and our assertion follows from (1.5). $\square$

We require a subsidiary result on the radical of a principal indecomposable $s\ell(2)$-module. Recall that the subspace M_γ of a $u(s\ell(2))$-module M relative to $\gamma \in \mathcal{X}$ is defined via

$$M_\gamma := \{m \in M \ ; \ hm = \gamma(h)m\}.$$

If $M_\gamma \neq (0)$, then γ is called a *weight* and M_γ is the corresponding *weight space*.

For $\lambda \in \mathcal{X}$ such that $\lambda(h) \neq -1$, we let $\hat{P}(\lambda)$ be the principal indecomposable $u(s\ell(2))$-module with top $S(\lambda)$. By results of [26, 42] each weight space of $P(\lambda)$ has dimension 2. Let $i \in \{0, \ldots, p-2\}$ be given such that $\lambda(h) \equiv i \mod(p)$. Since λ' is not a weight of $S(\lambda) \cong \mathrm{Top}(\hat{P}(\lambda))$, we have $\dim_k \mathrm{Rad}(\hat{P}(\lambda))_{\lambda'} = 2$.

We consider the coinduced module $Z'_{s\ell(2)}(\lambda) := \mathrm{Hom}_{u(kf\oplus kh)}(u(s\ell(2)), k_\lambda)$. Thanks to [41, (II.3.7)] we have $Z'_{s\ell(2)}(\lambda) \cong u(s\ell(2)) \otimes_{u(kf\oplus kh)} k_{-\lambda'}$. Moreover, [41, (II.3.8)] shows that $\mathrm{Soc}(Z'_{s\ell(2)}(\lambda)) \cong S(\lambda)$.

Lemma 4.2 *Suppose that $\lambda(h) \neq -1$. Then $e\mathrm{Rad}(\hat{P}(\lambda))_{\lambda'} \neq (0)$.*

Proof. By general theory (cf. [41, (II.11.4)]), the projective $u(s\ell(2))$-module $\hat{P}(\lambda)$ possesses a Z'-filtration. Since $\mathrm{Soc}(Z'(\lambda)) = S(\lambda) = \mathrm{Soc}(\hat{P}(\lambda))$ we thus have $Z'_{s\ell(2)}(\lambda) \subset \hat{P}(\lambda)$. Recall that $Z'_{s\ell(2)}(\lambda)$ has length 2, whence $Z'_{s\ell(2)}(\lambda) \subset \mathrm{Rad}(\hat{P}(\lambda))$. Hence we also have $Z'_{s\ell(2)}(\lambda)_{\lambda'} \subset \mathrm{Rad}(\hat{P}(\lambda))_{\lambda'}$. However, the isomorphism $Z'_{s\ell(2)}(\lambda) \cong u(s\ell(2))_{u(kf\oplus kh)}k_{-\lambda'}$ shows that the λ'-weight space is $k(e^{p-2-i}\otimes 1)$. Since this vector is not annihilated by e, our Lemma follows. $\square$

Proposition 4.3 *Suppose that $\lambda(h) \neq -1$. Then the module $J^{2m-1}P(\lambda)/J^{2m+1}P(\lambda)$ is indecomposable for $m \in \{1, \ldots, p-1\}$. In particular, the heart $H(P(\lambda)) := JP(\lambda)/\mathrm{Soc}(P(\lambda))$ is indecomposable.*

Proof. We proceed in several steps. Since $P(\lambda)$ has a simple top, the exact sequence of (4.1) induces a short exact sequence

$$(0) \longrightarrow Z(\lambda') \longrightarrow JP(\lambda) \longrightarrow JZ(\lambda) \longrightarrow (0).$$

We first claim

(a) *The above exact sequence induces a short exact sequence*

$$(\dagger) \qquad (0) \longrightarrow Z(\lambda')/J^2Z(\lambda') \xrightarrow{\gamma} JP(\lambda)/J^3P(\lambda) \xrightarrow{\eta} JZ(\lambda)/J^3Z(\lambda) \longrightarrow (0).$$

Since (†) is obtained from the above sequence by tensoring with $u(s\ell(2)_\psi)/J^2$, it suffices to verify the injectivity of γ. From (1.6) and (3.1(3)) we have $\ell(JP(\lambda)/J^3P(\lambda)) = 4$ and $\ell(JZ(\lambda)/J^3Z(\lambda)) = 2 = \ell(Z(\lambda')/J^2Z(\lambda'))$. Thus, $2 = \ell(\ker\eta) = \ell(\operatorname{im}\gamma)$, so that γ is injective.

In view of (3.2) the right-hand term of (†) is isomorphic to $Z(\lambda')/J^2Z(\lambda')$. Recall from (3.1) that $J^2Z(\zeta) = v_0Z(\zeta)$ and $Z(\zeta)/v_0Z(\zeta) \cong Z_{s\ell(2)}(\zeta)$ for $\zeta \in \{\lambda, \lambda'\}$. Accordingly, the sequence (†) gives rise to an exact sequence

$$(\dagger\dagger) \qquad (0) \longrightarrow Z_{s\ell(2)}(\lambda') \longrightarrow JP(\lambda)/J^3P(\lambda) \longrightarrow Z_{s\ell(2)}(\lambda') \longrightarrow (0)$$

of $u(s\ell(2))$-modules.

(b) *The exact sequence (††) of $u(s\ell(2))$-modules is an almost split sequence.*
We put $M(\lambda) := JP(\lambda)/J^3P(\lambda)$ and assume that the exact sequence

$$(0) \longrightarrow Z_{s\ell(2)}(\lambda') \longrightarrow M(\lambda) \xrightarrow{\eta} Z_{s\ell(2)}(\lambda') \longrightarrow (0)$$

splits. Thanks to (1.6) the simple modules $S(\lambda)$ and $S(\lambda')$ each occur in $M(\lambda)$ with multiplicity 2. Consequently, every weight space of $M(\lambda)$ has dimension 2. By virtue of our current assumption the weight space $M(\lambda)_{\lambda'}$ is generated by two highest weight vectors, so that $eM(\lambda)_{\lambda'} = (0)$. The canonical surjection $P(\lambda) \longrightarrow \hat{P}(\lambda)$ induces a surjection $M(\lambda) \longrightarrow \mathrm{Rad}(\hat{P}(\lambda))$ that sends $M(\lambda)_{\lambda'}$ onto $\mathrm{Rad}(\hat{P}(\lambda))_{\lambda'}$. Consequently, we also have $e\mathrm{Rad}(\hat{P}(\lambda))_{\lambda'} = (0)$, which contradicts (4.2). As a result, the sequence (††) does not split.

Owing to (1.1) the algebra $u(s\ell(2))$ is symmetric, and [3, (4.12.8)] entails the identity $\mathrm{DTr} = \Omega^2_{u(s\ell(2))}$. Since $\Omega^2_{u(s\ell(2))}(Z_{s\ell(2)}(\lambda')) \cong Z_{s\ell(2)}(\lambda')$, and $\mathrm{End}_{u(s\ell(2))}(Z_{s\ell(2)}(\lambda')) \cong k$, an application of [2, (V.2.4)] shows that the sequence (††) is either split exact or almost split. In view of our above observation, the former alternative does not apply.

Since the non-projective Verma modules of $u(s\ell(2))$ lie at ends of homogeneous tubes (cf. [23, (6.1)]), claim (b) implies the indecomposablility of $M(\lambda)$. We have therefore established the first part of our proposition for $m = 1$.

Now suppose that $1 < m \leq p-1$. From (1.6) we recall that $M(\lambda) = JP(\lambda)/v_0JP(\lambda)$. Left multiplication by v_0^{m-1} thus gives a surjection $M(\lambda) \longrightarrow v_0^{m-1}JP(\lambda)/v_0^mJP(\lambda)$, and (1.6) shows that the latter module is isomorphic to $J^{2m-1}P(\lambda)/J^{2m+1}P(\lambda)$. By the same token, we have $\ell(J^{2m-1}P(\lambda)/J^{2m+1}P(\lambda)) = 4 = \ell(M(\lambda))$, so that $J^{2m-1}P(\lambda)/J^{2m+1}P(\lambda) \cong M(\lambda)$.

We finally show that $H(P(\lambda))$ is indecomposable. Suppose that $H(P(\lambda)) = X \oplus Y$. Then we have $M(\lambda) = H(P(\lambda))/J^2H(P(\lambda)) \cong X/J^2X \oplus Y/J^2Y$. By the above we may assume that $X = J^2X$, whence $X = (0)$. This concludes the proof of our proposition. □

Theorem 4.4 *Let $\mathcal{B} \subset u(s\ell(2)_n)$ be a block with two simple modules. Then $\mathcal{B}$ is wild.*

Proof. Since $\psi_n(f) = v_0$ Proposition 1.3 applies and Gabriel's Theorem (cf. [3, (4.1.7)]) shows that the algebra $\mathcal{B}$ is Morita equivalent to the bound quiver algebra $\Lambda := k[Q]/I$, where Q is the quiver

$$
\begin{tikzcd}
1 \arrow[r, shift left=3, "x_1"] \arrow[r, shift left=1, "y_1"] & 2, \arrow[l, shift left=1, "x_2"] \arrow[l, shift left=3, "y_2"]
\end{tikzcd}
$$

and I is an admissible ideal in the path algebra $k[Q]$ of Q. We consider the factor algebra $\mathcal{A} := \Lambda/J^3$ of Λ by the cube of its Jacobson radical J. Then $\mathcal{A}$ admits a Galois covering $F : R \longrightarrow R/G$ (in the sense of [4, 30]), where $R = k[\tilde{Q}]/\tilde{I}$ is a locally bounded k-category given by the locally finite quiver $\tilde{Q}$ of the form

$$
\cdots\ i \overset{x_i}{\underset{y_i}{\rightrightarrows}} i+1 \overset{x_{i+1}}{\underset{y_{i+1}}{\rightrightarrows}} i+2 \overset{x_{i+2}}{\underset{y_{i+2}}{\rightrightarrows}} i+3 \cdots
$$

The group G is the infinite cyclic group generated by the square of the canonical shift that sends the vertex i to the vertex $i+1$ and the arrows x_i and y_i to x_{i+1} and y_{i+1}, respectively. The ideal $\tilde{I}$ is the admissible ideal in the path category $k[\tilde{Q}]$ given by all elements $u \in k[\tilde{Q}]$ with $F(u) \in I$. Let $F_\lambda : \mathrm{mod}R \longrightarrow \mathrm{mod}R/G$ be the push-down functor associated to the Galois covering F. We denote by $P_R(1)$ and $P_{\mathcal{A}}(1)$ the indecomposable projective R-module and the indecomposable projective $\mathcal{A}$-module corresponding to the vertex 1, respectively. Then we have $P_{\mathcal{A}}(1) = F_\lambda(P_R(1))$ as well as $M := \mathrm{Rad}(P_{\mathcal{A}}(1)) = F_\lambda(N)$, where $N := \mathrm{Rad}(P_R(1))$ (cf. [4, Prop. 3.2]).

We let Q' be the full subquiver of $\tilde{Q}$ of the form

$$
1 \overset{x_1}{\underset{y_1}{\rightrightarrows}} 2 \overset{x_2}{\underset{y_2}{\rightrightarrows}} 3
$$

and put $I' := k[Q'] \cap \tilde{I}$. Then $\Gamma := k[Q']/\tilde{I}'$ is the one-point extension

$$
\Gamma = \begin{pmatrix} k & 0 \\ N & H \end{pmatrix}
$$

of the path algebra H of the Kronecker quiver

$$
2 \overset{x_2}{\underset{y_2}{\rightrightarrows}} 3
$$

by the radical N of the indecomposable projective Γ-module $P_\Gamma(1) = P_R(1)$ (cf. [2, p.71]). Thanks to (4.3) M is an indecomposable $\mathcal{A}$-module of dimension 4, so that N is an indecomposable R-module (H-module) of dimension 4. It follows from the structure of the Auslander-Reiten quiver of the Kronecker algebra (cf. [2, VII.7]) that N is an indecomposable, regular H-module of quasi-length 2 lying in a tube of rank 1. An application of [45, p.260] now shows that the algebra Γ is wild. Consequently, R is also wild, and [11, Proposition 2] yields the wildness of $\mathcal{A}$. As a result, the algebras Λ and $\mathcal{B}$ are wild as well. $\square$

5. Extensions of the Kronecker Algebra

Having collected the structural ingredients of the blocks of enveloping algebras, we will now employ methods from abstract representation theory to pass from the quiver and relations of $u(s\ell(2))$ to the corresponding data for $u(s\ell(2)_\psi)$. It was shown in [26] that the blocks of $u(s\ell(2))$ possessing two simple modules are given by the following quiver

$$\begin{tikzcd} 1 \arrow[r, shift left=3, "x_1"] \arrow[r, shift left=1, "y_1"] & 2 \arrow[l, shift left=1, "x_2"] \arrow[l, shift left=3, "y_2"] \end{tikzcd}$$

and relations $x_1y_2 = y_1x_2$, $y_2x_1 = x_2y_1$, $x_1x_2 = 0$, $y_1y_2 = 0$, $x_2x_1 = 0$, $y_2y_1 = 0$. We denote this basic algebra by $\mathcal{K}$ and observe that $\mathcal{K}$ is the trivial extension $H \ltimes D(H)$ of the path algebra H of the Kronecker quiver by its minimal injective cogenerator $D(H) := \mathrm{Hom}_k(H, k)$. The block we are interested in is a connected algebra Λ_0 with Jacobson radical J such that $\mathcal{K} = \Lambda_0/I$ for some ideal I satisfying $J^3 \subset I \subset J^2$. In particular, Λ_0 and $\mathcal{K}$ have the same Gabriel quiver, and we denote the idempotents of Λ_0 by e_1, e_2. We choose pre-images of the arrows x_i and y_i under the canonical projection $\Lambda_0 \longrightarrow \mathcal{K}$, which, by abuse of notation, will also be denoted x_i and y_i. Thus, the elements $x_i,\ y_i \in e_{i+1}Je_i \setminus e_{i+1}J^2e_i$, with $i \in \{1, 2\}$ and addition of the indices interpreted mod(2), enjoy the following properties:

(1) $\{x_i + e_{i+1}J^2e_i,\ y_i + e_{i+1}J^2e_i\}$ is a basis of $e_{i+1}Je_i/e_{i+1}J^2e_i$.
(2) $y_{i+1}x_i - x_{i+1}y_i \in I$, but $y_{i+1}x_i,\ x_{i+1}y_i \notin I$.
(∗) $x_{i+1}x_i,\ y_{i+1}y_i \in I$.

We require that, in addition to (1) and (2), the generators x_i and y_i satisfy the following crucial conditions:

(3) $x_{i+1}x_i,\ y_{i+1}y_i \in J^3 \qquad 1 \le i \le 2$.
(4) $\mathrm{Top}(Jx_i)$ and $\mathrm{Top}(Jy_i)$ are simple for $1 \le i \le 2$.

Lemma 5.1 *We have*

$$J^\ell x_i = \begin{cases} k(y_{i+1}x_i)^{\frac{\ell+1}{2}} + J^{\ell+1}x_i & \text{if } \ell \text{ is odd} \\ kx_i(y_{i+1}x_i)^{\frac{\ell}{2}} + J^{\ell+1}x_i & \text{if } \ell \text{ is even} \end{cases}$$

$$J^\ell y_i = \begin{cases} k(x_{i+1}y_i)^{\frac{\ell+1}{2}} + J^{\ell+1}y_i & \text{if } \ell \text{ is odd} \\ ky_i(x_{i+1}y_i)^{\frac{\ell}{2}} + J^{\ell+1}y_i & \text{if } \ell \text{ is even.} \end{cases}$$

In particular, the modules Λ_0x_i and Λ_0y_i are uniserial.

Proof. In virtue of property (2) we have

$$y_{i+1}x_i \notin J^2x_i \quad \text{and} \quad x_{i+1}y_i \notin J^2y_i.$$

In view of (4) these elements therefore generate Jx_i and Jy_i, respectively, and we obtain

$$Jx_i = \Lambda_0y_{i+1}x_i = ky_{i+1}x_i + Jy_{i+1}x_i = ky_{i+1}x_i + J^2x_i$$

as well as

$$Jy_i = kx_{i+1}y_i + J^2y_i.$$

Since $Je_i = kx_i + ky_i + J^2e_i$ induction on ℓ in conjunction with (3) shows that

$$J^\ell x_i = \begin{cases} k(y_{i+1}x_i)^{\frac{\ell+1}{2}} + J^{\ell+1}x_i & \text{if } \ell \text{ is odd} \\ kx_i(y_{i+1}x_i)^{\frac{\ell}{2}} + J^{\ell+1}x_i & \text{if } \ell \text{ is even} \end{cases}$$

as well as

$$J^\ell y_i = \begin{cases} k(x_{i+1}y_i)^{\frac{\ell+1}{2}} + J^{\ell+1}y_i & \text{if } \ell \text{ is odd} \\ ky_i(x_{i+1}y_i)^{\frac{\ell}{2}} + J^{\ell+1}y_i & \text{if } \ell \text{ is even.} \end{cases}$$

Consequently, the modules Λ_0x_i and Λ_0y_i are uniserial. $\square$

Let Λ be a self-injective algebra. Recall that Λ is *biserial* if for every indecomposable projective Λ-module P the heart $H(P) := \mathrm{Rad}(P)/\mathrm{Soc}(P)$ is a direct sum of at most two uniserial modules. Following [51, p.174] we say that Λ is *special biserial* if Λ is Morita equivalent to the bound quiver algebra $k[Q]/I$, were the bound quiver (Q, I) satisfies the following conditions:

(SB1) Each vertex of Q is the starting point and end point of at most two arrows.

(SB2) For any arrow α of Q, there is at most one arrow β and one arrow γ such that $\alpha\beta,\ \gamma\alpha \notin I$.

By [51, Lemma 1] special biserial algebras are biserial. Note that the algebra $\mathcal{K}$ is special biserial.

Proposition 5.2 *Suppose that Λ_0 is symmetric of Loewy length $2m+1$, $m \geq 2$, and such that $J^\ell e_i/J^{\ell+1}e_i \cong \mathrm{Top}(\Lambda_0 e_{i+\ell}) \oplus \mathrm{Top}(\Lambda_0 e_{i+\ell})$ for $i \in \{1, 2\}$ and $1 \leq \ell \leq 2m-1$. Then there exist generators x_i, $y_i \in \Lambda_0$ of J satisfying* (1) *through* (4) *and such that*

$$x_{i+1}x_i = 0 = y_{i+1}y_i \qquad 1 \leq i \leq 2.$$

In particular, the algebra Λ_0 is special biserial.

Proof. Proceeding in several steps we first claim that

(i) *$x_2x_1 \in J^2x_1$ and $y_2y_1 \in J^2y_1$.*

Owing to (5.1) we have $Jx_1 = ky_2x_1 + J^2x_1$. The assumption $x_2x_1 \notin J^2x_1$ thus entails the existence of a nonzero element $\lambda \in k$ such that $x_2x_1 \equiv \lambda y_2x_1 \mod J^2x_1$. In view of (3) we thus have

$$y_2x_1 \in J^3,$$

which contradicts property (2). This establishes the first statement, and the second claim follows analogously.

(ii) *We may choose the x_i and y_i such that $x_2x_1 = 0 = y_2y_1$.*

Thanks to (i) we can find elements $u, v \in J^2e_2$ such that

$$x_2x_1 = ux_1 \ \ ; \ \ y_2y_1 = vy_1.$$

Since $J^2e_2/J^3e_2 \cong \mathrm{Top}(\Lambda_0 e_2) \oplus \mathrm{Top}(\Lambda_0 e_2)$ we have $e_1 J^2 e_2 = e_1 J^3 e_2$, so that $u, v \in J^3 e_2$. We put $x_2' := x_2 - u$ and $y_2' := y_2 - v$. As $u, v \in J^3 \subset I$ lie in the kernel of the canonical projection $\Lambda_0 \longrightarrow \mathcal{K}$, properties (1) through (3) hold for x_1, x_2', y_1, y_2'. Moreover, we have

$$x_2' x_1 = (x_2 - u)x_1 = 0 \ \ ; \ \ y_2' y_1 = (y_2 - v)y_1 = 0.$$

Thus, it remains to show that the tops of Jx_2' and Jy_2' are simple. Since $u \in J^3$ we have $y_1 x_2' \equiv y_1 x_2 \ \mathrm{mod}(J^3)$, and property (2) now yields $y_1 x_2' \in Jx_2' \setminus J^2 x_2'$. We claim that $x_1 x_2' \in J^2 x_2'$. This will imply that Jx_2' is generated by $y_1 x_2'$, and hence the top of Jx_2' is simple. Observe that x_2' and y_2 generate Je_2, and consequently $J^3 e_2 = J^2 x_2' + J^2 y_2$. Thanks to (3) we have $x_1 x_2' \in J^3 e_2$, and so $x_1 x_2' = a x_2' + b y_2$ for some elements $a, b \in J^2 e_1$. The relation $x_2' x_1 = 0$ now implies

$$(by_2)x_1 = (ax_2' + by_2)x_1 = x_1(x_2' x_1) = 0.$$

By virtue of (5.1) the modules $\Lambda_0 x_1$ and $\Lambda_0 y_2$ are uniserial. Hence there is $i \in \mathbb{N}$ such that, for the left annihilator $\mathrm{ann}(x_1)$ of x_1 in Λ_0, we have $\mathrm{ann}(x_1) \cap \Lambda_0 y_2 = J^i y_2$. This yields $J^i y_2 x_1 = (0)$. In view of (5.1) we have $Jx_1 = k y_2 x_1 + J^2 x_1$, whence $\Lambda_0 y_2 x_1 = Jx_1$. It follows that

$$J^{i+1} x_1 = J^i y_2 x_1 = (0),$$

and uniseriality implies $i + 1 \geq \ell(\Lambda_0 x_1)$. Our current assumption on the Loewy layers of the projective indecomposable modules $\Lambda_0 e_i$ entails $\ell(\Lambda_0 x_1) = 2m = \ell(\Lambda_0 y_2)$. We therefore conclude that $by_2 \in \mathrm{ann}(x_1) \cap \Lambda_0 y_2 \subset J^{2m-1} y_2 = \mathrm{Soc}(\Lambda_0 y_2)$.

Since $\Lambda_0 x_2$ has Loewy length $2m$, and $x_2' = x_2 - u$ with $u \in J^3$ the module $\Lambda_0 x_2'$ also has Loewy length $2m$. Clearly, we also have $\mathrm{Soc}(\Lambda_0 y_2) = \mathrm{Soc}(\Lambda_0 e_2) = \mathrm{Soc}(\Lambda_0 x_2')$. Hence $by_2 \in \mathrm{Soc}(\Lambda_0 x_2') \subset J^2 x_2'$, so that $x_1 x_2' \in J^2 x_2'$, as required. The proof that $y_1 y_2' \in J^2 y_2'$, and hence Jy_2' has simple top, is completely analogous.

We shall assume from now on that the x_i and y_i are chosen such that $x_2 x_1 = 0 = y_2 y_1$.

(iii) $x_1 x_2$, $y_1 y_2 \in \mathrm{Soc}(\Lambda_0)$.

From (5.1) we readily obtain

$$x_1 x_2 \in e_2 J x_2 \subset y_1 \Lambda_0.$$

Thus, $x_1(x_1x_2) = x_1^2 x_2 = 0$, $x_2(x_1 x_2) = (x_2 x_1)x_2 = 0$, $y_1(x_1 x_2) = (y_1 x_1)x_2 = 0$ and $y_2(x_1 x_2) \in y_2 y_1 \Lambda_0 = (0)$. As a result, $J(x_1 x_2) = (0)$, so that $x_1 x_2 \in \mathrm{Soc}(\Lambda_0)$. The proof for $y_1 y_2$ is completely analogous.

Since Λ_0 is a symmetric algebra, there exists a linear form $\varphi : \Lambda_0 \longrightarrow k$ satisfying $\varphi(ab) = \varphi(ba)$ for all $a, b \in \Lambda_0$ and such that $\ker \varphi$ does not contain any non-zero left ideals (see [3, (1.6)]). By (iii) the space $k x_1 x_2$ is a left ideal, which, owing to $x_2 x_1 = 0$, is annihilated by φ. Consequently, $x_1 x_2 = 0$. By the same token, we have $y_1 y_2 = 0$. This establishes the asserted properties of the generators. The special biseriality of Λ_0 is a direct consequence of these relations. $\square$

Example. The assumption (3) and the symmetry are essential for the special biseriality of Λ_0. Let $\mathcal{C}$ be the algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left=4, "x_1"] \arrow[r, shift left=1.5, "y_1"] & 2, \arrow[l, shift left=1.5, "x_2"'] \arrow[l, shift left=4, "y_2"'] \end{tikzcd}$$

and relations $x_1y_2 = y_1x_2 = x_1x_2 = y_1y_2$, $y_2x_1 = x_2y_1$, $x_2x_1 = 0$, $y_2y_1 = 0$. Then $\mathcal{C}$ is a weakly symmetric biserial radical cube zero algebra which is not special biserial.

6. Special Biseriality of the Algebra $\mathbf{u(s\ell(2)_s)}$

In this section we are going to apply the results of section 5 to study the representation type of the blocks of $u(s\ell(2)_s)$ containing two simple modules. Thus, we assume that $\psi(e) = 0 = \psi(f)$ as well as $\psi(h) = v_0$ and consider a block $\mathcal{B}(\lambda)$ associated to a linear form $\lambda \in \mathcal{X}_\psi$ that satisfies $\lambda(h) \neq -1$. The basic algebra of such a block is readily seen to afford generators x_i, y_i enjoying properties (1) and (2) of section 5. The verification of conditions (3) and (4) requires a detailed analysis of the module $M(\lambda) := JP(\lambda)/J^3P(\lambda)$. Note that (1.6) implies $v_0M(\lambda) = (0)$, so that $M(\lambda)$ also has the structure of a $u(s\ell(2))$-module. Recall the isomorphism $Z'_{s\ell(2)}(\lambda) \cong u(s\ell(2))_{u(kf\oplus kh)}k_{-\lambda'}$. Since $\psi(f) = 0$, the space $B^-_\psi := k(h+v_0)\oplus kf$ is a p-subalgebra of $s\ell(2)_s$. Given $\lambda \in \mathcal{X}_\psi$, we put $Z'(\lambda) := u(s\ell(2)_s) \otimes_{u(B^-_\psi)} k_{-\lambda'}$.

Proposition 6.1 *Suppose that $\lambda(h) \neq -1$. Then we have $M(\lambda) \cong Z_{s\ell(2)}(\lambda') \oplus Z'_{s\ell(2)}(\lambda)$.*

Proof. Thanks to (3.1(3)) there is a surjection $P(\lambda) \longrightarrow Z(\lambda)$, which induces a surjection $JP(\lambda)/J^3P(\lambda) \longrightarrow JZ(\lambda)/J^3Z(\lambda)$. Owing to (3.2) and (3.1(2),(3)) the latter module is isomorphic to $Z(\lambda')/J^2Z(\lambda') \cong Z_{s\ell(2)}(\lambda')$. Let N be the kernel of the resulting surjection $\eta : M(\lambda) \longrightarrow Z_{s\ell(2)}(\lambda')$. It readily follows from (1.6) that $\ell(N) = 2$. If N was semisimple, then $N = \mathrm{Soc}(M(\lambda)) = JM(\lambda)$. Since $Z_{s\ell(2)}(\lambda')$ is not semisimple, this is impossible. Hence N is uniserial. We thus have an exact sequence

$$(*) \qquad (0) \longrightarrow N \longrightarrow M(\lambda) \longrightarrow Z_{s\ell(2)}(\lambda') \longrightarrow (0)$$

whose extreme terms are uniserial modules of length 2.

Note that the linear map $\gamma : s\ell(2)_s \longrightarrow s\ell(2)_s$ given by

$$\gamma(e) = f \;\; ; \;\; \gamma(f) = e \;\; ; \;\; \gamma(h) = -h \;\; ; \;\; \gamma(v_0) = -v_0$$

is an automorphism of the restricted Lie algebra $s\ell(2)_s$. By abuse of notation we will denote the corresponding automorphism of $u(s\ell(2)_s)$ also by γ. For an arbitrary $u(s\ell(2)_s)$-module M we let $M^{(\gamma)}$ be the $u(s\ell(2)_s)$-module with underlying k-space M, and action given by

$$u \cdot m := \gamma^{-1}(u)m \qquad \forall\ u \in u(s\ell(2)_s),\ m \in M.$$

Note that $M \mapsto M^{(\gamma)}$ is an auto-equivalence of the category $\mathrm{mod}(u(s\ell(2)_s))$ of finite dimensional $u(s\ell(2)_s)$-modules. Since each simple $u(s\ell(2)_s)$-module S is uniquely determined by its dimension, we have

$$S^{(\gamma)} \cong S.$$

Hence $P(\lambda)^{(\gamma)} \cong P(\lambda)$ and $M(\lambda)^{(\gamma)} \cong M(\lambda)$, with the latter isomorphism following from the fact that $M(\lambda)$ is a "Loewy factor" of $P(\lambda)$. Let ω be the automorphism of $s\ell(2)$ that is induced by γ. Then we have an isomorphism

$$M(\lambda)^{(\omega)} \cong M(\lambda)$$

of $u(s\ell(2))$-modules. According to [40, (2.2)] this implies that the rank variety $\mathcal{V}_{s\ell(2)}(M(\lambda))$ is invariant under ω, i.e., $\omega(\mathcal{V}_{s\ell(2)}(M(\lambda))) = \mathcal{V}_{s\ell(2)}(M(\lambda))$.

Since every indecomposable $\mathcal{K}$-module of length 2 is periodic (cf. [2, (VII.7)]), we conclude the periodicity of N and $Z_{s\ell(2)}(\lambda')$. Hence both modules have complexity 1, so that their rank varieties are one-dimensional. It now follows from [27, (2.2)] that the rank varieties of these indecomposable modules are in fact lines. We have $\mathcal{V}_{s\ell(2)}(Z_{s\ell(2)}(\lambda')) = ke$ (cf. [20, (3.4)]), and there is $x_0 \in \mathcal{V}_{s\ell(2)}$ with $\mathcal{V}_{s\ell(2)}(N) = kx_0$. Since $M(\lambda)$ is not projective, the sequence $(*)$ readily yields

$$\{0\} \neq \mathcal{V}_{s\ell(2)}(M(\lambda)) \subset kx_0 \cup ke.$$

The assumption $kx_0 = ke$ thus implies $\mathcal{V}_{s\ell(2)}(M(\lambda)) = ke$, which contradicts the ω-invariance of $\mathcal{V}_{s\ell(2)}(M(\lambda))$. Hence $kx_0 \cap ke = (0)$. An application of [27, (2.1)] gives

$$\mathcal{V}_{s\ell(2)}(Z_{s\ell(2)}(\lambda')^* \otimes_k N) = \mathcal{V}_{s\ell(2)}(Z_{s\ell(2)}(\lambda')) \cap \mathcal{V}_{s\ell(2)}(N) = (0),$$

so that $Z_{s\ell(2)}(\lambda')^* \otimes_k N$ is injective. Observing [3, (3.1.8)] we obtain

$$\mathrm{Ext}^1_{u(s\ell(2))}(Z_{s\ell(2)}(\lambda'), N) \cong \mathrm{Ext}^1_{u(s\ell(2))}(k, Z_{s\ell(2)}(\lambda')^* \otimes_k N) = (0),$$

and the sequence $(*)$ splits.

It remains to identify the module N. From (1.6) we see that N has dimension p, with $\mathrm{Top}(N) \cong S(\lambda')$ and $\mathrm{Soc}(N) \cong S(\lambda)$. Since $(*)$ splits, [27, (2.1)] yields $\mathcal{V}_{s\ell(2)}(M(\lambda)) = ke \cup \mathcal{V}_{s\ell(2)}(N)$. The ω-invariance then implies

$$ke \cup \mathcal{V}_{s\ell(2)}(N) = kf \cup \omega(\mathcal{V}_{s\ell(2)}(N)),$$

whence $\mathcal{V}_{s\ell(2)}(N) = kf$. Thus, $N|_{u(kf)}$ is not projective, so that $\dim_k \mathrm{Soc}_{u(kf)}(N) > \frac{1}{p}\dim_k N = 1$ (cf. [5, (2.2)] or [20, (3.1)]). Since the $u(kf)$-socle of the simple $u(s\ell(2))$-module $S(\zeta)$ is the one-dimensional weight space $S(\zeta)_{-\zeta}$, it follows that $\mathrm{Soc}_{u(kf)}(N) \not\subset \mathrm{Soc}(N)$. Observing $\mathrm{Top}(N) \cong S(\lambda')$ we conclude the existence of an element $v \in N_{-\lambda'} \cap \mathrm{Soc}_{u(kf)}(N)$ that generates N. Accordingly, the map $u \otimes \alpha \mapsto \alpha uv$ induces a surjection $Z'_{s\ell(2)}(\lambda) \longrightarrow N$, which, by equality of dimensions, is in fact an isomorphism. □

Let $\lambda \in \mathcal{X}_{\psi_s}$ be given such that $\lambda(h) \neq -1$. The block $\mathcal{B}(\lambda)$ has two simple modules $S(1) := S(\lambda)$ and $S(2) := S(\lambda')$. We denote the corresponding principal indecomposable modules by $P(1)$ and $P(2)$. The Verma modules are labelled in the same fashion. The Hopf algebra $u(s\ell(2)_s)$ is isomorphic to its opposite algebra $u(s\ell(2)_s)^{\mathrm{op}}$. Note that this isomorphism preserves the dimensions of the simple modules, so that $\mathcal{B} \cong \mathcal{B}^{\mathrm{op}}$ for every block $\mathcal{B} \subset u(s\ell(2)_s)$. It now follows from general theory (cf. [3, (2.2.6)]) that $\Lambda_0 := \mathrm{End}_{\mathcal{B}(\lambda)}(P(1)\oplus$

$P(2)$) is the basic algebra of $\mathcal{B}(\lambda)$. By the same token, we have $\Lambda_0 = e\mathcal{B}(\lambda)e$ for a suitably chosen idempotent e of $\mathcal{B}(\lambda)$. Let J be the Jacobson radical of $u(s\ell(2)_s)$. Then $A := \Lambda_0/eJ^3e$ is isomorphic to $\mathrm{End}_{\mathcal{B}(\lambda)}(\tilde{P})$, where $\tilde{P} = \tilde{P}(1) \oplus \tilde{P}(2)$, and $\tilde{P}(i) := P(i)/J^3P(i)$. Likewise, $\mathcal{K} \cong \Lambda_0/eIe \cong \mathrm{End}_{\mathcal{B}(\lambda)}(\hat{P})$, with $\hat{P} = \hat{P}(1) \oplus \hat{P}(2)$, and $\hat{P}(i) := P(i)/IP(i)$, is the basic algebra of the block $\hat{\mathcal{B}}(\lambda)$ of $u(s\ell(2))$ affording the simple modules $S(1)$ and $S(2)$. Thanks to (6.1) we have

$$J\tilde{P}(i) = Z_{s\ell(2)}(i+1) \oplus Z'_{s\ell(2)}(i),$$

where the addition is interpreted mod(2). Since $Z_{s\ell}(i)$ and $Z'_{s\ell(2)}(i)$ have tops $S(i)$ and $S(i+1)$, respectively, we can define A-linear maps

$$\tilde{x}_i : \tilde{P}(i+1) \longrightarrow Z_{s\ell(2)}(i+1) \subset J\tilde{P}(i) \quad ; \quad \tilde{y}_i : \tilde{P}(i+1) \longrightarrow Z'_{s\ell(2)}(i) \subset J\tilde{P}(i).$$

By setting $\tilde{x}_i(\tilde{P}(i)) = (0) = \tilde{y}_i(\tilde{P}(i))$ we view $\tilde{x}_i$ and $\tilde{y}_i$ as elements of A satisfying

$$\mathrm{im}\,\tilde{x}_i = Z_{s\ell(2)}(i+1) \quad ; \quad \mathrm{im}\,\tilde{y}_i = Z'_{s\ell(2)}(i).$$

Recall the automorphism $\gamma \in \mathrm{Aut}_p(s\ell(2)_s)$ that was employed in the proof of (6.1). Direct computation shows that $Z(\lambda)^{(\gamma)} \cong Z'(\lambda')$. It thus follows from (3.1(3)) that $Z'(\lambda)$ is a p^2-dimensional uniserial module of length $2p$ with simple top $S(\lambda')$ and simple socle $S(\lambda)$. By the same token we have $Z'(\lambda)/v_0Z'(\lambda) \cong Z'_{s\ell(2)}(\lambda)$.

In view of (3.1(2)) and the foregoing remarks there exist elements $x_i,\, y_i \in \Lambda_0$ with $\mathrm{im}\, x_i = Z(i+1) \subset JP(i)$, and $\mathrm{im}\, y_i = Z'(i) \subset JP(i)$, and whose residue classes coincide with $\tilde{x}_i$, and $\tilde{y}_i$, respectively. By construction, the subalgebra Q generated by these elements satisfies $Q + \mathrm{Rad}(\Lambda_0)^2 = \Lambda_0$, and an application of [3, (1.2.8)] shows that $Q = \Lambda_0$.

Now let $\pi : \tilde{P} \longrightarrow \hat{P}$ be the canonical projection, and observe that $\ker \pi = v_0\tilde{P}$. We denote by $\hat{x}_i,\, \hat{y}_i \in \Lambda_0/eIe$ the residue classes of $\tilde{x}_i$ and $\tilde{y}_i$, respectively. By definition, we have $\hat{x}_i \circ \pi = \pi \circ \tilde{x}_i$ and $\hat{y}_i \circ \pi = \pi \circ \tilde{y}_i$ for $1 \leq i \leq 2$.

Lemma 6.2 *The elements $x_i,\, y_i \in \Lambda_0$ satisfy conditions* (1) *through* (4) *of section 5.*

Proof. We begin by verifying the following claim:

(a) $v_0\tilde{P}(i+1) \cap Z_{s\ell(2)}(i) = (0) = v_0\tilde{P}(i+1) \cap Z'_{s\ell(2)}(i+1)$.

It follows from (1.5) and (1.6) that $v_0\tilde{P}(i+1)$ is a simple submodule of $J\tilde{P}(i+1)$. Accordingly, the assumption $v_0\tilde{P}(i+1) \cap Z_{s\ell(2)}(i) \neq (0)$ entails $v_0\tilde{P}(i+1) \subset Z_{s\ell(2)}(i)$. The canonical projection thus induces a decomposition

$$J\hat{P}(i+1) \cong Z_{s\ell(2)}(i)/v_0\tilde{P}(i+1) \oplus Z'_{s\ell(2)}(i+1),$$

which contradicts the indecomposability of $J\hat{P}(i+1)$.

(b) *We have* $\ker(\hat{x}_i|_{\hat{P}(i+1)}) \cong Z_{s\ell(2)}(i)$ *and* $\ker(\hat{y}_i|_{\hat{P}(i+1)}) \cong Z'_{s\ell(2)}(i+1)$.

Directly from (a) we obtain $\mathrm{im}\,\hat{x}_i \cong \mathrm{im}\,\tilde{x}_i$ and $\mathrm{im}\,\hat{y}_i \cong \mathrm{im}\,\tilde{y}_i$. Hence the map $\hat{x}_i|_{\hat{P}(i+1)} : \hat{P}(i+1) \longrightarrow Z_{s\ell(2)}(i+1)$ is surjective. The Z-filtration of $\hat{P}(i+1)$ (cf. [41, (II.11.4)]) provides a surjection $f : \hat{P}(i+1) \longrightarrow Z_{s\ell(2)}(i+1)$ with $\ker f \cong Z_{s\ell(2)}(i)$. Thus, both maps define a projective cover of $Z_{s\ell(2)}(i+1)$, and there exists an automorphism $\zeta : \hat{P}(i+1) \longrightarrow \hat{P}(i+1)$ with $f \circ \zeta = \hat{x}_i|_{\hat{P}(i+1)}$. Its restriction to $\ker(\hat{x}_i|_{\hat{P}(i+1)})$ is the asserted isomorphism.

The proof of the second statement proceeds analogously by using the modules $Z'_{s\ell(2)}$ in place of the $Z_{s\ell(2)}$'s.

(c) *The elements $\hat{x}_i$, $\hat{y}_i$ satisfy the standard relations of the algebra $\mathcal{K}$.*

We begin by verifying $\hat{x}_{i+1} \circ \hat{x}_i = 0$. The assumption $\hat{x}_{i+1} \circ \hat{x}_i \neq 0$ implies $\operatorname{im} \hat{x}_i \not\subset \ker(\hat{x}_{i+1}|_{\hat{P}(i)})$. From (a) and (b) we see that $\operatorname{im} \hat{x}_i \cong Z_{s\ell(2)}(i+1)$ and $\ker(\hat{x}_{i+1}|_{\hat{P}(i)}) \cong Z_{s\ell(2)}(i+1)$ are generated by highest weight vectors of highest weight μ corresponding to $i+1$. Since the weight space $\mathrm{Rad}(\hat{P}(i))_\mu$ is two-dimensional, we conclude that $e\mathrm{Rad}(\hat{P}(i))_\mu = (0)$. As this contradicts (4.2), the desired relations follow. The relations $\hat{y}_{i+1} \circ \hat{y}_i = 0$ can be proven similarly by replacing the $Z_{s\ell(2)}$'s by $Z'_{s\ell(2)}$'s.

We continue by showing that $\hat{y}_{i+1} \circ \hat{x}_i \neq 0$. Assume that $\hat{y}_{i+1} \circ \hat{x}_i = 0$. Then we have $\operatorname{im} \hat{x}_i \subset \ker(\hat{y}_{i+1}|_{\hat{P}(i)})$. Since both modules have length 2, we have equality, and (b) gives rise to

$$Z_{s\ell(2)}(i+1) \cong \operatorname{im} \hat{x}_i = \ker(\hat{y}_{i+1}|_{\hat{P}(i)}) \cong Z'_{s\ell(2)}(i).$$

Since $\mathcal{V}_{s\ell(2)}(Z_{s\ell(2)}(i+1)) = ke$ while $\mathcal{V}_{s\ell(2)}(Z'_{s\ell(2)}(i)) = kf$, we have reached a contradiction. The same arguments show that $\hat{x}_{i+1} \circ \hat{y}_i \neq 0$.

By the above we have non-zero maps $\hat{y}_{i+1} \circ \hat{x}_i, \hat{x}_{i+1} \circ \hat{y}_i : \hat{P}(i+1) \longrightarrow \mathrm{Soc}(\hat{P}(i+1))$. Schur's Lemma now provides scalars $\alpha_i \in k \setminus \{0\}$ such that

$$\hat{y}_{i+1} \circ \hat{x}_i = \alpha_i \; \hat{x}_{i+1} \circ \hat{y}_i \qquad\qquad 1 \le i \le 2.$$

By re-scaling we may assume that $\alpha_2 = 1$.

Since $\mathcal{K}$ is symmetric, there exists a linear form $\varphi \in \mathcal{K}^*$ whose kernel contains no non-zero left ideals and such that $\varphi(ab) = \varphi(ba)$ for every $a, b \in \mathcal{K}$. In particular, $\mathcal{K}\hat{y}_2\hat{x}_1 = k\hat{y}_2\hat{x}_1$ does not belong to $\ker \varphi$, so that $\varphi(\hat{y}_2\hat{x}_1) \neq 0$. We thus obtain, observing $\hat{y}_1\hat{x}_2 = \hat{x}_1\hat{y}_2$,

$$\varphi(\hat{y}_2\hat{x}_1) = \alpha_1\varphi(\hat{x}_2\hat{y}_1) = \alpha_1\varphi(\hat{y}_1\hat{x}_2) = \alpha_1\varphi(\hat{x}_1\hat{y}_2) = \alpha_1\varphi(\hat{y}_2\hat{x}_1).$$

Thus, $\alpha_1 = 1$, and $\hat{x}_i$, $\hat{y}_i$ are the standard generators of $\mathcal{K}$.

(d) *We have $\tilde{x}_{i+1} \circ \tilde{x}_i = 0 = \tilde{y}_{i+1} \circ \tilde{y}_i$.*

Since $\hat{x}_{i+1} \circ \hat{x}_i = 0$, we have $\operatorname{im}(\tilde{x}_{i+1} \circ \tilde{x}_i) \subset v_0\tilde{P}(i+1)$. As we also have $\operatorname{im}(\tilde{x}_{i+1} \circ \tilde{x}_i) \subset \operatorname{im} \tilde{x}_{i+1} \subset Z_{s\ell(2)}(i)$, claim (a) gives the desired result.

Note that claims (c) and (d) imply the validity of conditions (1) through (3) of section 5 for the generators x_i, y_i of Λ_0. It remains to verify condition (4). Let $U \subset P$ be a uniserial $\mathcal{B}(\lambda)$-module, $z : P \longrightarrow U$ a surjection. Since P is injective, we have an injection

$$\mathrm{Hom}_{\mathcal{B}(\lambda)}(U, P) \hookrightarrow \Lambda_0 \;\; ; \;\; f \mapsto f \circ z,$$

whose image coincides with $\Lambda_0 z$. By the same token, there exists an exact sequence

$$(0) \longrightarrow \mathrm{Hom}_{\mathcal{B}(\lambda)}(P/\mathrm{Soc}(P), P) \longrightarrow \mathrm{Hom}_{\mathcal{B}(\lambda)}(P, P) \longrightarrow \mathrm{Hom}_{\mathcal{B}(\lambda)}(\mathrm{Soc}(P), P) \longrightarrow (0),$$

so that $\mathrm{Rad}(\Lambda_0) \subset \mathrm{Hom}_{\mathcal{B}(\lambda)}(P/\mathrm{Soc}(P), P)$. Since the right-hand space is a left ideal of Λ_0 consisting of nilpotent endomorphisms, we have in fact equality.

Let f be an element of $\mathrm{Rad}(\Lambda_0)\mathrm{Hom}_{\mathcal{B}(\lambda)}(U, P)$. Then $f = \sum_{i=1}^{r} g_i \circ h_i$ with $g_i \in \mathrm{Rad}(\Lambda_0)$ and $h_i \in \mathrm{Hom}_{\mathcal{B}(\lambda)}(U, P)$. Thus, $f(\mathrm{Soc}(U)) = (0)$ and we have

$$\mathrm{Rad}(\mathrm{Hom}_{\mathcal{B}(\lambda)}(U, P)) \subset \mathrm{Hom}_{\mathcal{B}(\lambda)}(U/\mathrm{Soc}(U), P).$$

Conversely, let $f : U \longrightarrow P$ be given such that $f(\mathrm{Soc}(U)) = (0)$. Since U has a simple socle, we can find $\mathcal{B}(\lambda)$-modules Q_1, Q_2 such that $P = Q_1 \oplus Q_2$, $U \subset Q_1$, and $\mathrm{Soc}(U) = \mathrm{Soc}(Q_1)$. We consider the map $\bar{f} : U \oplus Q_2 \longrightarrow P$ sending $u + q$ to $f(u)$. As P is injective, there exists a homomorphism $\psi : P \longrightarrow P$ such that $\psi \circ \iota = \bar{f}$, where $\iota : U \oplus Q_2 \hookrightarrow P$ is the natural embedding. Since $\mathrm{Soc}(P) = \mathrm{Soc}(U \oplus Q_2)$, we see that $\psi \in \mathrm{Rad}(\Lambda_0)$. Consequently,

$$f = \bar{f}|_U = \psi \circ \iota|_U \in \mathrm{Rad}(\Lambda_0)\mathrm{Hom}_{\mathcal{B}(\lambda)}(U, P).$$

Induction now shows

$$\mathrm{Rad}(\Lambda_0)^i \mathrm{Hom}_{\mathcal{B}(\lambda)}(U, P) = \mathrm{Hom}_{\mathcal{B}(\lambda)}(U/\mathrm{Soc}^i(U), P) \quad \forall\, i \geq 1,$$

where $\mathrm{Soc}^i(U)$ denotes the i-th term of the socle series of U. Since U is uniserial, we conclude that $\Lambda_0 z \cong \mathrm{Hom}_{\mathcal{B}(\lambda)}(U, P)$ is a uniserial Λ_0-module.

Recall that the $\mathcal{B}(\lambda)$-modules $Z(i)$ and $Z'(i+1)$ are uniserial. The foregoing observations thus imply the uniseriality of $\Lambda_0 x_i$ and $\Lambda_0 y_i$. In particular, condition (4) holds. □

By way of illustration we record an immediate consequence, which will be generalized and elaborated on in the succeeding section.

Theorem 6.3 *Let $\mathcal{B} \subset u(s\ell(2)_s)$ be a block with two simple modules. Then $\mathcal{B}$ is special biserial. In particular, $\mathcal{B}$ is tame.*

Proof. By the introductory remarks of section 3 we have $(\psi(\mathcal{V}_{s\ell(2)}))_p = kv_0$. Let $e \in \mathcal{B}$ be an idempotent, such that $\Lambda_0 := e\mathcal{B}e$ is the basic algebra of $\mathcal{B}$. In view of (1.3) the canonical projection $\pi : u(s\ell(2)_s) \longrightarrow u(s\ell(2))$ sends $\mathcal{B}$ onto a block $\hat{\mathcal{B}}$ of $u(s\ell(2))$. By the same token, $\mathcal{K} \cong \pi(\Lambda_0) = \pi(e)\hat{\mathcal{B}}\pi(e)$ is the basic algebra of $\hat{\mathcal{B}}$. Owing to (6.2) the algebra Λ_0 affords generators x_i, y_i that satisfy conditions (1) through (4) of section 5. Since $u(s\ell(2)_s)$ is symmetric, we may now apply (1.6) and (5.2) to see that Λ_0 is special biserial. In virtue of [53, (2.4)] (see also [12, (5.2)] for a proof via covering techniques) this implies the tameness of Λ_0. It follows that $\mathcal{B}$ also enjoys these properties. □

Remarks. (1). In view of (6.3) and the remark following (1.3) the algebra $u(s\ell(2)_s)$ is tame. Note that $B := ke \oplus kv_0$ is a p-subalgebra of $s\ell(2)_s$ with restricted enveloping algebra $u(B) \cong k[X, Y]/(X^p, Y^p)$. As $p \geq 3$, this algebra has a factor algebra $k[X, Y]/(X^3, X^2Y, XY^2, Y^3)$ and hence is wild (see [14, (I.10.10(b)]). Thus, in contrast to the modular representation theory of finite groups, subgroups of tame infinitesimal groups may be wild.

(2). We now have a complete picture of the representation theory of the family $(u(s\ell(2))_\psi)$ of p^4-dimensional algebras. There is a dense orbit of special biserial algebras corresponding to the semisimple orbit $G \cdot h$, and two wild orbits of degenerations of these algebras that correspond to the nilpotent orbits $G \cdot 0 = \{0\}$ and $G \cdot e$.

7. Tame Principal Blocks and their AR-Quivers

In this section we establish our first main result concerning tame blocks of enveloping algebras of restricted Lie algebras. The tame blocks occurring are obtained from the algebra $\mathcal{K}$ by "lengthening" some of the relations. Specifically, for $n \in \mathbb{N}_0$, we let $\mathcal{K}(n)$ be the algebra that is given by the quiver

$$1 \underset{\underset{y_2}{\longleftarrow}}{\overset{\overset{x_1}{\longrightarrow}}{\overset{y_1}{\longrightarrow}}} \underset{x_2}{\longleftarrow}\; 2,$$

and relations $(x_1y_2)^{p^n} = (y_1x_2)^{p^n}$, $(y_2x_1)^{p^n} = (x_2y_1)^{p^n}$, $x_1x_2 = 0$, $y_1y_2 = 0$, $x_2x_1 = 0$, $y_2y_1 = 0$.

Note that $\mathcal{K} = \mathcal{K}(0)$ is the trivial extension of the Kronecker algebra defined in section 5.

Given an arbitrary restricted Lie algebra L, we denote by $T(L)$ the largest toral ideal of L. Note that $T(L)$ is contained in the center $C(L)$ of L. The p-unipotent radical of L will be denoted $\mathrm{rad}_p(L)$. We put $n(L) := \dim \mathrm{rad}_p(L)$.

Theorem 7.1 *Let $(L, [p])$ be a restricted Lie algebra of characteristic $p \geq 3$. Then the following statements are equivalent:*

(1) *The principal block $\mathcal{B}_0(L)$ is tame.*

(2) *Each block $\mathcal{B} \subset u(L/T(L))$ is either Morita equivalent to the truncated polynomial algebra $k[X]/(X^{p^{n(L)}})$ or to $\mathcal{K}(n(L))$. In particular, $\mathcal{B}_0(L)$ is of the latter form.*

Proof. (1) $\Rightarrow$ (2). By [25, (1.1)] we have $\mathcal{B}_0(L) \cong \mathcal{B}_0(L/T(L))$, so that we may assume without loss of generality that $T(L) = (0)$. Now [25, (6.4)] yields $L \cong s\ell(2)_\psi$ for a suitably chosen p-semilinear map $\psi : s\ell(2) \longrightarrow V$, whose target is an n-dimensional nil-cyclic restricted Lie algebra $V = (kv_0)_p$ with $v_0^{[p]^n} = 0$. Consequently, $n = n(L)$. According to (1.1), the algebra $u(L)$ is symmetric.

The remark following (1.3) together with the obvious analogue of (1.5) for $S(p-1)$ implies that the block $\mathcal{B}(p-1)$ is Morita equivalent to $k[X]/(X^{p^n})$.

As before, we let J be the Jacobson radical of $u(L)$, and put $I := u(L)v_0$ as well as $I' := u(L)v_0^p$. We consider an arbitrary block $\mathcal{B} \subset u(L)$ containing two simple modules. Since $\mathcal{B}_0(L)$ is tame, a consecutive application of (1.2), (1.4) and (1.6) gives the inclusions $I'\mathcal{B} \subset J^{2p}\mathcal{B} \subset J^3\mathcal{B} \subset I\mathcal{B} \subset J^2\mathcal{B}$.

Let $L' := L/(kv_0^{[p]})_p$, so that $u(L') \cong u(L)/I'$. According to (2.2) we have $L' \cong s\ell(2)_0$, $s\ell(2)_n$ or $s\ell(2)_s$. By [25, (1.1(3))] the principal block $\mathcal{B}_0(L')$ is tame, so that by (1.2) and (4.4) the latter alternative applies.

We let $\mathcal{B}'$ and $\hat{\mathcal{B}}$ be the images of $\mathcal{B}$ under the canonical projections $u(L) \longrightarrow u(L')$ and $u(L) \longrightarrow u(s\ell(2))$, respectively. From the structure of the quivers of $u(L)$, $u(L')$, and $u(s\ell(2))$, it follows that $\mathcal{B}'$ and $\hat{\mathcal{B}}$ are blocks with two simple modules. Let Λ_0, Λ_0' and $\mathcal{K}$ be the basic algebras of $\mathcal{B}$, $\mathcal{B}'$ and $\hat{\mathcal{B}}$, respectively. Then we have surjections

$$\Lambda_0 \xrightarrow{\pi} \Lambda_0' \;\; ; \;\; \Lambda_0 \longrightarrow \mathcal{K}.$$

From (6.2) we obtain the existence of generators x_i', y_i' of Λ_0' satisfying conditions (1) through (4) of section 5.

Since $\ker\pi \subset \mathrm{Rad}(\Lambda_0)^{2p}$ any choice of pre-images x_i, $y_i \in \Lambda_0$ gives rise to generators that enjoy properties (1) through (3). Setting $J := \mathrm{Rad}(\Lambda_0)$ and $J' := \mathrm{Rad}(\Lambda_0')$ we continue by verifying the simplicity of Jx_i/J^2x_i. To that end we first show that $J^{2p} \cap Jx_i \subset J^2x_i$.

Note that $J^{2p} \cap Jx_i = J^{2p}e_i \cap Jx_i$. We have $J^{2p}e_i = J^{2p-1}(Je_i) = J^{2p-1}x_i + J^{2p-1}y_i$. Since $J^{2p-1}x_i \subset J^2x_i$, it is enough to show that $J^{2p-1}y_i \cap Jx_i \subset J^2x_i$. Assume $ax_i = by_i$ for some $a \in Je_{i+1}$ and $b \in J^{2p-1}e_{i+1}$. Then the images $a' := \pi(a)$, $b' := \pi(b) \in \Lambda_0'$ satisfy $a'x_i' = b'y_i'$. Observe that $\Lambda_0'x_i'$, $\Lambda_0'y_i' \subset \Lambda_0'e_i$ are uniserial submodules of length $2p$ having socle $\mathrm{Soc}(\Lambda_0'e_i)$. Since $b' \in (J')^{2p-1}$, we have $b'y_i' \in \mathrm{Soc}(\Lambda_0'y_i')$, and hence $a'x_i' \in \mathrm{Soc}(\Lambda_0'y_i') = \mathrm{Soc}(\Lambda_0'x_i') = (J')^{2p-1}x_i$. Accordingly, we may assume $a' \in (J')^{2p-1}$, so that $a \in J^{2p-1} + \ker\pi \subset J^{2p-1}$. Consequently, $ax_i \in J^{2p-1}x_i \subset J^2x_i$, as desired.

Since $J^{2p} \cap Jx_i \subset J^2x_i$ the canonical surjection $\mathrm{Top}(Jx_i) \longrightarrow \mathrm{Top}(J'x_i')$ induced by π is injective. Consequently, $\mathrm{Top}(Jx_i)$ is simple. The simplicity of $\mathrm{Top}(Jy_i)$ follows analogously.

Owing to (1.6) Λ_0 is a symmetric algebra satisfying the conditions of (5.2). Consequently, we may assume that

$$x_{i+1}x_i = 0 = y_{i+1}y_i \qquad 1 \le i \le 2.$$

Property (2) ensures the existence of a natural number $m \in \mathbb{N}$ such that

$$(x_{i+1}y_i)^{p^m} = (y_{i+1}x_i)^{p^m} \qquad 1 \le i \le 2.$$

Let m be minimal subject to the above relations. Since the elements $x_{i+1}y_i - y_{i+1}x_i$ belong to the kernel I of the projection $\Lambda_0 \longrightarrow \mathcal{K}$ and $I^{p^n} = (0)$ we have $m \le n$. Thanks to (5.1) we have

$$J^{2p^m}x_i = (0) = J^{2p^m}y_i,$$

so that each principal indecomposable Λ_0-module has Loewy length $\le 2p^m + 1$. Owing to (1.6) the Loewy length of each principal indecomposable Λ_0-module is $t = 2p^n + 1$, so that $n \le m$. Consequently, $m = n$ and the bound quiver algebra $\mathcal{K}(n)$ surjects onto Λ_0. The formulae of (5.1) show that $\mathcal{K}(n)$ has dimension $8p^n$, and the remark following (1.5) implies that the above surjection is in fact an isomorphism.

(2) $\Rightarrow$ (1). The given algebra is special biserial. Hence [53, (2.4)] implies that $\mathcal{B}_0(L)$ is tame. $\square$

Remark. Since the Cartan matrix of $\mathcal{K}(n)$ is singular, it is not a basic algebra of a group algebra of a finite group.

Given a self-injective algebra Λ, we denote by $\Gamma_s(\Lambda)$ its *stable Auslander-Reiten quiver*. By definition, $\Gamma_s(\Lambda)$ is a directed graph whose vertices are the isoclasses $[M]$ of the non-projective indecomposable Λ-modules M, and whose arrows are given by the irreducible morphisms. The graph $\Gamma_s(\Lambda)$ also possesses an automorphism τ, the so-called *Auslander-Reiten translation* that reflects homological properties of Λ-modules. In fact, if Λ is symmetric, then $\tau([M]) = [\Omega^2_\Lambda(M)]$ for every $[M] \in \Gamma_s(\Lambda)$. We refer the reader to [2, 3] for further details.

Recall that the Auslander-Reiten theory of the trivial extension algebra $\mathcal{K}$ of the path algebra of the Kronecker quiver by its minimal injective cogenerator is well understood. The quiver $\Gamma_s(\mathcal{K})$ has infinitely many components of type $Z[A_\infty]/(\tau)$ (homogeneous tubes) and 2 components of Euclidean type $\mathbb{Z}[\tilde{A}_{12}]$. The general AR-theory of self-injective special biserial algebras can be found in [17].

Let Λ be a tame algebra. For each $d \in \mathbb{N}$ we let $\mu_\Lambda(d)$ be the minimum number of $(\Lambda, k[X])$-bimodules $Q_1, \ldots, Q_{\mu_\Lambda(d)}$ giving rise to the one-parameter families of elements of $\mathrm{ind}_\Lambda(d)$. The algebra Λ is said to have *polynomial growth* if there exists a natural number m such that $\mu_\Lambda(d) \leq d^m$ for every $d \geq 1$ (see [50]). Moreover, we refer to Λ as being *domestic* if there is a natural number m such that $\mu_\Lambda(d) \leq m$ for every $d \geq 1$ (see [8, (5.7)])

Corollary 7.2 *Let $(L, [p])$ be a restricted Lie algebra such that $\mathcal{B}_0(L)$ is tame.*

(1) *If $\mathcal{B}_0(L)$ is domestic, then $L/T(L) \cong s\ell(2)$.*

(2) *If $L/T(L) \not\cong s\ell(2)$, then $\mathcal{B}_0(L)$ is not of polynomial growth, and $\Gamma_s(\mathcal{B}_0(L))$ is the disjoint union of infinitely many components of type $\mathbb{Z}[A_\infty]/(\tau)$ and infinitely many components of type $\mathbb{Z}[A_\infty^\infty]$.*

Proof. According to (1.1) the block $\mathcal{B}_0(L)$ is symmetric. Thus, [19, (2.5)] shows that all tubes of the stable Auslander-Reiten quiver have rank 1. In view of (7.1) and [17, (2.1)] the domesticity of $\mathcal{B}_0(L)$ implies that $\Gamma_s(\mathcal{B}_0(L))$ contains a component of type $\mathbb{Z}[\tilde{A}_{12}]$. By [14, (IV.3.8.3)] this readily implies that every principal indecomposable $\mathcal{B}_0(L)$-module has length 4. In view of (1.5) we have $n(L) = 0$, whence $L/T(L) \cong s\ell(2)$.

If $L/T(L) \not\cong s\ell(2)$, then $n(L) \geq 1$, and [17, (2.2)] gives the shape of the AR-quiver along with the fact that $\mathcal{B}_0(L)$ is not of polynomial growth. $\square$

Remark. The representation theory of infinitesimal group schemes affords an analogue of Webb's Theorem [54] for finite groups (cf. [16, 19, 22]). For $p \geq 3$ the actual occurrence of components of type $\mathbb{Z}[A_\infty^\infty]$ was previously not known. A consecutive application of [21, (2.1)] and [21, (4.1)] shows that the hearts of the principal indecomposable modules belonging to simple modules of complexity ≥ 3 are indecomposable. The above results show that decomposability of the hearts of the principal indecomposables is concomitant to the tameness of the corresponding blocks.

We continue with an observation concerning the position of Verma modules within the stable Auslander-Reiten quiver of $u(s\ell(2)_s)$. As a direct consequence of (3.1) we obtain that each Verma module $Z(\lambda)$ (with $\lambda(h) \neq -1$) belongs to a homogeneous tube.

Corollary 7.3 *Suppose that $\lambda(h) \neq -1$.*

(1) *Each $[Z(\lambda)]$ lies at the end of a homogeneous tube.*

(2) *$[Z_{s\ell(2)}(\lambda)]$ is the unique vertex of minimal dimension of a component of type $\mathbb{Z}[A_\infty^\infty]$.*

Proof. (1). We have noted in (3.1) that $\mathcal{V}_{s\ell(2)_s}(Z(\lambda)) = ke$. Let M be an indecomposable module whose isoclass belongs to the component of $Z(\lambda)$. In view of [19, (5.2)] we then have $\mathcal{V}_{s\ell(2)_s}(M) = ke$, so that $M|_{u(kf \oplus kv_0)}$ is projective. Consequently, $\dim_k M$ is a multiple of $p^2 = \dim_k Z(\lambda)$, so that $Z(\lambda)$ has minimal dimension in its component. This readily implies our assertion.

(2). Since the $u(s\ell(2)_s)$-module $Z_{s\ell(2)}(\lambda)$ is readily seen to be isomorphic to the induced module $u(s\ell(2)_s) \otimes_{u(kh\oplus ke\oplus kv_0)} k_\lambda$, [20, (3.4)] implies $\mathcal{V}_{s\ell(2)_s}(Z_{s\ell(2)}(\lambda)) \subset ke \oplus kv_0$. Note that every element $x \in ke \oplus kv_0$ operates on $Z_{s\ell(2)}(\lambda)$ via its image $\bar{x} \in ke \subset s\ell(2)$. In view of $ke \subset \mathcal{V}_{s\ell(2)}(Z_{s\ell(2)}(\lambda))$ we obtain $ke \oplus kv_0 \subset \mathcal{V}_{s\ell(2)_s}(Z_{s\ell(2)}(\lambda))$. It thus follows from (7.1) and the proof of (7.2) that $Z_{s\ell(2)}(\lambda)$ belongs to a component of type $\mathbb{Z}[A_\infty^\infty]$.

Now let M be another vertex of the component containing $Z_{s\ell(2)}(\lambda)$. Then $\mathcal{V}_{s\ell(2)_s}(M) = \mathcal{V}_{s\ell(2)_s}(Z_{s\ell(2)}(\lambda))$ (cf. [19, (5.2)]) so that $M|_{u(kf)}$ is projective. Hence $\dim_k M$ is a multiple of $p = \dim_k Z_{s\ell(2)}(\lambda)$. The unicity of $[Z_{s\ell(2)}(\lambda))]$ is a direct consequence of [33, Prop.3]. □

Theorem 7.4 *Let $(L, [p])$ be a restricted Lie algebra of characteristic $p \geq 3$. Then the following statements are equivalent:*

(1) *$\mathcal{B}_0(L)$ is tame.*

(2) *$L \cong s\ell(2)_\psi$, where $\psi : s\ell(2) \longrightarrow C(L)$ is a p-semilinear map satisfying $\psi(e), \psi(f) \in C(L)^{[p]}$ and $C(L) = k\psi(h) + C(L)^{[p]}$.*

(3) *$L/C(L)^{[p]} \cong s\ell(2), s\ell(2)_s$ and $\dim_k C(L)/C(L)^{[p]} \leq 1$.*

Proof. (1) ⇒ (2). Suppose that $\mathcal{B}_0(L)$ is tame. Owing to [25, (6.4)] we have $L \cong s\ell(2)_\psi$, where $\psi : s\ell(2) \longrightarrow C(L)$ is a p-semilinear map. By the same token, there exists a decomposition $C(L) = V \oplus T(L)$, with a nil-cyclic algebra $V = (kv_0)_p$. If $V = (0)$, then $C(L) = C(L)^{[p]}$, and we are done. Alternatively, we write $\psi = \psi_V + \psi_T$ as a sum of two p-semilinear maps $\psi_V : s\ell(2) \longrightarrow V$ and $\psi_T : s\ell(2) \longrightarrow T(L)$. As in (2.1) we see that each pair $(g, \mu) \in \mathrm{SL}(2) \times \mathrm{Aut}_p(V)$ defines an isomorphism

$$\hat{g}_\mu : s\ell(2)_\psi \longrightarrow s\ell(2)_{\mu\circ(g\cdot\psi_V)+(g\cdot\psi_T)} \quad ; \quad x + v + t \mapsto (\mathrm{Ad}\, g)(x) + \mu(v) + t.$$

of restricted Lie algebras. Let $\bar{\psi} : s\ell(2) \longrightarrow C(L)/C(L)^{[p]}$ be the p-semilinear map obtained by composing ψ with the canonical projection $C(L) \xrightarrow{\pi} C(L)/C(L)^{[p]}$. Then $L/C(L)^{[p]} \cong s\ell(2)_{\bar{\psi}}$ is a one-dimensional central extension of $s\ell(2)$ with nil-cyclic center. By [25, (1.1)] the principal block $\mathcal{B}_0(s\ell(2)_{\bar{\psi}})$ is tame, and a consecutive application of (2.2), (4.4) and (6.3) ensures the existence of a pair $(g, \alpha) \in \mathrm{SL}(2) \times k^\times$ such that $\zeta := \alpha(g \cdot \bar{\psi})$ satisfies

$$\zeta(e) = 0 = \zeta(f) \; ; \; \zeta(h) = v_0 + C(L)^{[p]}.$$

Suppose that $\dim_k V = n$, so that $\{v_0, v_0^{[p]}, \ldots, v_0^{[p]^{n-1}}\}$ is a basis of V. Let $\mu_\alpha : V \longrightarrow V$ be the automorphism given by

$$\mu_\alpha(\sum_{i=0}^{n-1} \beta_i v_0^{[p]^i}) = \sum_{i=0}^{n-1} \alpha^{p^i} \beta_i v_0^{[p]^i}.$$

Then we have $s\ell(2)_\psi \cong s\ell(2)_{\mu_\alpha\circ(g\cdot\psi_V)+(g\cdot\psi_T)}$ and

$$\pi \circ (\mu_\alpha \circ (g \cdot \psi_V) + (g \cdot \psi_T)) = \pi \circ \mu_\alpha \circ (g \cdot \psi_V) = \zeta.$$

Consequently, the p-semilinear map $\mu_\alpha \circ (g \cdot \psi_V) + (g \cdot \psi_T)$ satisfies (2).

(2) ⇒ (3). Trivial

(3) $\Rightarrow$ (1). We decompose the abelian Lie algebra $C(L) = V \oplus T(L)$ into its p-unipotent and toral part. Then $C(L)^{[p]} = V^{[p]} \oplus T(L)$, so that $\dim_k V/V^{[p]} \leq 1$. Hence $V = (0)$ or $V = (kv_0)_p$ is nil-cyclic.

Suppose that $L/C(L)^{[p]} \cong s\ell(2)$. Then $L/C(L)^{[p]}$ is simple, so that $C(L) = C(L)^{[p]}$. Consequently, $C(L) = T(L)$ and [25, (1.1)] implies that $\mathcal{B}_0(L) \cong \mathcal{B}_0(L/T(L)) \cong \mathcal{B}_0(s\ell(2))$ is tame.

Next, we assume that $L/C(L)^{[p]} \cong s\ell(2)_s$, and put $L' := L/T(L)$. If $V' \subset L'$ denotes the image of V under the canonical projection $L \longrightarrow L'$, then $L'/V'^{[p]} \cong L/C(L)^{[p]} \cong s\ell(2)_s$. Let Λ' be the basic algebra of $\mathcal{B}_0(L')$. Since $\mathcal{B}_0(s\ell(2)_s) \cong \mathcal{B}_0(L')/\mathcal{B}_0(L')V'^{[p]}$, there exists an ideal $I \subset \mathrm{Rad}(\Lambda')^{2p}$ such that $\Lambda'' := \Lambda'/I$ is the basic algebra of $\mathcal{B}_0(s\ell(2)_s)$. It now follows from (6.3) and (7.1) that Λ' possesses generators satisfying conditions (1) through (4) of section 5. As L is a central extension of $s\ell(2)$ (1.1) shows that Λ' is symmetric. Hence (5.2) applies and Λ' is special biserial. Owing to [53, (2.4)] the algebras Λ' and $\mathcal{B}_0(L')$ are tame, and (1) now follows from [25, (1.1)]. $\square$

8. Tame Blocks of Reduced Enveloping Algebras

In the foregoing section we have classified the blocks of the algebra $u(L/T(L))$ under the assumption that the principal block $\mathcal{B}_0(L)$ of the enveloping algebra $u(L)$ is tame. As we shall see below, the remaining blocks of $u(L)$ occur in certain Frobenius algebras associated to the restricted Lie algebra $(L, [p])$.

Let $(L, [p])$ be a restricted Lie algebra with ordinary universal enveloping algebra $\mathcal{U}(L)$. Given a linear form $\chi \in L^*$, the factor algebra

$$u(L, \chi) := \mathcal{U}(L)/(\{x^p - x^{[p]} - \chi(x)^p 1 \ ; \ x \in L\})$$

is called the *χ-reduced enveloping algebra* of L. This algebra is known to be a Frobenius algebra of dimension $p^{\dim_k L}$ (cf. [52, (V.3.1),(V.4.3)]). Note that $u(L, 0) = u(L)$ is the restricted enveloping algebra of L. Rank varieties may also be defined in this context, and they possess the same properties as the ones considered so far (see [29, §6]). Thus, given a $u(L, \chi)$-module M, we define

$$\mathcal{V}_L(M) := \{x \in \mathcal{V}_L \ ; \ M|_{u(kx, \chi|_{kx})} \text{ is not free}\} \cup \{0\}.$$

We consider the character group $X(T(L))$ of the toral radical $T(L) \subset L$ that consists of all linear forms $\gamma \in T(L)^*$ such that $\gamma(t^{[p]}) = \gamma(t)^p$ for every $t \in T(L)$. Note that $X(T(L))$ naturally identifies with the set of algebra homomorphisms $u(T(L)) \longrightarrow k$. For $\gamma \in X(T(L))$ we put

$$\mathcal{A}_\gamma := u(L)/(\{t - \gamma(t)1 \ ; \ t \in T(L)\}).$$

Lemma 8.1 *There is an isomorphism $u(L) \cong \bigoplus_{\gamma \in X(T(L))} \mathcal{A}_\gamma$ of k-algebras.*

Proof. Since tori are rigid, $T(L) \subset C(L)$ is contained in the center of L. Thus, $u(T(L))$ is a semisimple subalgebra of the center $\mathcal{Z}(L)$ of the restricted enveloping algebra $u(L)$.

Since the set $\{k_\gamma \ ; \ \gamma \in X(T(L))\}$ is a full set of representatives of the isomorphism classes of simple $u(T(L))$-modules, we have a decomposition

$$u(T(L)) = \bigoplus_{\gamma \in X(T(L))} ke_\gamma,$$

where the e_γ are primitive, orthogonal idempotents such that

$$ae_\gamma = \gamma(a)e_\gamma \qquad \forall \ a \in u(T(L)), \ \gamma \in X(T(L)).$$

Thus, each ideal $I_\gamma := u(L)e_\gamma$ of $u(L)$ is generated by a central idempotent, and we have

$$u(L) = \bigoplus_{\gamma \in X(T(L))} I_\gamma.$$

Given $\gamma_0 \in X(T(L))$, we let $\pi : u(L) \longrightarrow \mathcal{A}_{\gamma_0}$ be the canonical projection. If $ue_{\gamma_0} \in \ker \pi \cap I_{\gamma_0}$ for some $u \in u(L)$, then we have

$$ue_{\gamma_0} = \sum_{t \in T(L)} a_t(t - \gamma_0(t)1),$$

with all but finitely many $a_t \in u(L)$ being zero. Consequently,

$$ue_{\gamma_0} = ue_{\gamma_0}^2 = \sum_{t \in T(L)} a_t(t - \gamma_0(t)1)e_{\gamma_0} = 0,$$

so that $\pi|_{I_{\gamma_0}}$ is injective.

Let γ be an element of $X(T(L)) \setminus \{\gamma_0\}$. Then we have $\gamma_0(e_\gamma)e_{\gamma_0} = e_\gamma e_{\gamma_0} = 0$, so that $\gamma_0(e_\gamma) = 0$. Since $\pi(a) = \gamma_0(a)\pi(1)$ for every element $a \in u(T(L))$, it follows that

$$\pi(e_\gamma) = \gamma_0(e_\gamma)\pi(1) = 0.$$

Consequently, $\pi(I_{\gamma_0}) = \pi(u(L)) = \mathcal{A}_{\gamma_0}$, so that $\pi|_{I_{\gamma_0}}$ is an isomorphism. This implies

$$u(L) \cong \bigoplus_{\gamma \in X(T(L))} I_\gamma \cong \bigoplus_{\gamma \in X(T(L))} \mathcal{A}_\gamma,$$

as desired. $\square$

Let $L := H \oplus C$ be direct sum of Lie algebras such that $[C, C] = (0)$. We assume H and C to be restricted, and consider the extension L_ψ that is given by the p-semilinear map $\psi : H \longrightarrow C$. Thus, L_ψ has a p-map defined by

$$(h + c)^{[p]} := h^{[p]} + \psi(h) + c^{[p]} \qquad \text{for all } h \in H, \ c \in C.$$

Lemma 8.2 *Let $\lambda \in C^*$ be a linear form, and denote by $\chi \in H^*$ the unique linear form satisfying $\chi(h)^p = (\lambda \circ \psi)(h)$ for every $h \in H$. Then there exists an isomorphism $u(H, \chi) \cong u(L_\psi)/(\{c - \lambda(c)1 \ ; \ c \in C\})$.*

Proof. Let $A_\lambda := u(L_\psi)/(\{c - \lambda(c)1 \ ; \ c \in C\})$. Directly from [52, (I.9.7)] we obtain $\dim_k A_\lambda = p^{\dim_k H} = \dim_k u(H,\chi)$. Now let $\zeta : H \longrightarrow A_\lambda$ be the composite of the canonical maps $H \longrightarrow L_\psi$, $L_\psi \longrightarrow u(L_\psi)$, and $u(L_\psi) \longrightarrow A_\lambda$. Then we have

$$\zeta([a,b]) = \zeta(a)\zeta(b) - \zeta(b)\zeta(a) \quad \forall\ a,b \in H.$$

Since

$$\zeta(h)^p - \zeta(h^{[p]}) - \chi(h)^p 1 = h^{[p]} + \psi(h) - h^{[p]} - \chi(h)^p 1 = \psi(h) - (\lambda \circ \psi)(h)1 = 0$$

for every element $h \in H$, the universal property of $u(H,\chi)$ (see [52, (V.3)]) ensures that ζ lifts to a homomorphism $\zeta : u(H,\chi) \longrightarrow A_\lambda$ of k-algebras. Owing to [52, (I.9.7)] the map ζ is surjective and therefore, by equality of dimensions, bijective. $\square$

Our analysis of the representation type of the blocks is based on the following facts concerning isomorphism classes of reduced enveloping algebras. From the universal property it readily follows that any isomorphism $f : L \longrightarrow L'$ of restricted Lie algebras induces isomorphisms

$$u(L,\chi) \cong u(L', \chi \circ f^{-1})$$

of reduced enveloping algebras for every $\chi \in L^*$. Thus, the coadjoint action

$$\mathrm{SL}(2) \longrightarrow \mathrm{GL}(s\ell(2)^*) \ \ ; \ \ g \cdot \lambda := \lambda \circ \mathrm{Ad}(g^{-1})$$

induces isomorphisms

$$u(s\ell(2), \hat{\chi}) \cong u(s\ell(2), g \cdot \hat{\chi})$$

for $g \in \mathrm{SL}(2)$ and $\hat{\chi} \in s\ell(2)^*$. Owing to [29, §2] there are three isomorphism types of reduced enveloping algebras of $s\ell(2)$:

1. $\hat{\chi} = 0$, i.e., $u(s\ell(2)_\psi, \hat{\chi})$ is the restricted enveloping algebra,
2. $\hat{\chi}$ is of *semisimple type*, i.e., $\hat{\chi}$ is conjugate to a linear form $\hat{\chi}_s$ satisfying $\hat{\chi}_s(e) = 0 = \hat{\chi}_s(f)$, $\hat{\chi}_s(h) \neq 0$, and
3. $\hat{\chi}$ is of *nilpotent type*, i.e., $\hat{\chi}$ is conjugate to the linear form $\hat{\chi}_n$ satisfying $\hat{\chi}_n(e) = 0 = \hat{\chi}_n(h)$, $\hat{\chi}_n(f) = 1$.

In view of our projected applications we continue by considering the following set-up. We are given a Lie algebra $s\ell(2)_\psi$, defined by a p-semilinear map $\psi : s\ell(2) \longrightarrow V$ whose target is an n-dimensional nil-cyclic Lie algebra $(0) \neq V = (kv_0)_p$. Let $\chi \in s\ell(2)_\psi^*$ be a linear form such that $\chi(V) = (0)$, and denote by $\hat{\chi} \in s\ell(2)^*$ the linear form induced by χ.

Given $g \in \mathrm{SL}(2)$ the map

$$\hat{g} : s\ell(2)_\psi \longrightarrow s\ell(2)_{g\cdot\psi} \ \ ; \ \ x + v \mapsto \mathrm{Ad(g)}(x) + v$$

is an isomorphism of restricted Lie algebras that induces isomorphisms

$$u(s\ell(2)_\psi, \chi) \cong u(s\ell(2)_{g\cdot\psi}, g \cdot \chi),$$

where $g \cdot \chi := \chi \circ \hat{g}^{-1}$. Accordingly, we shall henceforth consider only the following linear forms

1. The form $\chi = 0$, i.e., $u(s\ell(2), \chi)$ is the restricted enveloping algebra,

2. the *semisimple* forms χ_s given by $\chi_s(ke \oplus kf \oplus V) = (0)$ and $\chi_s(h) \neq 0$, and

3. the *nilpotent* form χ_n given by $\chi_n(ke \oplus kh \oplus V) = (0)$ and $\chi_n(f) = 1$.

In either case $I := u(s\ell(2)_\psi, \chi)v_0$ is a nilpotent ideal such that $u(s\ell(2)_\psi, \chi)/I \cong u(s\ell(2), \hat{\chi})$. In particular, the algebras $u(s\ell(2)_\psi, \chi)$ and $u(s\ell(2), \hat{\chi})$ have the same simple modules.

Owing to [29, §2] the simple $u(s\ell(2)_\psi, \chi)$-modules are indexed by highest weights $\lambda \in \mathcal{X}(\hat{\chi})$, where

$$\mathcal{X}(\hat{\chi}) := \{\lambda \in (kh \oplus ke)^* \ ; \ \lambda(h)^p - \lambda(h^{[p]}) = \hat{\chi}(h)^p, \, \lambda(e) = 0\}.$$

When convenient we will view elements of $\mathcal{X}(\hat{\chi})$ as linear forms of the space $kh \oplus ke \oplus V$ by setting $\lambda(V) = (0)$. As before we will write $S(\lambda)$ and $P(\lambda)$ for the simple module associated to λ and its projective cover, respectively.

Given a p-semilinear map $\psi : s\ell(2) \longrightarrow V$, we let $\hat{\psi} : s\ell(2) \longrightarrow V/V^{[p]}$ the p-semilinear map obtained by composing ψ with the canonical projection.

Replacing [42] by [29, (2.2),(2.3)] in the arguments of the proof of (1.5) we obtain the following result:

Lemma 8.3 *Let* $\chi \in \{\chi_s, \chi_n\}$.

(1) *The module* $v_0^i P(\lambda)/v_0^{i+1}P(\lambda)$ *is the projective cover of the* $u(s\ell(2), \hat{\chi})$*-module* $S(\lambda)$ *for* $0 \leq i \leq p^n - 1$.

(2) *If* $\chi = \chi_n$ *and* $\lambda(h) = -1$, *or if* $\chi = \chi_s$, *then* $\dim_k P(\lambda) = p^{n+1}$ *and* $\ell(P(\lambda)) = p^n$.

(3) *If* $\chi = \chi_n$ *and* $\lambda(h) \neq -1$, *then* $\dim_k P(\lambda) = 2p^{n+1}$ *and* $\ell(P(\lambda)) = 2p^n$. $\square$

We consider the number

$$d(\psi) := \dim_k ke \cap \ker \hat{\psi} \in \{0, 1\}.$$

Lemma 8.4 *Given simple* $u(s\ell(2)_\psi, \chi_n)$*-modules* S *and* T, *we have*

$$\dim_k \operatorname{Ext}^1_{u(s\ell(2)_\psi, \chi_n)}(S, T) = \begin{cases} 0 & \text{if } S \not\cong T \\ 1 + d(\psi) & \text{if } S \cong T \not\cong S(p-1) \\ 1 & \text{if } S \cong T \cong S(p-1) \end{cases} .$$

In particular, each block of $u(s\ell(2)_\psi, \chi_n)$ *is primary.*

Proof. As in (1.3) we put $M := \operatorname{Hom}_k(S, T)$, and observe that M is a $u(s\ell(2)_\psi)$-module that is annihilated by v_0 (cf. [52, (V.2.7)]). Thanks to [29, (5.1)] there are isomorphisms $\operatorname{Ext}^i_{u(s\ell(2), \hat{\chi}_n)}(S, T) \cong H^i(u(s\ell(2)), M)$ and $\operatorname{Ext}^i_{u(s\ell(2)_\psi, \chi_n)}(S, T) \cong H^i(u(s\ell(2)_\psi), M)$ for every $i \geq 0$. There thus results an exact sequence

$$(*) \qquad (0) \longrightarrow \operatorname{Ext}^1_{u(s\ell(2),\hat{\chi}_n)}(S,T) \longrightarrow \operatorname{Ext}^1_{u(s\ell(2)_\psi,\chi_n)}(S,T) \longrightarrow H^1(u(V), M)^{s\ell(2)_\psi}$$
$$\longrightarrow \operatorname{Ext}^2_{u(s\ell(2),\hat{\chi}_n)}(S,T).$$

Moreover, the arguments of (1.3) show that

$$(**) \qquad\qquad H^1(u(V),M)^{s\ell(2)_\psi} \cong \mathrm{Hom}_{u(s\ell(2)_\psi,\chi_n)}(S,T).$$

Thus, if $S \not\cong T$, then [29, (2.3)] and Schur's Lemma imply the vanishing of the first and third term, so that $\mathrm{Ext}^1_{u(s\ell(2)_\psi,\chi_n)}(S,T) = (0)$. This already implies that each block of $u(s\ell(2)_\psi,\chi_n)$ is primary.

If $S = S(p-1) = T$, then [29, (2.3)] yields $\mathrm{Ext}^1_{u(s\ell(2),\hat{\chi}_n)}(S,S) = (0) = \mathrm{Ext}^2_{u(s\ell(2),\hat{\chi}_n)}(S,S)$, whence

$$\dim_k \mathrm{Ext}^1_{u(s\ell(2)_\psi,\chi_n)}(S,S) = 1.$$

Similarly, we have

$$1 \le \dim_k \mathrm{Ext}^1_{u(s\ell(2)_\psi,\chi_n)}(S,S) \le 2$$

in the remaining cases.

In order to compute the dimension of the latter space, we consider the Lie algebra $L := s\ell(2)_\psi/V^{[p]}$. Then L is a one-dimensional central extension of $s\ell(2)$. Let $\chi'_n \in L^*$ be the linear form induced by χ_n, and denote by P a projective cover of S. As before, we let J be the Jacobson radical of $u(s\ell(2)_\psi,\chi)$. Since $v_0^{[p]} \in J^p \subset J^2$ we obtain, setting $I' := u(s\ell(2)_\psi,\chi_n)v_0^p$,

$$\begin{aligned}\mathrm{Ext}^1_{u(s\ell(2)_\psi,\chi_n)}(S,S) &\cong \mathrm{Hom}_{u(s\ell(2)_\psi,\chi_n)}(JP/J^2P,S) \cong \mathrm{Hom}_{u(L,\chi'_n)}((JP/I'P)/(J^2P/I'P),S)\\ &\cong \mathrm{Ext}^1_{u(L,\chi'_n)}(S,S).\end{aligned}$$

Hence we may assume that $\dim_k V = 1$ and $\psi = \hat{\psi}$.

Thanks to [29, (2.3)] there exists $\lambda \in \mathcal{X}(\hat{\chi}_n)$ with $\lambda(h) \neq -1$ and such that

$$S \cong u(s\ell(2),\hat{\chi}_n) \otimes_{u(kh\oplus ke,\hat{\chi}_n|_{kh\oplus ke})} k_\lambda \cong u(s\ell(2)_\psi,\chi_n) \otimes_{u(B_\psi,\chi_n|_{B_\psi})} k_\lambda,$$

where we set $B_\psi := kh \oplus ke \oplus kv_0$ and $\lambda(v_0) = 0$. An application of [20, (3.4)] yields

$$\{0\} \neq \mathcal{V}_{s\ell(2)_\psi}(S) \subset \mathcal{V}_{ke\oplus kv_0}.$$

If $d(\psi) = 1$, then $\psi(e) = 0$, and the arguments of (7.3) may now be adopted verbatim to see that $\mathcal{V}_{s\ell(2)_\psi}(S) = kv_0 \oplus ke$ is two-dimensional. Hence the block $\mathcal{B} \subset u(s\ell(2)_\psi,\chi_n)$ belonging to S is not representation-finite. Since $\mathcal{B}$ is primary, this implies $\dim_k \mathrm{Ext}^1_{u(s\ell(2)_\psi,\chi_n)}(S,S) \ge 2$, as desired.

Alternatively, $\psi(e) \neq 0$, so that $\mathcal{V}_{ke\oplus kv_0} = kv_0$. Hence S is periodic and $\mathcal{B}$ has finite representation type (cf. [19, (3.2)]). In particular, $\dim_k \mathrm{Ext}^1_{u(s\ell(2)_\psi,\chi_n)}(S,S) = 1$. $\square$

Suppose that $V = kv_0$ and $\psi(e) = 0$. Then $B := k(h+\psi(h)) \oplus ke$ is a p-subalgebra of $s\ell(2)_\psi$, and every highest weight $\lambda \in \mathcal{X}(\hat{\chi}_n)$ gives rise to a one-dimensional $u(B,\chi_n|_B)$-module k_λ. We consider the Verma modules

$$Z(\lambda) := u(s\ell(2)_\psi,\chi_n) \otimes_{u(B,\chi_n|_B)} k_\lambda \quad \text{and} \quad Z_{s\ell(2)}(\lambda) := u(s\ell(2),\hat{\chi}_n) \otimes_{u(B,\hat{\chi}_n|_B)} k_\lambda.$$

Thanks to [29, (2.3)] the module $Z_{s\ell(2)}(\lambda)$ is simple. In view of (8.3) the block $\mathcal{B}(\lambda) \subset u(s\ell(2)_\psi,\chi_n)$ associated to λ is primary. The arguments of (3.1) now yield the first two parts of the following result.

Lemma 8.5 *Suppose that $V = kv_0$, $\psi(e) = 0$ as well as $\lambda(h) \neq -1$.*

(1) *We have isomorphisms $v_0^i Z(\lambda)/v_0^{i+1} Z(\lambda) \cong Z_{s\ell(2)}(\lambda)$ for $0 \leq i \leq p-1$.*

(2) *The module $Z(\lambda)$ is uniserial of length p with $v_0^i Z(\lambda) = J^i Z(\lambda)$ for $0 \leq i \leq p-1$.*

(3) *$\Omega_{u(s\ell(2)_\psi,\chi_n)}(Z(\lambda)) \cong Z(\lambda)$.*

Proof. (3). By the PBW-Theorem $u(s\ell(2)_\psi, \chi_n)$ is a free left and right module of rank p^2 over the subalgebra $u(B, \chi_n|_B) \cong u(B)$. Accordingly, the functor $M \mapsto u(s\ell(2)_\psi, \chi_n) \otimes_{u(B,\chi_n|_B)} M$ is exact, and Frobenius reciprocity (cf. [3, (2.8.4)] ensures that it sends projectives to projectives. Directly from the definition of the Heller operator we therefore obtain (cf. [6, (4.4)])

$$\Omega_{u(s\ell(2)_\psi,\chi_n)}(Z(\lambda)) \oplus P \cong u(s\ell(2)_\psi, \chi_n) \otimes_{u(B,\chi_n|_B)} \Omega_{u(B,\chi_n|_B)}(k_\lambda)$$

for some projective module P. Observing that $u(B, \chi_n|_B) \otimes_{u(k(h+\psi(h)),\chi_n|_{k(h+\psi(h))})} k_\lambda$ is the projective cover of k_λ, we see that $\Omega_{u(B,\chi_n|_B)}(k_\lambda)$ has a composition series with composition factors $k_{\lambda+2i\zeta}$ for $1 \leq i \leq p-1$. Here $\zeta : kh \oplus ke \oplus kv_0 \longrightarrow k$ denotes the linear map sending h to 1 and all other basis vectors to zero. Thus, the module $u(s\ell(2)_\psi, \chi_n) \otimes_{u(B,\chi_n|_B)} \Omega_{u(B,\chi_n|_B)}(k_\lambda)$ is filtered by modules $Z(\lambda + 2i\zeta)$ with $1 \leq i \leq p-1$.

By (1) and [29, (2.3)] $Z(\lambda)$ and $Z(\lambda')$ are the only Verma modules belonging to the primary block $\mathcal{B}(\lambda)$. Among these, only $Z(\lambda')$ occurs as a filtration factor (with multiplicity one). Set $M := u(s\ell(2)_\psi, \chi_n) \otimes_{u(B,\chi_n|_B)} \Omega_{u(B,\chi_n|_B)}(k_\lambda)$, and let $V \subset M$ be the first module of the filtration containing $Z(\lambda')$ as a filtration factor. There results an exact sequence

$$(0) \longrightarrow V' \longrightarrow V \longrightarrow Z(\lambda') \longrightarrow (0).$$

Since the filtration factors of V' all belong to blocks different from $\mathcal{B}(\lambda)$, the sequence splits. Thus, $Z(\lambda')$ is a submodule of M, and we have an exact sequence

$$(0) \longrightarrow Z(\lambda') \longrightarrow M \longrightarrow M/Z(\lambda') \longrightarrow (0).$$

Arguing as above, we conclude that $Z(\lambda')$ is a direct summand of M. Consequently, there exists a $u(s\ell(2)_\psi, \chi_n)$-module Q such that

$$Z(\lambda') \oplus Q \cong M \cong \Omega_{u(s\ell(2)_\psi,\chi_n)}(Z(\lambda)) \oplus P.$$

Thus, $Z(\lambda')$ and $\Omega_{u(s\ell(2)_\psi,\chi_n)}(Z(\lambda))$ are the unique non-projective parts of M belonging to $\mathcal{B}(\lambda)$. The theorem of Krull-Remak-Schmidt now yields $\Omega_{u(s\ell(2)_\psi,\chi_n)}(Z(\lambda)) \cong Z(\lambda')$.

It remains to show that $Z(\lambda) \cong \hat{Z}(\lambda')$. To that end we let $i \in \{0, \ldots, p-2\}$ be an element such that $\lambda(h) \cong i \bmod(p)$. From the Cartan-Weyl identities it follows that

$$\varrho : Z(\lambda') \longrightarrow Z(\lambda) \;\; ; \;\; u \otimes 1 \mapsto uf^{i+1} \otimes 1$$

is a homomorphism of $u(s\ell(2)_\psi, \chi_n)$-modules. Since $\chi_n(f) = 1$, we have

$$f^p = f^{[p]} + 1 = \psi(f) + 1,$$

proving that $\operatorname{im} \varrho + v_0 Z(\lambda) = Z(\lambda)$. In view of (2) this readily yields the surjectivity of ϱ. As both modules have dimension p^2, ϱ is an isomorphism. □

For the proof of our main result we also require the following Lemma concerning the Loewy layers of certain principal indecomposable modules. Given a highest weight $\lambda \in \mathcal{X}(\hat{\chi}_n)$ with $\lambda(h) \neq -1$, we consider the associated principal indecomposable module $P(\lambda)$, the block $\mathcal{B}(\lambda) \subset u(s\ell(2)_\psi, \chi_n)$ as well as the algebra $\mathcal{A}(\lambda) := \mathcal{B}(\lambda)/v_0^2\mathcal{B}(\lambda)$. In the sequel we will determine the structure of the module $M(\lambda) := JP(\lambda)/J^3P(\lambda)$. As usual, $J \subset u(s\ell(2)_\psi, \chi_n)$ denotes the Jacobson radical. Recall that $v_0 \in J$.

Lemma 8.6 *Suppose that $\hat{\psi}(e) = 0$ and $\lambda(h) \neq -1$. Then we have $\ell(J^iP(\lambda)/J^{i+1}P(\lambda)) = 2$ for $1 \leq i \leq p^n - 1$. In particular, $M(\lambda)$ is an $\mathcal{A}(\lambda)$-module of length 4.*

Proof. By (8.3) and [29, (2.3)] the modules $v_0^iP(\lambda)/v_0^{i+1}P(\lambda)$ are uniserial of length 2 for $0 \leq i \leq p^n - 1$. Accordingly, we have proper inclusions

$$v_0P(\lambda) \subset JP(\lambda) \subset P(\lambda).$$

Since $\ell(JP(\lambda)/v_0P(\lambda)) = 1$, the module $v_0P(\lambda)$ is a maximal submodule of $JP(\lambda)$, so that $J^2P(\lambda) \subset v_0P(\lambda)$. Thanks to (8.4) the module $JP(\lambda)/J^2P(\lambda)$ has length 2, implying $\ell(v_0P(\lambda)/J^2P(\lambda)) = 1$. Since $v_0P(\lambda)/v_0^2P(\lambda)$ is uniserial of length 2 and $v_0^2P(\lambda) \subset J^2P(\lambda)$, we thus obtain

$$J^2P(\lambda) = Jv_0P(\lambda) = v_0JP(\lambda).$$

An easy induction argument now yields

$$J^iP(\lambda) = Jv_0^{i-1}P(\lambda) \quad \forall\, i \geq 1.$$

Suppose that $i \in \{1, \dots, p^n - 1\}$. Since $v_0^{i-1}P(\lambda)/v_0^iP(\lambda)$ and $v_0^iP(\lambda)/v_0^{i+1}P(\lambda)$ are uniserial modules of length 2, the above identity yields $\ell(J^iP(\lambda)/v_0^iP(\lambda)) = 1 = \ell(v_0^iP(\lambda)/J^{i+1}P(\lambda))$. Thus, $\ell(J^iP(\lambda)/J^{i+1}P(\lambda)) = 2$, and $M(\lambda)$ is an $\mathcal{A}(\lambda)$-module of length 4. $\square$

Lemma 8.7 *Suppose that $\hat{\psi}(e) = 0$ and $\lambda(h) \neq -1$. Then $\mathcal{A}(\lambda)$ is Morita equivalent to $k[X, Y]/(X^2, Y^2)$. In particular, $\mathcal{A}(\lambda)$ is symmetric.*

Proof. We consider the principal indecomposable module $\hat{P}(\lambda) := P(\lambda)/v_0^2P(\lambda)$ of the primary algebra $\mathcal{A}(\lambda)$. The proof of (8.6) shows that

$$(0) \subset J^2\hat{P}(\lambda) \subset v_0\hat{P}(\lambda) \subset J\hat{P}(\lambda) \subset \hat{P}(\lambda)$$

is a composition series of $\hat{P}(\lambda)$.

Accordingly, $\ell(\hat{P}(\lambda)) = 4$, and we note that $\mathrm{Soc}(\hat{P}(\lambda))$ is annihilated by v_0. Since $P(\lambda)$ is free over $k[v_0] \cong k[X]/(X^p)$, it follows that $\mathrm{Soc}(\hat{P}(\lambda))$ is contained in the uniserial module $v_0\hat{P}(\lambda) \cong v_0P(\lambda)/v_0^2P(\lambda)$. Thus, $\mathrm{Soc}(\hat{P}(\lambda))$ is simple, and the primary algebra $\mathcal{A}(\lambda)$ is self-injective (cf. [3, §1.6]).

By the above, the left multiplication by v_0 defines a central generator of the Jacobson radical of the local algebra $\Lambda := \mathrm{End}_{\mathcal{A}(\lambda)}(\hat{P}(\lambda))$. It now follows from (8.4) that Λ is a commutative algebra. By [14, (I.3.4)] Λ is a four-dimensional, local, symmetric algebra, and [14, (III.3)] now implies $\Lambda \cong k[X, Y]/(X^2, Y^2)$. $\square$

Lemma 8.8 *Suppose that $\hat{\psi}(e) = 0$. If $\lambda \in \mathcal{X}(\hat{\chi}_n)$ is a highest weight with $\lambda(h) \neq -1$, then $M(\lambda)$ is indecomposable.*

Proof. Since $v_0^{[p]} \in J^3$, we may assume $V = kv_0$ and $\psi = \hat{\psi}$. In particular, the Verma modules $Z(\lambda)$ are at our disposal. According to (8.5(2)) the $\mathcal{A}(\lambda)$-module $\hat{Z}(\lambda) := Z(\lambda)/v_0^2 Z(\lambda)$ is uniserial of length 2.

In view of (8.5(3)) we have an exact sequence

$$(*) \qquad (0) \longrightarrow Z(\lambda) \longrightarrow P(\lambda) \longrightarrow Z(\lambda) \longrightarrow (0).$$

The exact sequence $(*)$ induces

$$(0) \longrightarrow Z(\lambda) \longrightarrow JP(\lambda) \longrightarrow JZ(\lambda) \longrightarrow (0),$$

which in turn gives rise to an exact sequence

$$Z(\lambda)/J^2 Z(\lambda) \longrightarrow M(\lambda) \longrightarrow JZ(\lambda)/J^3 Z(\lambda) \longrightarrow (0).$$

In virtue of (8.5(1)) the first and third term are isomorphic to $\hat{Z}(\lambda)$. Since $\ell(M(\lambda)) = 4$, and $\ell(\hat{Z}(\lambda)) = 2$, the left-hand arrow is injective, so that we finally obtain an exact sequence

$$(**) \qquad (0) \longrightarrow \hat{Z}(\lambda) \longrightarrow M(\lambda) \xrightarrow{\omega} \hat{Z}(\lambda) \longrightarrow (0)$$

of $\mathcal{A}(\lambda)$-modules.

Similar arguments show that $(*)$ gives rise to

$$(0) \longrightarrow \hat{Z}(\lambda) \longrightarrow \hat{P}(\lambda) \longrightarrow \hat{Z}(\lambda) \longrightarrow (0),$$

whence $\hat{Z}(\lambda) \cong \Omega_{\mathcal{A}(\lambda)}(\hat{Z}(\lambda))$.

We put $\hat{u}(s\ell(2)_\psi, \chi_n) := u(s\ell(2)_\psi, \chi_n)/u(s\ell(2)_\psi, \chi_n)v_0^2$. By the PBW-Theorem the set $\{f^i(h + \psi(h))^j e^r v_0^s \ ; \ 0 \leq i, j, r, s \leq p - 1\}$ is a basis of $u(s\ell(2)_\psi, \chi_n)$ over k. Abusing notation we will not distinguish between elements of $u(s\ell(2)_\psi, \chi_n)$ and $\hat{u}(s\ell(2)_\psi, \chi_n)$. Thus, $\{f^i(h + \psi(h))^j e^r v_0^s \ ; \ 0 \leq i, j, r \leq p - 1 \ , \ 0 \leq s \leq 1\}$ is a basis of $\hat{u}(s\ell(2)_\psi, \chi_n)$ over k, and the canonical projection $u(s\ell(2)_\psi, \chi_n) \longrightarrow \hat{u}(s\ell(2)_\psi, \chi_n)$ restricts to an embedding $u(B, \chi_n|_B) \hookrightarrow \hat{u}(s\ell(2)_\psi, \chi_n)$. Moreover, $\hat{u}(s\ell(2)_\psi, \chi_n)$ is a free left and right $u(B, \chi_n|_B)$-module of rank $2p$.

We proceed in several steps.

(i) *We have an isomorphism $\hat{u}(s\ell(2)_\psi, \chi_n) \otimes_{u(B,\chi_n|_B)} k_\lambda \cong \hat{Z}(\lambda)$.*

The canonical projection $u(s\ell(2)_\psi, \chi_n) \xrightarrow{\pi} \hat{u}(s\ell(2)_\psi, \chi_n)$ induces a surjective map

$$Z(\lambda) \longrightarrow \hat{u}(s\ell(2)_\psi, \chi_n) \otimes_{u(B,\chi_n|_B)} k_\lambda \ \ ; \ \ u \otimes 1 \mapsto \pi(u) \otimes 1,$$

whose kernel obviously contains $v_0^2 Z(\lambda)$. Since $\dim_k \hat{Z}(\lambda) = \dim_k \hat{u}(s\ell(2)_\psi, \chi_n) \otimes_{u(B,\chi_n|_B)} k_\lambda$, the above map induces the desired isomorphism.

(ii) *The sequence $(**)$ does not split.*

Assume the contrary and let $i \in \{0, \dots, p-2\}$ be a natural number such that $\lambda(h) \equiv i \bmod(p)$. From the Cartan-Weyl identities (cf. [52, (I.1.3)]) and (i) it follows that

$$\mathrm{Soc}_{u(B,\chi_n|_B)}(\hat{Z}(\lambda)) = k(1 \otimes 1) \oplus k(v_0 \otimes 1) \oplus k(f^{i+1} \otimes 1) \oplus k(f^{i+1}v_0 \otimes 1).$$

Thus, the splitting of $(**)$ implies $\dim_k \mathrm{Soc}_{u(B,\chi_n|_B)}(M(\lambda)) = 8$.

Thanks to (8.6) the module $V := v_0^2 P(\lambda)/J^3 P(\lambda) \subset M(\lambda)$ is simple. Consequently, $V \cong Z_{s\ell(2)}(\lambda)$ and the above arguments yield $\dim_k \mathrm{Soc}_{u(B,\chi_n|_B)}(V) = 2$. Owing to (8.3) $\hat{P}(\lambda)$ is a projective $\hat{u}(s\ell(2)_\psi, \chi_n)$-module of dimension $4p$. By our above observations, it is also projective over the Nakayama algebra $u(B, \chi_n|_B)$. As noted above, the principal indecomposable $u(B, \chi_n|_B)$-modules are p-dimensional. Consequently, $\dim_k \mathrm{Soc}_{u(B,\chi_n|_B)}(\hat{P}(\lambda)) = 4$, so that the dimension of the module $\mathrm{Soc}_{u(B,\chi_n|_B)}(JP(\lambda)/v_0^2P(\lambda))$ is bounded by 4. The exact sequence

$$(0) \longrightarrow V \longrightarrow M(\lambda) \longrightarrow JP(\lambda)/v_0^2P(\lambda) \longrightarrow (0)$$

gives rise to the exact sequence

$$(0) \longrightarrow \mathrm{Soc}_{u(B,\chi_n|_B)}(V) \longrightarrow \mathrm{Soc}_{u(B,\chi_n|_B)}(M(\lambda)) \longrightarrow \mathrm{Soc}_{u(B,\chi_n|_B)}(JP(\lambda)/v_0^2P(\lambda)),$$

whence $\dim_k \mathrm{Soc}_{u(B,\chi_n|_B)}(M(\lambda)) \leq 2+4 = 6$, a contradiction. As a result, the sequence $(**)$ does not split.

Now let $\zeta : \hat{Z}(\lambda) \longrightarrow \hat{Z}(\lambda)$ be a non-zero, non-isomorphism. Since $\mathrm{Soc}(M(\lambda)) \not\subset \ker \omega$, there exists a map $\mu : \hat{Z}(\lambda) \longrightarrow \mathrm{Soc}(M(\lambda))$ so that $\omega \circ \mu \neq 0$. By Schur's Lemma we thus have $\zeta = \alpha(\omega \circ \mu) = \omega \circ (\alpha\mu)$ for some scalar $\alpha \in k$. Consequently, ζ factors through ω. As $\mathcal{A}(\lambda)$ is symmetric, we have $\mathrm{DTr}(\hat{Z}(\lambda)) \cong \Omega^2_{\mathcal{A}(\lambda)}(\hat{Z}(\lambda)) \cong \hat{Z}(\lambda)$. In view of (ii) and the foregoing observation [2, (V.2.2)] implies that the sequence $(**)$ is almost split. Since the periodic length 2 module $\hat{Z}(\lambda)$ is located at the mouth of a homogeneous tube, it follows that $M(\lambda)$ is indecomposable. $\square$

Lemma 8.9 *Suppose that $\hat{\psi}(e) = 0$. If $\lambda \in \mathcal{X}(\hat{\chi}_n)$ is a highest weight with $\lambda(h) \neq -1$, then the algebra $\mathcal{B}(\lambda)/J^3\mathcal{B}(\lambda)$ is Morita equivalent to $k[X,Y]/(Y^2, X^2Y, X^3)$. Consequently, $\mathcal{B}(\lambda)$ is wild.*

Proof. Let Λ be the basic algebra of $\mathcal{B}(\lambda)/J^3\mathcal{B}(\lambda)$ and put $R := \mathrm{Rad}(\Lambda)$. The arguments employed in (8.7) show that Λ is commutative. It follows from (8.8) and (8.6) that R is indecomposable of length 4 with the top and socle of length 2. Moreover, there exists an exact sequence

$$(0) \longrightarrow U \longrightarrow R \longrightarrow U \longrightarrow (0),$$

where U is a uniserial submodule of R of length 2. Hence $U = \Lambda y$ for some element $y \in R\backslash R^2$. Since R is indecomposable, and R/R^2, R^2 are of length 2, there exists an element $x \in R \setminus R^2$ such that $R = \Lambda x + \Lambda y$, Λx is of length 3, and $\mathrm{Soc}(\Lambda y) = \mathrm{Soc}(\Lambda x) \cap \mathrm{Soc}(\Lambda y) = kxy = kyx$. Clearly, then $x^2 \neq 0$. We claim that $y^2 = 0$. Suppose $y^2 \neq 0$. Then $y^2 = \alpha xy$ for some non-zero element $\alpha \in k$. Take $y' := y - \alpha x$. Then $R = \Lambda y' + \Lambda y$ and $yy' = y'y = (y - \alpha x)y = 0$. Then $\Lambda y'$ and Λy are uniserial modules of length 2 with $\mathrm{Soc}(\Lambda y') = k(y')^2$ and $\mathrm{Soc}(\Lambda y) = ky^2 = kxy$. Moreover, $(y')^2 = \alpha^2x^2 - \alpha xy \notin \mathrm{Soc}(\Lambda y)$, because $x^2 \notin \mathrm{Soc}(\Lambda y)$. Therefore, we obtain $R = \Lambda y' \oplus \Lambda y$, which contradicts the indecomposability of R. Thus, $y^2 = 0$, and Λ is isomorphic to $k[X,Y]/(Y^2, X^2Y, X^3)$. It follows from [14, (I.10.10(d))] that this algebra is wild, and consequently $\mathcal{B}(\lambda)$ is also wild. □

Remark. It follows directly from (8.9) that the algebra $u(s\ell(2)_s, \chi_n)$ is wild. Thus, the restricted Lie algebra $s\ell(2)_s$ possesses reduced enveloping algebras whose representation theory is more complicated than that of its restricted enveloping algebra. This phenomenon does not arise for Lie algebras with representation-finite enveloping algebras (see [19, (4.3)]).

We are now in a position to establish the main result on the block structure of restricted enveloping algebras with tame principal block. Recall that $n(L) = \dim_k \operatorname{rad}_p(L)$ is the dimension of the p-unipotent radical of the restricted Lie algebra $(L, [p])$.

Theorem 8.10 *Let $(L, [p])$ be a restricted Lie algebra of characteristic $p \geq 3$ such that $\mathcal{B}_0(L)$ is tame. Then each block $\mathcal{B} \subset u(L)$ is either Morita equivalent to $k[X]/(X^{p^{n(L)}})$, $k[X]/(X^{2p^{n(L)}})$, $\mathcal{K}(n(L))$, or $\mathcal{B}$ is wild. In particular, each tame block of $u(L)$ has exactly 2 simple modules.*

Proof. Thanks to [25, (6.4)] we have an isomorphism $L/C(L) \cong s\ell(2)$. By the same token, the center $C(L)$ decomposes into a direct sum

$$C(L) = V \oplus T(L)$$

of restricted Lie algebras with a nil-cyclic Lie algebra $V = (kv_0)_p$ of dimension $n = n(L)$. In view of the general observations made in section 1, we thus have a decomposition

$$L = s\ell(2) \oplus V \oplus T(L)$$

of Lie algebras. The p-map on L is induced by those of the three summands, and a p-semilinear map $\psi : s\ell(2) \longrightarrow C(L)$, which we write as a sum $\psi = \psi_V + \psi_T$ of two p-semilinear maps $\psi_V : s\ell(2) \longrightarrow V$ and $\psi_T : s\ell(2) \longrightarrow T(L)$.

We consider the restricted Lie algebra $H := s\ell(2) \oplus V$ whose p-map is given by

$$(x+v)^{[p]} := x^{[p]} + \psi_V(x) + v^{[p]} \qquad \forall\ x \in s\ell(2),\ v \in V.$$

If $\psi'_T : H \longrightarrow T(L)$ denotes the p-semilinear map defined by

$$\psi'_T(x+v) := \psi_T(x) \qquad \forall\ x \in s\ell(2),\ v \in V,$$

then $L_{\psi'_T} = H \oplus T(L)$ is the originally given restricted Lie algebra $(L, [p])$.

For an element $\gamma \in X(T(L))$, we let $\chi_\gamma \in H^*$ be the linear form satisfying

$$\chi_\gamma(h)^p = (\gamma \circ \psi'_T)(h) \quad \text{for every } h \in H.$$

In particular, we have $V \subset \ker \chi_\gamma$. A consecutive application of (8.1) and (8.2) shows that there exists an isomorphism

$$u(L) \cong \bigoplus_{\gamma \in X(T(L))} u(H, \chi_\gamma)$$

of k-algebras.

In particular, the blocks of $u(L)$ are preceisely the blocks of the algebras $u(H, \chi_\gamma)$, where γ ranges over $X(T(L))$.

Now let $\mathcal{B} \subset u(H, \chi_\gamma)$ be a block.

Suppose that $\chi_\gamma = 0$. By [25, (1.1)] the principal block $\mathcal{B}_0(H)$ is tame. Since $n(H) = n(L) = n$ Theorem 7.1 now shows that $\mathcal{B}$ is either Morita equivalent to $\mathcal{K}(n(L))$ or to $k[X]/(X^{p^{n(L)}})$.

Assume $\chi_\gamma \neq 0$ to be of semisimple type. In this case [29, (2.2)] ensures that $u(s\ell(2), \hat{\chi}_\gamma)$ is semisimple, so, for $n(L) \neq 0$, the Jacobson radical $I = J$ of $u(H, \chi_\gamma)$ is generated by the central element v_0. Morita's Theorem [9, (62.26)] then ensures that $u(H, \chi_\gamma)$ is a Nakayama algebra. Consequently, $\mathcal{B}$ also enjoys this property. By (8.3) the block $\mathcal{B}$ has Loewy length $p^{n(L)}$ and is thereby Morita equivalent to $k[X]/(X^{p^{n(L)}})$.

Alternatively, we may assume that $\chi_\gamma = \chi_n$ is of nilpotent type. Owing to [29, (2.3)] each block of $u(s\ell(2), \hat{\chi}_n)$ is a Nakayama algebra of Loewy length ≤ 2. Thus, for $n(L) = 0$, the block $\mathcal{B}$ is Morita equivalent to $k[X]/(X^\ell)$ $\ell \in \{1, 2\}$.

We assume $n(L) \neq 0$. If $d(\psi_V) = 0$, then (8.4) shows that the block $\mathcal{B}$ is a primary Nakayama algebra. By (8.3) its Loewy length is $p^{n(L)}$ or $p^{2n(L)}$, so that $\mathcal{B}$ is a full matrix ring over $k[X]/(X^{p^{n(L)}})$ or $k[X]/(X^{p^{2n(L)}})$.

We finally assume $d(\psi_V) = 1$, and let $\lambda \in \mathcal{X}(\hat{\chi}_n)$ be the highest weight corresponding to $\mathcal{B}$. If $\lambda(h) = -1$, then (8.3) and (8.4) imply that $\mathcal{B}$ is Morita equivalent to $k[X]/(X^{p^{n(L)}})$. Alternatively, (8.8) yields the indecomposability of $JP(\lambda)/J^3P(\lambda)$. Consequently, $\mathcal{B}$ is wild in this case. $\square$

Let $L = s\ell(2)_\psi$ be a central extension of $s\ell(2)$. As before we write $\psi = \psi_V + \psi_T$. For a linear form $\chi \in L^*$ with $\chi(C(L)) = (0)$ we put

$$d(\psi, \chi) := \dim \mathcal{V}_{s\ell(2)} \cap \ker \hat{\chi} \cap \ker \hat{\psi}_V.$$

Note that $d(g \cdot \psi, g \cdot \chi) = d(\psi, \chi)$ for every element $g \in \mathrm{SL}(2)$. Since $\mathcal{V}_{s\ell(2)} \cap \ker \hat{\chi}_n = ke$, we have $d(\psi, \chi_n) = d(\psi_V)$.

Corollary 8.11 *Let $L = s\ell(2)_\psi$ be a central extension of $s\ell(2)$ with center $C(L) = V \oplus T(L)$, $V = (kv_0)_p$. Then the following statements are equivalent:*

(1) *$u(L)$ is tame.*

(2) (a) *$\hat{\psi}_V \in (k^\times \times \mathrm{SL}(2)) \cdot \psi_s$ is of semisimple type, and*

(b) *$d(\chi_\gamma, \psi_V) = 0$ for every $\gamma \in X(T(L))$ such that χ_γ is of nilpotent type.*

Proof. (1) $\Rightarrow$ (2). Condition (a) is a direct consequence of (7.4) and (b) follows from the proof of (8.10).

(2) $\Rightarrow$ (1). If L_{ψ_V} denotes the restricted Lie algebra $s\ell(2) \oplus V$ with p-map given by ψ_V, then (8.1) and (8.2) imply $u(L) \cong \bigoplus_{\gamma \in X(T(L))} u(L_{\psi_V}, \chi_\gamma)$. According to (7.4) condition (a) implies that $\mathcal{B}_0(L)$ is tame. The proof of (8.10) in conjunction with condition (b) shows that each algebra $u(L_{\psi_V}, \chi_\gamma)$ is tame. Consequently, $u(L)$ is tame as well. $\square$

Examples. Consider the 5-dimensional restricted Lie algebra $L := s\ell(2) \oplus kv_0 \oplus kt_0$ with center $C(L) = kv_0 \oplus kt_0$.

(1). We define a p-map on L via

$$e^{[p]} := 0 \ \ ; \ \ h^{[p]} := h + v_0 \ \ ; \ \ f^{[p]} := t_0 \ \ ; \ \ v_0^{[p]} := 0 \ \ ; \ \ t_0^{[p]} = t_0.$$

In view of (7.4) the principal block $\mathcal{B}_0(L)$ is tame. In fact, it is Morita equivalent to $\mathcal{K}(1)$. Let $\gamma : kt_0 \longrightarrow k$ be given by $\gamma(t_0) = 1$. Note that $\psi_V = \hat{\psi}_V = \psi_s$ and that $\hat{\chi}_\gamma \in s\ell(2)^*$ coincides with $\hat{\chi}_n$. Since $d(\psi_V) = 1$, the algebra $u(L)$ is wild.

(2). Now define a p-map on L via

$$e^{[p]} := v_0 \ \ ; \ \ h^{[p]} := h \ \ ; \ \ f^{[p]} := v_0 + t_0 \ \ ; \ \ v_0^{[p]} := 0 \ \ ; \ \ t_0^{[p]} = t_0.$$

Then ψ_V is of semisimple type and the principal block $\mathcal{B}_0(L)$ is tame. Let $\gamma : kt_0 \longrightarrow k$ be given by $\gamma(t_0) \in \mathrm{GF}(p)\backslash\{0\}$. As before, $\hat{\chi}_\gamma \in s\ell(2)^*$ is of nilpotent type. Since $d(\chi_\gamma, \psi_V) = 0$, the algebra $u(L)$ is tame.

Corollary 8.12 *Let $(L, [p])$ be a restricted Lie algebra of characteristic $p \geq 3$. Then the following statements are equivalent:*

(1) *$u(L, \chi)$ is tame for every $\chi \in L^*$.*

(2) *$L/T(L) \cong s\ell(2)$.*

Proof. (1) $\Rightarrow$ (2). Since $u(L)$ is tame, (7.4) implies that $L/C(L)^{[p]} \cong s\ell(2)$, $s\ell(2)_s$. In the former case we have (2) while in the latter there exists a linear form $\chi \in L^*$ which is zero on $C(L)^{[p]}$ and equal to χ_n on $s\ell(2)_s$. Accordingly, $u(L, \chi)$ surjects onto the wild algebra $u(s\ell(2)_s, \chi_n)$.

(2) $\Rightarrow$ (1). This follows from the proof of (8.10). □

9. The Krull-Gabriel Dimension of u(L)

Given a k-algebra Λ, we denote by $\mathrm{mod}(\Lambda)$ and $\mathcal{C}(\Lambda)$ the categories of finite dimensional left Λ-modules and finitely presented covariant functors from $\mathrm{mod}(\Lambda)$ to $\mathrm{mod}(k)$, respectively. We will write $\mathcal{A}/\mathcal{S}$ for the abelian quotient of an abelian category $\mathcal{A}$ by a Serre subcategory $\mathcal{S}$ (cf. [35, (1.11)] and [49, (I.2)]). We put $\mathcal{C}_{-1} := 0$ and define $\mathcal{C}_n$ to be the category of all functors $F \in \mathcal{C}$ that are sent to an object of finite length by the canonical quotient functor $\mathcal{C} \longrightarrow \mathcal{C}/\mathcal{C}_{n-1}$. Following [31] we define the *Krull-Gabriel dimension* $\mathrm{KG}(\Lambda) \in \mathbb{N} \cup \{\infty\}$ via

$$\mathrm{KG}(\Lambda) := \min\{n \in \mathbb{N} \cup \{\infty\} \ ; \ \mathcal{C}_n = \mathcal{C}\}.$$

It was shown in [1, (3.14)] that $\mathrm{KG}(\Lambda) = 0$ if and only if Λ is of finite representation type.

As an application of our classification of tame restricted Lie algebras, we have the following result on the Krull-Gabriel dimension of restricted enveloping algebras:

Theorem 9.1 *Let $(L, [p])$ be a restricted Lie algebra of characteristic $p \geq 3$. Then the following statements are equivalent:*

(1) $\mathrm{KG}(u(L)) < \infty$.

(2) *$L/T(L) \cong s\ell(2)$, or $L/T(L)$ is the semidirect sum of a torus of dimension ≤ 1 and a nil-cyclic ideal.*

(3) $\mathrm{KG}(u(L)) \in \{0, 2\}$.

Proof. (1) $\Rightarrow$ (2). Suppose that $u(L)$ has finite Krull-Gabriel dimension. Owing to [32] this implies that $u(L)$ is of finite or tame representation type. In the former case, the classification of representation-finite restricted Lie algebras (cf. [19, (4.3)]) shows that $L/T(L)$ is a semidirect sum of torus of dimension ≤ 1 and a nil-cyclic ideal. If $u(L)$ is tame, then $\mathcal{B}_0(L)$ also enjoys this property. Thanks to (7.4) we thus have $L/C(L)^{[p]} \cong s\ell(2)$ or $s\ell(2)_s$. In the latter case (7.2) shows that $\mathcal{B}_0(L)$ is not of polynomial growth, so that the string algebra $\mathcal{B}_0(L)/\mathrm{Soc}(\mathcal{B}_0(L))$ is non-domestic. In view of [46, Prop.2] this algebra has infinite Krull-Gabriel dimension. Thus, $\mathrm{KG}(u(L)) = \infty$, and we have reached a contradiction. It follows that the Lie algebra $L/C(L)^{[p]} \cong s\ell(2)$ is simple, so that $C(L) = C(L)^{[p]} = T(L)$.

(2) $\Rightarrow$ (3). If L is given as in (2), then (7.4), the proof of (8.9) and [19, (4.3)] show that $u(L)$ is of tame or finite representation type. In the latter case [1, (3.14)] yields $\mathrm{KG}(u(L)) = 0$. Alternatively, we have $L/T(L) \cong s\ell(2)$ and [25, (1.1)] implies $\mathcal{B}_0(L) \cong \mathcal{B}_0(s\ell(2))$. Since $\mathcal{B}_0(s\ell(2))$ is the trivial extension $H \ltimes D(H)$ of the Kronecker algebra H, an application of [32, (3.9)] gives $\mathrm{KG}(\mathcal{B}_0(s\ell(2))) = 2$. Owing to (8.9) all other blocks of $u(L)$ are Nakayama algebras, so that $u(L)$ has Krull-Gabriel dimension 2.

(3) $\Rightarrow$ (1). Trivial $\square$

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