Representation dimension and quasi-hereditary algebras

by

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We study Auslander's representation dimension of Artin algebras, which is by definition the minimal projective dimension of coherent functors on modules which are both generators and cogenerators. We show the following statements: (1) if an Artin algebra A is stably hereditary, then the representation dimension of A is at most 3. (2) If two Artin algebras are stably equivalent of Morita type, then they have the same representation dimension. Particularly, if two self-injective algebras are derived equivalent, then they have the same representation dimension. (3) Any incidence algebra of a finite partially ordered set over a field has finite representation dimension. Moreover, we use results on quasi-hereditary algebras to show that (4) the Auslander algebra of a Nakayama algebra has finite representation dimension.

1 Introduction

Among Artin algebras the class of representation finite Artin algebras are much better understood in the representation theory. To investigate the connection of arbitrary Artin algebras with representation finite Artin algebras, the representation dimension is introduced by Auslander in [1]. " It is hoped that this notion gives a reasonable way of measuring how far an Artin algebra is from being representation finite type" [1, p.134].

The key ingredients in the notion of the representation dimension are the coherent functors and their homological dimensions. Recently, Hartshorne in [11] reveals an important application of coherent functors to the study of Rao modules in algebraic space curves. This implies that coherent functors are very useful and desired to be investigated further.

Unfortunately, in the last three decades there is not much progress on the representation dimension. It is still a mysterious subject in the representation theory. One does not even know whether the representation dimension of a finite dimensional algebra over a field is finite. To enrich our knowledge on representation dimension, we study in this paper the question of the following type: Suppose two Artin algebras A and B have certain good connection (for example, they are stably equivalent, or B is a quotient of A), how is the relationship of their representation dimensions ? Secondly, we want to relate the investigation of the representation dimension to that of quasi-hereditary algebras. As one of the main results in this paper, we shall prove in section three that if an Artin algebra is stably hereditary , then its representation

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dimension is at most 3. In particular, we reobtain a result of Auslander and Reiten in [4] which says that if an Artin algebra is stably equivalent to a hereditary algebra, then its representation dimension is bounded by 3. Along this direction we shall consider in section four the stable equivalence of Morita type between two Artin algebras. In this case we will demonstrate that if there is a stable equivalence of Morita type between two Artin algebras A and B, then A and B have the same representation dimension. In particular, if two algebras are derived equivalent, then their trivial extensions have the same representation dimensions. Section five is devoted to the consideration of representation dimensions of a self-injective algebra and its factor algebras

Since quasi-hereditary algebras have finite global dimension, we can use results on quasi-hereditary algebras to get some upper bounds for the representation dimension. In section six we shall prove that the Auslander algebra of a Nakayama algebra has finite representation dimension, and that the algebra of the form $\mathrm{End}(A \oplus X)$ with A a self-injective algebra and X a semi-colocal A-module has finite representation dimension. Finally, we shall show in section seven that the incidence algebra of an arbitrary finite partially ordered set over a field always has a finite representation dimension.

Throughout this paper we work with Artin algebras. Sometimes we assume a strong condition that the algebra considered is a finite dimensional k-algebra over a fixed field k. We always assume that our algebras have the identity element. By a module we mean a finitely generated left module. The global dimension of an algebra A is denoted by gl.dim(A). By D we denote the usual duality and by A-mod the category of all (finitely generated left) A-modules. Given two homomorphisms $f : L \longrightarrow M$ and $g : M \longrightarrow N$, the composition of f and g is a homomorphism from L to N and is denoted in the paper by fg.

2 Preliminaries

Given an Artin algebra A, that is, A is a ring whose center is an Artin ring and over the center A is a finitely generated module, we say that A has dominant dimension greater than or equal to n, denoted by dom.dim $(A) \geq n$, if there is an exact sequence

$$0 \longrightarrow {}_A A \longrightarrow X_1 \longrightarrow X_2 \longrightarrow ...$$

of A-modules such that X_i is projective and injective for $i = 1, ..., n$. We denote by $I_0(A)$ the module X_1.

For a representation finite Artin algebra, Auslander proved that the endomorphism algebra of the direct sum of all non-isomorphic indecomposable modules has global dimension at most two and dominant dimension at least two. More precisely, Auslander proved the following theorem, which motivated him to introduce the notion of representation dimension, as a way of measuring how far an Artin algebra is from being representation finite type.

Theorem 2.1 *Suppose A is an Artin algebra with gl.dim $(A) \leqslant 2$. If P is a projective and injective A-module, then $End_A(P)$ has representation finite type. Further, up to Morita equivalence, all Artin algebras of representation finite type are obtained in this way.*

The representation dimension is defined as follows.

Definition 2.2 *Let A be an Artin algebra. Consider an Artin algebra Λ of dominant dimension at least two such that $End_\Lambda(I_0(\Lambda))$ is Morita equivalent to A. Then the representation* *dimension of A is defined to be the minimum of the global dimension of all possible Λ, and denoted by rep.dim (A).*

In fact, Auslander also proved in [1] that the above definition is equivalent to the following definition:

$$\text{rep.dim}(A) = \inf\{\text{gl.dim } (\text{End}_A(M)) \mid M \text{ is a generator-cogenerator}\}$$

Note that an A-module M is called a generator-cogenerator if every indecomposable projective module and also every indecomposable injective module is isomorphic to a summand of M.

Note that for a semisimple algebra its representation dimension is zero by definition. It is easy to see that there is no algebra whose representation dimension is 1.

The following lemma collects some known results on the representation dimension which we shall need in the sequel.

Lemma 2.3 *Let A be a non-semisimple k-algebra. Then*

(1) rep.dim$(A) = 2$ if and only if A is representation finite.

(2) If A is self-injective algebra, then rep.dim$(A) \leqslant LL(A)$, where $LL(A)$ stands for the Loewy length of A.

(3) If gl.dim $(A) \leqslant 1$, then rep.dim $(A) \leqslant 3$.

(4) If the radical square of A is zero, then rep.dim $(A) \leqslant 3$.

(5) Let $T_2(A)$ denote the 2×2 triangular matrix algebra over A, then rep.dim $(T(A)) \leqslant$ rep.dim$(A) + 2$.

(6) If A and B are two algebras over a perfect field k, then

$$\textit{rep.dim } (A \otimes_k B) \leqslant \textit{rep.dim } (A) + \textit{ rep.dim } (B).$$

The statements (1)–(4) were proved in [1], and the statement (5) was shown in [10, p. 115]. (6) was proved in [19].

Finally, let us recall a result of Auslander which is useful for computing the global dimension of the endomorphism algebra of a given module.

Let M be an A-module. We denote by add(M) the full subcategory of A-mod whose objects are isomorphic to direct summands of direct sums of finite copies of M.

Let $\mathcal{C}$ be a skeletally small category. We denote by $\mathcal{C}^{op}$ the opposite category of $\mathcal{C}$ and by Funct$(\mathcal{C}^{op}, Ab)$ the abelian category of all functors from $\mathcal{C}^{op}$ to the category Ab of abelian groups. Let $\widehat{\mathcal{C}}$ be the full subcategory of Funct$(\mathcal{C}^{op}, Ab)$ consisting of all coherent functors G, that is, those functors G for which there is a morphism $C_1 \longrightarrow C_2$ in $\mathcal{C}$ such that the sequence

$$(,C_1) \longrightarrow (,C_2) \longrightarrow G \longrightarrow 0$$

is exact. Here and in the sequel we denote by $(,C)$ the Hom functor $\text{Hom}_{\mathcal{C}}(,C) : \mathcal{C}^{op} \longrightarrow Ab$ for $C \in \mathcal{C}$.

The following lemma is proved in [1].

Lemma 2.4 *Let M be in A-mod. Then the category $\widehat{add(M)}$ and End(M)-mod are equivalent. In particular, gl.dim$(End_A(M))$ = gl.dim$(\widehat{add(M)})$.*

Finally, let us remark that representation dimension is invariant under Morita equivalences and that rep.dim(A) = rep.dim (A^{op}) for all Artin algebras, where A^{op} stands for the opposite algebra of A.

3 Stably hereditary algebras

We know from 2.3 that for a hereditary algebra A one has rep.dim$(A) \leqslant 3$. Moreover, it is shown in [4] that if an algebra is stably equivalent to a hereditary algebra then its representation dimension is also bounded by 3. In fact, we prove that this is true for stably hereditary algebras, a class of algebras which are more general than that of algebras stably equivalent to hereditary algebras.

Let us first recall some definitions and notation.

Given an Artin algebra A. We define the stable category A-$\underline{\text{mod}}$ of the algebra A as follows: the objects are the same as those of the module category A-mod, and the morphisms between two objects M and N are given by $\underline{\text{Hom}}_A(M,N)= \text{Hom}_A(M,N)/R(M,N)$, where $R(M,N)$ is the subgroup of $\text{Hom}_A(M,N)$ consisting of the homomorphisms from M to N which factor through a projective A-module.

Definition 3.1 *Let A and B be two Artin algebras over a field k. We say that A and B are stably equivalent if A-$\underline{mod}$ and B-$\underline{mod}$ are equivalent.*

In [5], algebras which are stably equivalent to hereditary algebras are investigated in details. The following is a characterization of these algebras:

Lemma 3.2 *([3, 4]) Let A be an Artin algebra. Then A is stably equivalent to a hereditary algebra if and only if the following two conditions hold:*

(1) Each indecomposable submodule of an indecomposable projective module is projective or simple;

(2) For each non-projective simple submodule L of a projective module there is an injective module Q with $L \subseteq Q/rad(Q)$.

Note that if an algebra A is stably equivalent to a hereditary algebra , then the opposite algebra A^{op} of A is also stably equivalent to a hereditary algebra. From this observation and the above lemma, we have the following obvious fact which is a part of the dual statement of Lemma 3.2.

Lemma 3.3 *If A is stably equivalent to a hereditary algebra, then each indecomposable factor module of an indecomposable injective module is injective or simple.*

Motivated by these characterizitions of algebras being stably equivalent to hereditary algebras, we introduce the following notion.

Definition 3.4 *Let A be an Artin algebra. We say that A is* **stably hereditary** *if each indecomposable submodule of an indecomposable projective module is either projective or simple, and each indecomposable factor module of an indecomposable injective module is either injective or simple.*

Clearly, algebras which are stably equivalent to hereditary algebras are stably hereditary, but the converse is not true, we shall see an example at the end of this section.

Now let us prove the following main result of this section.

Theorem 3.5 *If an Artin algebra A is stably hereditary, then rep.dim$(A) \leqslant 3$.*

Proof. We define $V := A \oplus D(A_A) \oplus A/rad(A)$. Then V is clearly a generator-cogenerator for A-mod. We shall prove that for each A-module M there is an exact sequence

$$0 \longrightarrow V_2 \longrightarrow V_1 \longrightarrow M \longrightarrow 0$$

with $V_i \in \mathrm{add}(V)$ such that for any module $X \in \mathrm{add}(V)$ the following sequence

$$0 \longrightarrow (X, V_2) \longrightarrow (X, V_1) \longrightarrow (X, M) \longrightarrow 0$$

is exact, where we denote by (X, M) the A-homomorphism set from X to M.

For the given module M, we denote by M' the sum of the maximal injective submodule of M and the socle of M. The canonical inclusion from M' to M and the canonical surjection from M to M/M' are denoted by μ and π respectively. Let $h : P \longrightarrow M/M'$ be a projective cover of M/M'. Then there is a lifting $g : P \longrightarrow M$ such that $h = g\pi$. Let Ω be the kernel of h. Then we have the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega \arrow[d] \arrow[r, equal] & \Omega \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, equal] & M' \oplus P \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & M' \arrow[r, "\mu"] & M \arrow[r, "\pi"] \arrow[d] & M/M' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We define $f : M' \oplus P \longrightarrow M$ by $(m', p) \longmapsto m' + (p)g$, where $m' \in M', p \in P$, and the image of p under g is denoted by $(p)g$. It is clear that f is a surjective map and that M' is a direct sum of an injective module and a semisimple module. Thus M' belongs to $\mathrm{add}(V)$. If X is projective or simple, then the morphism from $(X, M' \oplus P) \longrightarrow (X, M)$ induced from f is surjective by the definition of M'. Now let X be an indecomposable injective A-module and let $\varphi : X \longrightarrow M$ be a non-zero homomorphism . By the definition of stably hereditary algebras, the image of φ is either injective, or simple, thus lies in M'. This means that φ factors through f. Thus for any X in $\mathrm{add}(V)$ the morphism from $(X, M' \oplus P) \longrightarrow (X, M)$ induced by f is surjective.

To show that the sequence

$$0 \longrightarrow \Omega \longrightarrow M' \oplus P \xrightarrow{f} M \longrightarrow 0$$

is a desired one, we need only to show that Ω lies in $\mathrm{add}(V)$. Let $\Omega = \Omega_1 \oplus \Omega_2$, where each indecomposable direct summand of Ω_1 is projective or injective, and each indecomposable direct summand X of Ω_2 is neither projective, nor injective. If we decompose P into a direct sum of indecomposable projective modules, say $P_1 \oplus P_2 \oplus ... \oplus P_m$, then the image of X in P_j must be simple by the definition of stably hereditary algebras. Thus the image of Ω_2 under the canonical inclusion $\Omega_2 \hookrightarrow P$ is contained in the socle of P. This implies that Ω_2 itself is semisimple, thus lies in $\mathrm{add}(V)$.

To finish the proof of Theorem 3.5, we take a coherent functor F in $\widehat{\mathrm{add}(V)}$. Then there is a morphism $g : X_1 \longrightarrow X_0$ with $X_i \in \mathrm{add}(V)$ such that the functor sequence

$$(-, X_1) \longrightarrow (-, X_0) \longrightarrow F \longrightarrow 0$$

is exact on add(V). Let M be the kernel of g. Then, by what we have proved, there exists an exact sequence

$$0 \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$$

with $X_2, X_3 \in \text{add}(V)$ such that for any module $X \in \text{add}(V)$ the following sequence

$$0 \longrightarrow (X, X_3) \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact. This shows that the exact sequence

$$0 \longrightarrow (, X_3) \longrightarrow (, X_2) \longrightarrow (, X_1) \longrightarrow (, X_0) \longrightarrow F \longrightarrow 0$$

of functors is exact on add(V). By 2.4, we have $\text{gl.dim}(\text{End}_A(V)) \leqslant 3$. Thus $\text{rep.dim}(A) \leqslant 3$. This finishes the proof.

As a corollary of the above theorem, we have the following result due to Auslander and Reiten [4, Proposition 4.7].

Corollary 3.6 *If an Artin algebra A is stably equivalent to a hereditary algebra, then rep.dim$(A) \leqslant 3$.*

We know that representation-finite type is invariant under stable equivalences. This means that if A is stably equivalent to a hereditary algebra B, then $\text{rep.dim}(A) = \text{rep.dim}(B)$.

We point out that a stably hereditary algebra may not be stably equivalent to a hereditary algebra. As is shown in [4], if A is the algebra $\begin{pmatrix} k & k & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$ with k a field and if S is the unique simple module which is neither projective, nor injective, then the endomorphism algebra of $A \oplus S$ is not stably equivalent to a hereditary algebra, but it is stably hereditary. In fact, we have the following general construction:

Proposition 3.7 *Let A be a stably hereditary algebra over algebraically closed field k and S a simple A-module. Then the one-point extension*

$$A[S] = \begin{pmatrix} A & S \\ 0 & k \end{pmatrix}$$

of A by S is again stably hereditary. In particular, if S is a simple A-module which is not a submodule of a projective module, then the endomorphism algebra $E := End_A(A \oplus S)$ of $A \oplus S$ is stably hereditary. In particular, rep.dim$(E) \leqslant 3$.

Proof. Let E denote the one-point extension of A by S. We denote by $L(\omega)$ the simple E-module which is not an A-module. Thus every indecomposable submodule of indecomposable projective modules is projective or simple. Now let Q be an indecomposable injective E-module which is not simple. Then either Q has no composition factor $L(\omega)$, or Q is the E-injective envelope of S. In the former case every indecomposable factor module of Q is injective since A is stably hereditary. In the latter case, $Q/Soc(Q)$ is a direct sum of $L(\omega)$ and an injective A-module, this implies that each indecomposable factor module of Q is either the simple module $L(\omega)$, or injective. Thus E is stably hereditary.

4 Stable equivalences of Morita type

There are evidences which suggest the following conjecture: If A and B are stably equivalent, then rep.dim(A) = rep.dim(B). In this section we show that this conjecture is true for stable equivalences of Morita type.

First recall from [7] that given two Artin algebras A and B, a stable equivalence $\phi : A - \underline{\mathrm{mod}} \longrightarrow B - \underline{\mathrm{mod}}$ is said to be of Morita type if there are bimodules ${}_AM_B$ and ${}_BN_A$ such that

(a) ${}_AM, {}_BN, M_B$ and N_A are projective modules; and $M \otimes_B N \simeq A \oplus P$ as A-bimodules for a projective A-bimodule P, and $N \otimes_A M \simeq B \oplus Q$ as B-bimodules for a projective B-bimodule Q; and

(b) The following diagram

$$\begin{array}{ccc} A - mod & \stackrel{N\otimes_A -}{\longrightarrow} & B - mod \\ \pi_A \big\downarrow & & \big\downarrow \pi_B \\ A - \underline{mod} & \stackrel{\phi}{\longrightarrow} & B - \underline{mod} \end{array}$$

is commutative as natural isomorphisms, where π_A denotes the canonical functor for A-mod to its stable category A-$\underline{\mathrm{mod}}$.

Note that the above ϕ is lifted to a Morita equivalence if and only if it is of Morita type with $P = 0 = Q$.

We shall prove that under the condition (a) the representation dimensions of A and B are equal to each other. For convenience, we say that two Artin algebras A and B are **Morita-type equivalent** if there are two bimodules ${}_AM_B$ and ${}_BN_A$ such that the condition (a) is fulfilled. The main result in this section is the following theorem.

Theorem 4.1 *Let A and B be two Artin algebras. If they are Morita-type equivalent, then rep.dim(A) = rep.dim(B).*

Proof. We define functors $T_M : B\text{-mod} \longrightarrow A\text{-mod}$ by $X \longmapsto M \otimes_B X$ and $T_N : A\text{-mod} \longrightarrow B\text{-mod}$ by $Y \longmapsto N \otimes_A Y$. Similarly, we have the functors T_P and T_Q. It is clear from (a) that $T_M \circ T_N \simeq \mathrm{id}_{A-mod} \oplus T_P$ and $T_N \circ T_M \simeq \mathrm{id}_{B-mod} \oplus T_Q$. Note also that the images of T_P and T_Q consist of projective modules (see [6, Corollary 3.3]) and that all tensor functors involved are exact.

(1) If I is an injective A-module, then so is the B-module $T_N(I)$. In fact, given an monomorphism $f : Y_1 \longrightarrow Y_2$ of B-modules and a homomorphism $g : Y_1 \longrightarrow T_N(I)$, we show that there is a morphism $g' : Y_2 \longrightarrow T_N(I)$ such that $g = fg'$. Since I is injective, there is a morphism $g_1 : T_M(Y_2) \longrightarrow I$ such that $T_M(f)g_1 = T_M(g)\binom{\mathrm{id}_I}{0}$. Note that $T_MT_N(I) = I \oplus T_P(I)$. By applying the functor T_N, we have

$$\begin{pmatrix} f & 0 \\ 0 & T_Q(f) \end{pmatrix} T_N(g_1) = \begin{pmatrix} g & 0 \\ 0 & T_Q(g) \end{pmatrix} \begin{pmatrix} \mathrm{id}_{T_N(I)} \\ 0 \end{pmatrix}.$$

If we rewrite $T_N(g_1)$ as $(g', h)^{tr}$, then $g = fg'$. This shows that $T_N(I)$ is injective.

(2) If V is a generator-cogenrator for A-mod, then $T_N(V)$ is a generator-cogenerator for B-mod. Indeed, we have a surjective morphism $V^m \longrightarrow M$, from this we get a surjective morphism $T_N(V^m) \longrightarrow T_N(M) = B \oplus Q$ which implies that B is a direct summand of $T_N(V)^m$.

Thus $T_N(V)$ is a generator for B-mod. To prove that $T_N(V)$ is a cogenerator for B-mod, we take an injective B-module Y and consider the A-module $T_M(Y)$. According to (1), $T_M(Y)$ is an injective A-module, and therefore it lies in $\mathrm{add}(V)$. This implies that $T_N(T_M(Y))$ belongs to $\mathrm{add}(T_N(V))$, that is, $Y \oplus Q \otimes Y$ lies in $\mathrm{add}(T_N(V))$.

(3) If $f : V_0 \longrightarrow X$ is a right $\mathrm{add}(V)$-approximation of X, then $T_N(f) : T_N(V_0) \longrightarrow T_N(X)$ is a right $\mathrm{add}(T_N(V))$-approximation of $T_N(X)$. Recall that given a full subcategory $\mathcal{C}$ of A-mod and a A-module M, a morphism $g : X \longrightarrow M$ with $X \in \mathcal{C}$ is called a right $\mathcal{C}$-approximation of M if for any $X' \in \mathcal{C}$ and morphism $g' : X' \longrightarrow M$ there is a morphism $h : X' \longrightarrow X$ such that $g' = hg$. To prove our statement, we take a morphism $g : Y \longrightarrow T_N(X)$ with $Y \in \mathrm{add}(T_N(V))$. We write $T_M(g) := (g_1, g_2) : T_M(Y) \longrightarrow X \oplus T_P(X)$. Then

$$\begin{pmatrix} g & 0 \\ 0 & T_Q(g) \end{pmatrix} = T_N T_M(g) = (T_N(g_1), T_N(g_2))$$

and $T_N(g_1)) = (g, 0)^{tr}$. Since $T_M(Y) \in \mathrm{add}(V)$, there exists a morphism $h : T_M(Y) \longrightarrow V_0$ such that $g_1 = hf$. Thus $T_N(g_1) = T_N(h)T_N(f)$ and $g = h_1 T_N(f)$, where $T_N(h) = (h_1, h_2)^{tr} : Y \oplus T_Q(Y) \longrightarrow T_N(V_0)$. This is what we want to show.

(4) Suppose V is a generator-cogenerator for A-mod such that $\mathrm{rep.dim}(A) = m$. Then $T_N(V)$ is a generator-cogenerator for B-mod. If F is a coherent functor in $\mathrm{add}\widehat{(T_N(V))}$, then there is a morphism $f : Y_1 \longrightarrow Y_0$ with $Y_1, Y_0 \in \mathrm{add}(T_N(V))$ such that $(, Y_1) \longrightarrow (, Y_0) \longrightarrow F \longrightarrow 0$ is exact on $\mathrm{add}(T_N(V))$. Let Y be the kernel of f. Clearly, $T_M(f)$ is a morphism in $\mathrm{add}(V)$, and the sequence

$$(, T_M(Y_1)) \longrightarrow (, T_M(Y_0)) \longrightarrow G \longrightarrow 0$$

provides us a coherent functor G in $\widehat{\mathrm{add}(V)}$. Since the global dimension of $\widehat{\mathrm{add}(V)}$ is m by 2.4, there is a projective resolution

$$0 \longrightarrow (, V_m) \longrightarrow ... \longrightarrow (, V_2) \longrightarrow (, T_M(Y_1)) \longrightarrow (, T_M(Y_0)) \longrightarrow G \longrightarrow 0$$

with $V_i \in \mathrm{add}(V)$. This provides us an exact sequence

$$0 \longrightarrow V_m \longrightarrow ... \longrightarrow V_2 \longrightarrow T_M(Y_1) \stackrel{T_M(f)}{\longrightarrow} T_M(Y_0).$$

Let X be the kernel of $T_M(f)$. Then we have an exact sequence $0 \longrightarrow V_m \longrightarrow ... \longrightarrow V_2 \longrightarrow X \longrightarrow 0$. Since the canonical surjection from V_2 to X is in fact a right $\mathrm{add}(V)$-approximation of X, the sequence $0 \longrightarrow (, V_m) \longrightarrow ... \longrightarrow (, V_2) \longrightarrow (, X) \longrightarrow 0$ is exact on $\mathrm{add}(V)$. From this and (3) we have another exact sequence

$$(*) \quad 0 \longrightarrow T_N(V_m) \longrightarrow ... \longrightarrow T_N(V_2) \longrightarrow T_N(X) \longrightarrow 0$$

such that $0 \longrightarrow (, T_N(V_m)) \longrightarrow ... \longrightarrow (, T_N(V_2)) \longrightarrow (, T_N(X)) \longrightarrow 0$ is exact on $\mathrm{add}(T_N(V))$. Note that $T_N(X) = \mathrm{Ker}(T_N T_M(f)) = \mathrm{Ker}(f) \oplus \mathrm{Ker}(T_Q(f))$. It follows that there is a mimimal projective resolution

$$0 \longrightarrow (, Y_m) \longrightarrow ... \longrightarrow (, Y_2) \longrightarrow (, \mathrm{Ker}(f)) \longrightarrow 0$$

of the functor $(, \mathrm{Ker}(f))$ with $Y_i \in \mathrm{add}(T_N(V))$. This yields the following exact sequence

$$0 \longrightarrow (, Y_m) \longrightarrow ... \longrightarrow (, Y_2) \longrightarrow (, Y_1) \longrightarrow (, Y_0) \longrightarrow F \longrightarrow 0$$

of functors on $\text{add}(T_N(V))$. So the global dimension of $\text{add}\widehat{(T_N(V))}$ is at most m. By 2.4 and the definition of the representation dimension, we have $\text{rep.dim}(B) \leqslant m$, that is, $\text{rep.dim}(B) \leqslant \text{rep.dim}(A)$. Similarly, we have $\text{rep.dim}(A) \leqslant \text{rep.dim}(B)$. Thus $\text{rep.dim}(A) = \text{rep.dim}(B)$.

Now let us deduce some consequences of the above result.

Recall that two algebras A and B are called **derived equivalent** if the bounded derived categories of A-mod and B-mod are equivalent as triangular categories. By [15], there is a stable equivalence of Morita type between self-injective algebras if they are derived equivalent. Thus we have the following consequence of 4.1.

Corollary 4.2 *Let A and B be self-injective algebras. If they are derived equivalent, then they has the same representation dimension.*

Recall that given a finite dimensional k-algebra A the trivial extension $T(A)$ of A is defined to be $A \oplus D(A)$ (as a vector space) with the multiplication:

$$(a, f)(b, g) = (ab, ag + fb) \text{ for } a, b \in A; f, g \in D(A).$$

It is known that $T(A)$ is always a symmetric algebra for any algebra A, thus it is self-injective.

Corollary 4.3 *If two algebras A and B are derived equivalent, then rep.dim $T(A) =$ rep.dim $T(B)$. In particular, If B is an endomorphism algebra of a tilting A-module, then rep.dim $T(A) =$ rep.dim $T(B)$.*

Proof. By [16], if A and B are derived equivalent, then the trivial extensions $T(A)$ and $T(B)$ are also derived equivalent. Since $T(A)$ is self-injective, we have that $\text{rep.dim}T(A) = \text{rep.dim}T(B)$ by 4.2.

Remarks (1) It is easy to find two algebras which are not Morita equivalent, but there is a stable equivalence of Morita type between them. For example, the trivial extension of the path algebra A of the quiver $\circ \xrightarrow{\alpha} \circ \xrightarrow{\beta} \circ$ is not Morita equivalent to the trivial extension of the algebra $B := A/(\alpha\beta)$, but there is a stable equivalence of Morita type between $T(A)$ and $T(B)$ since B is tilted from A.

(2) Representation dimension is not invariant under derived equivalences. For instance, we take a tame hereditary algebra A and a tilting module T which contains both indecomposable preprojective modules and preinjective modules as direct summands, then the endomorphism algebra B of T is clearly derived equivalent to A, but we have $\text{rep.dim}A = 3$ and $\text{rep.dim}B = 2$ by Lemma 2.3. Hence the self-injectivity in 4.2 is necessary.

(3) If an Artin algebra A is stably equivalent to a self-injecitve algebra B, then the algebra A itself is a direct sum of self-injective algebras and Nakayama algebras, this is proved in [14]. Thus, by Lemma 2.3, the representation dimension of A is finite. However, it is not known whether A and B have the same representation dimension.

5 Representation dimensions of self-injective algebras and their factor algebras

If A is a finite dimensional algebra over a field k, then, as we have known, the trivial extension $T(A)$ of A is a self-injective algebra with $T(A)/I \simeq A$ for an ideal I in $T(A)$, where $I^2 = 0$. Thus, to know whether A has finite representation dimension, it is useful to consider the relationship of the representation dimensions between A and its factor algebra A/I with $I^2 = 0$.

In this section we have the following result in this direction.

Theorem 5.1 *Let A be a self-injective algebra, and let n be the nilpotency index of the Jacobson radical N of A. If I is an ideal in A with $IN = 0$ (for example, an ideal contained in N^{n-1}), then rep.dim$(A) \leqslant$ rep.dim$(A/I) + 3$.*

Proof. We may assume that rep.dim$(A/I) = m < \infty$. Otherwise there is nothing to prove. Let V_0 be an A/I-module such that rep.dim$(A/I) =$ gl.dim $\mathrm{End}_{A/I}(V_0)$. Put $V = V_0 \oplus A$. We prove that for each A-module M there is an exact sequence

$$0 \longrightarrow M_{m+1} \longrightarrow ... \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

with all $M_i \in \mathrm{add}(V)$ such that the induced sequence

$$0 \longrightarrow (X, M_{m+1}) \longrightarrow ... \longrightarrow (X, M_1) \longrightarrow (X, M_0) \longrightarrow (X, M) \longrightarrow 0$$

is exact for all $X \in \mathrm{add}(V)$

If $M \in \mathrm{add}(V)$, then we simply define $M_0 = M$ and the identity map $M_0 \to M$ and we get a desired sequence.

Suppose M is not in $\mathrm{add}(V)$. If M is an A/I-module, then we have a minimal projective resolution for $(, M)$:

$$0 \longrightarrow (, M_m) \longrightarrow ... \longrightarrow (, M_1) \longrightarrow (, M_0) \longrightarrow (, M) \longrightarrow 0$$

with all $M_i \in \mathrm{add}(V_0)$. Since $A/I \in \mathrm{add}(V_0)$, we have an exact sequence of A/I-modules

$$0 \longrightarrow M_m \longrightarrow ... \longrightarrow M_1 \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

Clearly, this is a desired sequence since ${}_AA$ is a projective module. Now suppose M is not an A/I-module. We take $M' = \{m \in M \mid Im = 0\}$. Then M' is an A/I-module. Suppose that $l : M_0 \longrightarrow M'$ is a right minimal $\mathrm{add}(V_0)$-approximation of M' and $g : P \longrightarrow M/M'$ a projective cover of A-modules and $h : P \longrightarrow M$ a lifting such that $g = h\pi$, where π is the canonical homomorphism $M \longrightarrow M/M'$. Define $f : M_0 \oplus P \longrightarrow M$ by $(x, p) \mapsto l(x) + (p)h$. If $(x, p) \in \mathrm{Ker}(f)$ with $x \in M_0$ and $p \in P$ then $p \in \mathrm{Ker}(g)$. Since P is a projective cover, we have $\mathrm{Ker}(g) \subset NP$. Hence $\mathrm{Ker}(f)$ is an A/I-module because $I(x, p) \subset (Ix, I\mathrm{Ker}(g)) \subset (0, INP) = 0$. Now we show that for any $X \in \mathrm{add}(V)$ the induced map $(X, M_0 \oplus P) \longrightarrow (X, M)$ is surjective. If $X \in \mathrm{add}(V_0)$ then the image of any homomorphism from X to M is an A/I-module, thus lies in M'. This implies that the induced map is surjective. If X is a projective A-module, then there is nothing to show. So we have proved that for all $X \in \mathrm{add}(V)$ the induced map is surjective.

Now by the previous result we have an exact sequence

$$0 \longrightarrow M_{m+1} \longrightarrow ... \longrightarrow M_1 \longrightarrow \mathrm{Ker}(f) \longrightarrow 0$$

with all $M_i \in \mathrm{add}(V_0) \subset \mathrm{add}(V)$ such that for all $X \in \mathrm{add}(V)$ the sequence

$$0 \longrightarrow (X, M_{m+1}) \longrightarrow ... \longrightarrow (X, M_1) \longrightarrow (X, \mathrm{Ker}(f)) \longrightarrow 0$$

is exact. Hence the exact sequence

$$0 \longrightarrow (X, M_{m+1}) \longrightarrow ... \longrightarrow (X, M_1) \longrightarrow (X, M_0 \oplus P) \longrightarrow (X, M) \longrightarrow 0$$

for all $X \in \mathrm{add}(V)$, where all X_i are in $\mathrm{add}(V)$.

Now we establish that gl.dim $(\widehat{\mathrm{add}(V)}) \leq m+3$. Take a functor G in $\widehat{\mathrm{add}(V)}$. Then there is a morphism $f : M_1 \longrightarrow M_0$ in add(V) such that $(, M_1) \longrightarrow (, M_0) \longrightarrow G \longrightarrow 0$ is exact. Letting $M = \mathrm{Ker}(f)$, we know that there is an exact sequence

$$0 \longrightarrow X_{m+1} \longrightarrow ... \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

with all $X_i \in \mathrm{add}(V)$ such that the induced sequence

$$0 \longrightarrow (X, X_{m+1})... \longrightarrow (X, X_0) \longrightarrow (X, M) \longrightarrow 0$$

is exact for all $X \in \mathrm{add}(V)$. Thus the sequence

$$0 \longrightarrow (, X_{m+1}) \longrightarrow ... \longrightarrow (, X_1) \longrightarrow (, X_0) \longrightarrow (, M_1) \longrightarrow (, M_0) \longrightarrow G \longrightarrow 0$$

is exact in add(V). This shows that proj.dim $(G) \leqslant m+3$. By 2.4, we get gl.dim $(\mathrm{End}_A(V)) \leqslant m+3 = $ rep.dim $(A/I)+3$. Thus the proof is completed.

6 Two conjectures

The results in the previous sections support that we may make the following conjecture.

Conjecture 1 Let A be an Artin algebra. Then its representation dimension is finite.

The other evidence of making this conjecture is the following conjecture of Ringel:

Conjecture 2 Let A be a finite dimensional algebra over a field k. Then for every A-module M in A-mod there is a module M' such that the endomorphism algebra of $M \oplus M'$ is quasi-hereditary.

Recall from [8] that an ideal J in a finite dimensional algebra A is called a **heredity ideal** if (1) $J^2 = J$, (2) $J\mathrm{rad}(A)J = 0$, and (3) ${}_AJ$ is a projective A-module. The algebra A is said to be **quasi-hereditary** if there is a finite chain $J_0 = 0 \subseteq J_1 \subseteq J_2 \subseteq ... \subseteq J_n = A$ of ideals such that for each i the ideal J_i/J_{i-1} is a heredity ideal in A/J_{i-1}.

Quasi-hereditary algebras have finite global dimensions. Typical examples of quasi-hereditary algebras are Schur algebras, Brauer algebras and Birman-Wenzl algebras for the most choices of parameters (see [21]).

Let us introduce the following notion.

Definition 6.1 *Given a module M. A module M' is called a quasi-heredity complement to M if the endomorphism algebra of $M \oplus M'$ is quasi-hereditary.*

Conjecture 2 is true for a semilocal module M (see [12]). Recall that a module is called local if it has a simple top, and is called colocal if it has a simple socle. A module is called semilocal (or semicolocal) if it is a direct sum of local (or colocal) modules.

Lemma 6.2 *(1) If M is a semilocal module, then there is a module M' such that the endomorphism algebra of $M \oplus M'$ is quasi-hereditary. Thus each semilocal module has a quasi-heredity complement.*

(2) If M is a semicolocal module, then there is a module N' such that the endomorphism algebra of $M \oplus N'$ is quasi-hereditary. Thus each semicolocal module has a quasi-heredity complement.

The statement (1) was proved in [12]. (2) is a dual statement of (1). For the convenience of the reader we include here a proof. Let M be a semicolocal module. Put $X_A = D(M)$. Then, by the right-version of the statement (1), we can find a module Y_A such that $\mathrm{End}(X_A \oplus Y_A)$ and $\mathrm{End}(X_A \oplus Y_A)^{op}$ are quasi-hereditary. Since D is a duality, we have that $\mathrm{End}({}_AM \oplus D(Y_A)) \cong \mathrm{End}(D(X_A \oplus Y_A)) \cong \mathrm{End}(X_A \oplus Y_A)^{op}$ is quasi-hereditary.

Note that the conjecture 2 is also true for a Δ-good module over an $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra (see [20]).

The relationship of the two conjectures is the following trivial lemma.

Lemma 6.3 *If Conjecture 2 is true, then so is Conjecture 1.*

Proof. We take M to be the module $A \oplus D(A)$. Then there is a module M' such that $E = \mathrm{End}_A(M \oplus M')$ is quasi-hereditary by Conjecture 2. Since quasi-hereditary algebras have finite global dimension, the algebra E has finite global dimension. Then by definition we have that $\mathrm{rep.dim}(A) < \infty$ since $M \oplus M'$ is a generator and cogenerator for the A-module category. Thus Conjecture 1 holds true.

In fact, in order to confirm the conjecture 1, it is sufficient to find a quasi-heredity complement only for the module $M = A \oplus D(A)$.

Proposition 6.4 *Let A be an Artin algebra. If every indecomposable injective module is local, then the representation dimension of A is finite. Dually, if each indecomposable projective module is colocal, then the algebra A has finite representation dimension.*

Proof. We prove only the first statement. By assumption, $D(A)$ is a semilocal module, hence the module $M = A \oplus D(A)$ is semilocal. By (1) of Lemma 6.2, there is a module M' such that the endomorphism algebra of M is quasi-hereditary. This implies that the representation dimension of A is finite by the proof of Lemma 6.3.

As an application of Proposition 6.4, we have the following result.

Proposition 6.5 *Let A be an Artin algebra. If $\mathcal{X}$ be a finite family of colocal A-modules such that it contains each indecomposable projective module as well as each indecomposable injective module, then the endomorphism algebra of the direct sum of all modules in $\mathcal{X}$ has finite representation dimension.*

In particular, if A is a self-injective Artin algebra, and if X is a semi-colocal A-module, then the representation dimension of End$({}_AA \oplus_A X)$ is finite.

Proof. Let M denote the direct sum of all modules in $\mathcal{X}$ and Λ the endomorphism algebra of M. To prove that Λ has a finite representation dimension, we show that each indecomposable projective Λ-module has a simple socle. For this we take an indecomposable A-module X in $\mathcal{X}$ and its injective envelope $I(X)$. Then the Λ-module $\mathrm{Hom}_A(M, X)$ is a Λ-submodule of $\mathrm{Hom}_A(M, I(X))$. If $\mathrm{Hom}_A(M, I(X))$ is a colocal Λ-module, then so is the Λ-module $\mathrm{Hom}_A(M, X)$. But this is clearly true since $\mathrm{Hom}_A(M, I(X))$ is a projective-injective Λ-module by [2, Proposition 8.3]. Thus the representation dimension of Λ is finite by 6.4.

As an easy consequence we have the following fact.

Corollary 6.6 *Let A be an Artin algebra. If each indecomposable A-module is colocal, then the Auslander algebra of A has finite representation dimension. In particular, the Auslander algebra of a Nakayama algebra has finite representation dimension.*

Recall that given a representation finite Artin algebra A, the Auslander algebra of A is by definition the endomorphism algebra of the direct sum of all non-isomorphic indecomposable A-modules. An Artin algebra is called a Nakayama algebra if each indecomposable projective module as well as injective module has a unique composition series. Note that the Auslander algebra of a Nakayama algebra is not necessarily of representation finite type.

In fact, Proposition 6.4 provides a large class of algebras with finite representation dimension. For example, we can apply it also to certain incidence algebras. Recall that for a given finite partially ordered set S and a field k the incidence algebra kS over k is defined as a quotient of the path algebra kQ of the quiver Q modulo all commutative relations, where the quiver Q has the vertex set S, and for two vertices a and b there is an arrow from a to b if $a > b$ and there is no element c such that $a > c > b$.

Corollary 6.7 *Let S be a finite partially ordered set with a greatest element. Then for any field k, the incidence algebra of S over k has finite representation dimension. Dually, if S has a unique minimal element, then the incidence algebra over any field has finite representation dimension.*

Proof. Since rep.dim (A) = rep.dim (A^{op}), the second statement follows from the first one. For the first, one only need to know that the existence of the greatest element in S implies that each indecomposable injective module of the incidence algebra has a simple top, thus the corollary follows from 6.4.

Finally, Recall from [9] that given a commutative local self-injective algebra R over a field k one may construct quasi-hereditary algebras in the following way:

Let Λ be a poset of cardinality $n=$ dim R. For each $\lambda \in \Lambda$, assume that there exists a local ideal X_λ in R such that $X_\lambda \subset X_\mu$ if and only if $\lambda \leqslant \mu$. Put $X = \oplus_\lambda X_\lambda$, and let $A=$ $\mathrm{End}_R(X)$. Then it is shown that A is quasi-hereditary if and only if $\mathrm{rad}(X_\lambda) = \sum_{\nu>\lambda} X_\nu$ for all λ.

From Proposition 6.5 the following statement follows.

Corollary 6.8 *The quasi-hereditary algebra A constructed above has finite representation dimension.*

7 Incidence algebras

In this section we shall prove the following more general result on incidence algebras.

Theorem 7.1 *Let S be a finite partially ordered set and k an arbitrary field. Then the incidence algebra kS has finite representation dimension.*

Before we start to prove Thoerem 7.1, let us first recall the definition of directed algebra and then prove a key lemma on incidence algebras.

Let A be an algebra and $\mathcal{X}$ an additive full subcategory of A-mod. A path in $\mathcal{X}$ is a sequence of non-zero, non-isomorphic homomorphisms

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X_n$$

between indecomposable modules X_i in $\mathcal{X}$. A cycle in $\mathcal{X}$ is a path in $\mathcal{X}$ with $X_1 = X_n$. An algebra is called directed if the additive subcategory of projective modules is directed. For example, the incidence algebra of a finite partially ordered set is directed. Moreover, the following property holds true for incidence algebras.

Lemma 7.2 *Let S be a finite poset and $a, b \in S$. Let $Q(a)$ denote the indecomposable injective module corresponding to a, and let $P(b)$ denote the indecomposable projective module corresponding to b. If $Hom_{kS}(Q(a), P(b)) \neq 0$, then $Q(a) \cong P(b)$.*

Proof. Note that if $L(x)$ is a composition factor of $P(b)$ then $x \leq b$, and if it is the case then $[P(b) : L(x)] = 1$. The socle of $P(b)$ is isomorphic to a direct sum of simple modules which are of the form $L(x)$ with x a minimal element in S. If f is a non-zero homomorphism from $Q(a)$ to $P(b)$, then the restriction of f to the socle of $Q(a)$ is either zero or not zero. If it is not zero, then f is injective. Thus $Q(a) \cong P(a)$ since $Q(a)$ is injective and $P(a)$ is indecomposable. To prove the lemma, we need to exclude the case that f restricted to the socle of $Q(a)$ is zero. Suppose it is the case. Then the image of f in $P(b)$ is a submodule of $P(b)$. If U is a simple submodule of the socle of $\mathrm{Im}(f)$, then $U \cong L(x)$ for some minimal element x with $x \leq b$. This means that $Q(a)/\mathrm{Ker}(f)$ has a composition factor $L(x)$ with $x \neq a$. Thus $Q(a)$ has a composition factor isomorphic to $L(x)$ with $x > a$. This contradicts the minimality of x in S and therefore the restriction of f to the socle of $Q(a)$ is not zero. This finishes the proof of the lemma.

Proof of Theorem 7.1: Let Q be the direct sum of all non-isomorphic, non-projective indecomposable injective kS-modules. We consider the module $M := kS \oplus Q$ and its endomorphism algebra. By Lemma 7.2, $\mathrm{Hom}_{kS}(Q, kS) = 0$. Thus the endomorphism algebra of M is of the following form

$$\mathrm{End}_{kS}(M) \cong \begin{pmatrix} kS & Q \\ 0 & \mathrm{End}_{kS}(Q) \end{pmatrix}.$$

Since kS is a directed algebra, $\mathrm{End}_{kS}(Q)$ is also a directed algebra. Thus both kS and $\mathrm{End}_{kS}(Q)$ have finite global dimension. Hence $\mathrm{End}_{kS}(M)$ has finite global dimension by [13, p. 246]. Since M is a generator-cogenerator , the representation dimension of kS is finite by definition. Thus the proof of the theorem is completed.

Remark. The proof of Theorem 7.1 shows also that if an algebra is given by a quiver with only commutative relations (i.e. any two paths with the same starting and terminal vertex are equal) and if it is also a directed algebra, then its representation dimension is finite. Note that here we allow multiple arrows to occur in the quiver.

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