COHERENT FUNCTORS AND COVARIANTLY FINITE SUBCATEGORIES

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Let $\mathcal{A}$ be an additive category and suppose that every map in $\mathcal{A}$ has a cokernel. Following Auslander [2], a functor $F\colon \mathcal{A} \to \mathrm{Ab}$ into the category of abelian groups is *coherent* if there exists an exact sequence

$$(Y,-) \longrightarrow (X,-) \longrightarrow F \longrightarrow 0,$$

where (X,A) denotes the maps from X to A. The coherent functors form an abelian category which we denote by $\widehat{\mathcal{A}}$. Next we recall from [3] that a full subcategory $\mathcal{B}$ of $\mathcal{A}$ is *covariantly finite* if for every object $X \in \mathcal{A}$ there exists a map $X \to Y$ such that $Y \in \mathcal{B}$ and for every $Y' \in \mathcal{B}$ the induced map $(Y,Y') \to (X,Y')$ is surjective. In addition, we require that every direct factor of an object in $\mathcal{B}$ belongs to $\mathcal{B}$. We are interested in a classification of all covariantly finite subcategories. The following observation leads to a reformulation of this problem in terms of colocalizing subcategories. Recall from [7] that a full subcategory $\mathcal{S}$ of an abelian category $\mathcal{C}$ is a *Serre subcategory* if it is closed under subobjects, quotients, and extensions. A Serre subcategory $\mathcal{S}$ is *colocalizing* if the corresponding quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{S}$ has a left adjoint.

Proposition 1. *Let $\mathcal{A}$ be an additive category with cokernels. Then there exists a bijection between the class of covariantly finite subcategories of $\mathcal{A}$ and the class of colocalizing subcategories of $\widehat{\mathcal{A}}$. This bijection sends a covariantly finite subcategory $\mathcal{B}$ to $\{F \in \widehat{\mathcal{A}} \mid F|_{\mathcal{B}} = 0\}$ and a colocalizing subcategory $\mathcal{S}$ to $\{X \in \mathcal{A} \mid FX = 0$ for all $F \in \mathcal{S}\}$.*

In this paper we study certain covariantly finite subcategories of an additive category which is locally presentable in the sense of Gabriel and Ulmer [8]. Every module category is locally presentable, and the covariantly finite subcategories which arise in practice have the additional property that the corresponding colocalizing subcategory of coherent functors is generated by a single object. For example, recent work of Eklof and Trlifaj [6] which led to a proof of Enochs' Flat Cover Conjecture, is based on an analysis of such subcategories. The following result provides a characterization.

Theorem 2. *Let $\mathcal{A}$ be a locally presentable additive category. Then the following are equivalent for a full subcategory $\mathcal{B}$:*

1. *$\mathcal{B}$ is covariantly finite and there exists a functor $F\colon \mathcal{A} \to \mathrm{Ab}$ such that the coherent functors vanishing on $\mathcal{B}$ form the smallest colocalizing subcategory of $\widehat{\mathcal{A}}$ containing F.*
2. *There exists a set of coherent functors $F_i\colon \mathcal{A} \to \mathrm{Ab}$ such that $X \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if $F_iX = 0$ for all i.*
3. *There exists a set of sequences $U_i \to V_i \to W_i$ in $\mathcal{A}$ such that $X \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if the induced sequence $(W_i,X) \to (V_i,X) \to (U_i,X)$ is exact for all i.*
4. *There exists a regular cardinal α such that $\mathcal{B}$ is closed under taking products, α-filtered colomits, and α-pure subobjects.*

We follow Crawley-Boevey and call a subcategory *definable* if it satisfies the equivalent conditions of the preceding theorem. In [5], Crawley-Boevey used some model theory to prove a special case of Theorem 2 for module categories (the equivalence of (2) and (4) for $\alpha = \aleph_0$).

Next we ask for a classification of the definable subcategories of a locally presentable additive category $\mathcal{A}$. To this end denote for every regular cardinal α by $\mathcal{A}_\alpha$ the full subcategory of α-presentable objects in $\mathcal{A}$. This subcategory is closed under taking cokernels and α-coproducts (that is, coproducts of less than α factors). The inclusion $i\colon \mathcal{A}_\alpha \to \mathcal{A}$ extends to an exact functor $i^*\colon \widehat{\mathcal{A}_\alpha} \to \widehat{\mathcal{A}}$ sending $(X,-)$ to $(iX,-)$ and we write $\bar{F} = i^*F$ for each $F \in \widehat{\mathcal{A}_\alpha}$. The category $\widehat{\mathcal{A}_\alpha}$ has α-products and we denote by Σ_α the set of all Serre subcategories $\mathcal{S}$ of $\widehat{\mathcal{A}_\alpha}$ which are closed under α-products and admit a faithful exact and α-product preserving functor $\widehat{\mathcal{A}_\alpha}/\mathcal{S} \to \mathrm{Ab}$. Let $\Sigma = \bigcup_\alpha \Sigma_\alpha$ and define $\mathcal{S}_1 \sim \mathcal{S}_2$ for $\mathcal{S}_1 \in \Sigma_{\alpha_1}$ and $\mathcal{S}_2 \in \Sigma_{\alpha_2}$ with $\alpha_2 \geq \alpha_1$ if $\mathcal{S}_2$ is the smallest Serre subcategory in Σ_{α_2} which contains $\bar{F}|_{\mathcal{A}_{\alpha_2}}$ for all $F \in \mathcal{S}_1$.

Theorem 3. *Let $\mathcal{A}$ be a locally presentable additive category. Then there exists a bijection between $\Sigma/{\sim}$ and the class of definable subcategories of $\mathcal{A}$. It sends $\mathcal{S} \in \Sigma$ to $\{X \in \mathcal{A} \mid \bar{F}X = 0$ for all $F \in \mathcal{S}\}$.*

The proof of both theorems is based on a filtration of the category $D(\mathcal{A}) = \widehat{\mathcal{A}}^{\mathrm{op}}$ which seems to be of independent interest. In fact, a similar project has been carried out for triangulated categories by Neeman [10]. We may think of the Yoneda functor

$$\mathcal{A} \longrightarrow D(\mathcal{A}), \quad X \mapsto (X,-),$$

as the universal colimit preserving functor to a cocomplete abelian category with exact coproducts since any other functor $\mathcal{A} \to \mathcal{A}'$ with this property has a unique factorization

$$\mathcal{A} \longrightarrow D(\mathcal{A}) \xrightarrow{F} \mathcal{A}'$$

such that F is exact and preserves coproducts. However, the category $D(\mathcal{A})$ is too big. For instance, a complete analysis of all localizing subcategories involves certain set-theoretic problems; it is not possible within the usual axiomatic framework of Gödel-Bernays (see the remark at the end of Section 2). Therefore we introduce a filtration of $D(\mathcal{A})$ as follows.

Theorem 4. *Let $\mathcal{A}$ be a locally α-presentable additive category. Then the Yoneda functor $\mathcal{A} \to D(\mathcal{A})$ has a factorization*

$$\mathcal{A} \longrightarrow D_\alpha(\mathcal{A}) \longrightarrow D_{\alpha+1}(\alpha) \longrightarrow \cdots \longrightarrow D_\beta(\mathcal{A}) \longrightarrow \cdots \longrightarrow D(\mathcal{A})$$

where β runs through all regular cardinals ($\beta \geq \alpha$) and
1. $D(\mathcal{A}) = \bigcup_{\beta \geq \alpha} D_\beta(\mathcal{A})$;
2. *each $D_\beta(\mathcal{A})$ is a locally β-presentable abelian category with exact coproducts and exact β-filtered colimits;*
3. *each inclusion $D_\beta(\mathcal{A}) \to D(\mathcal{A})$ has an exact and coproduct preserving right adjoint $G_\beta\colon D(\mathcal{A}) \to D_\beta(\mathcal{A})$ which induces an equivalence $D(\mathcal{A})/\operatorname{Ker} G_\beta \to D_\beta(\mathcal{A})$.*

Note that the construction of $D_\beta(\mathcal{A})$ is classical for $\beta = \alpha = \aleph_0$. It goes back to Gruson and Jensen [9] for module categories, and was later extended by Simson [11] and Crawley-Boevey [4]. We hope to convince the reader that the case $\beta > \aleph_0$ is of some interest as well.

This article is organized as follows. In Section 1 we provide background material about locally presentable categories and functor categories. In Section 2 we study the filtration of $D(\mathcal{A})$. In Section 3 we characterize coherent functors $\mathcal{A} \to \mathrm{Ab}$. In Section 4 we describe covariantly finite subcategories in terms of colocalizing subcategories and prove a general criterion for a subcategory to be covariantly finite. In Section 5 we study definable subcategories and provide proofs for the main results of this paper. In the final Section 6 we explain how to make a functor exact, a problem which arises in the classification of definable subcategories.

1. Functor categories

Locally presentable categories. We recall from [8] some basic facts about locally presentable categories. Let $\mathcal{A}$ be a cocomplete additive category and α a regular cardinal (that is, α is not the sum of fewer than α cardinals, all smaller than α). A colimit of a functor $\mathcal{I} \to \mathcal{A}$ is *α-filtered* if

(F1) the isoclasses of objects in $\mathcal{I}$ form a set,
(F2) for every set I of less than α objects in $\mathcal{I}$ there exists an object $j \in \mathcal{I}$ and a map $i \to j$ for all $i \in I$, and
(F3) for every set Σ of less than α parallel maps $i \to j$ in $\mathcal{I}$ there exists a map $\tau\colon j \to k$ such that $\tau \circ \sigma_1 = \tau \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in \Sigma$.

An object $X \in \mathcal{A}$ is *α-presentable* if $(X, -)$ preserves α-filtered colomits and we denote by $\mathcal{A}_\alpha$ the full subcategory of α-presentable objects. The category $\mathcal{A}$ is *locally α-presentable* if the isoclasses of α-presentable objects form a set and every object in $\mathcal{A}$ is an α-filtered colimit of α-presentable objects. Note that a locally α-presentable category is locally β-presentable for every regular cardinal $\beta \geq \alpha$ and that $\mathcal{A} = \bigcup_\beta \mathcal{A}_\beta$. Finally, $\mathcal{A}$ is *locally presentable* if $\mathcal{A}$ is locally α-presentable for some regular cardinal α.

For example, $\aleph_0$-filtered colimits are precisely the usual filtered colimits. In a module category the $\aleph_0$-presentable objects are precisely the finitely presented modules. Therefore every module category is locally $\aleph_0$-presentable.

Left exact functors. We fix a regular cardinal α and a small additive category $\mathcal{C}$. Suppose that $\mathcal{C}$ has kernels and α-products (that is, products of less than α factors). A functor $F\colon \mathcal{C} \to \mathrm{Ab}$ is called *α-left exact* if it is left exact and preserves α-products. We denote by $(\mathcal{C}, \mathrm{Ab})$ the category of additive functors $\mathcal{C} \to \mathrm{Ab}$ and $\mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$ denotes the full subcategory of α-left exact functors.

Lemma 1.1. *Let $\mathcal{C}$ be a small additive category with kernels and α-products.*

(1) *Every α-filtered colimit of α-left exact functors $\mathcal{C} \to \mathrm{Ab}$ is again α-left exact.*
(2) *Every α-left exact functor $\mathcal{C} \to \mathrm{Ab}$ is an α-filtered colimit of representable functors.*
(3) *The inclusion $\mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab}) \to (\mathcal{C}, \mathrm{Ab})$ has a left adjoint $L\colon (\mathcal{C}, \mathrm{Ab}) \to \mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$.*

Proof. See Section 5 in [8]. □

Lemma 1.2. *Let $\mathcal{C}$ be a small additive category with kernels and α-products. Then $\mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$ is a locally α-presentable category and the representable functors are, up to isomorphism, precisely the α-presentable objects.*

Proof. We apply Lemma 1.1. The category $\mathcal{A} = \mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$ is cocomplete. In fact, colimits are first computed in $(\mathcal{C}, \mathrm{Ab})$, and then one applies the left adjoint of the inclusion $\mathcal{A} \to (\mathcal{C}, \mathrm{Ab})$. The representable functors are α-presentable since α-filtered colimits are computed in $(\mathcal{C}, \mathrm{Ab})$. Moreover, every α-left exact functor $\mathcal{C} \to \mathrm{Ab}$ is an α-filtered colimit of representable functors. Writing an arbitrary α-presentable object X as α-filtered colimit of representable functors $(C_i, -)$, one sees that X is a direct factor of $(C_i,-)$ for some i, and therefore isomorphic to $(C,-)$ for some direct factor C of C_i. $\square$

Next observe that the category $\mathcal{A}_\alpha$ is α-cocomplete since the α-presentable objects are closed under taking cokernels and α-coproducts in $\mathcal{A}$.

Lemma 1.3. *Let $\mathcal{A}$ be a locally α-presentable additive category. Then the functor*

$$\mathcal{A} \longrightarrow \operatorname{Lex}_\alpha(\mathcal{A}_\alpha^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto (-,X)|_{\mathcal{A}_\alpha},$$

is an equivalence.

Proof. Korollar 7.9 in [8]. $\square$

Exact functors. We fix a regular cardinal α and a small abelian category $\mathcal{C}$ with α-products. A functor $F\colon \mathcal{C} \to \mathrm{Ab}$ is called *α-exact* if it is exact and preserves α-products. The category of α-exact functors is denoted by $\operatorname{Ex}_\alpha(\mathcal{C}, \mathrm{Ab})$.

Lemma 1.4. *Let $\mathcal{C}$ be a small abelian category with α-products and let $\mathcal{A} = \operatorname{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$.*

1. *$\mathcal{A}$ is a cocomplete abelian category and the functor $\mathcal{C} \to \mathcal{A}$, $X \mapsto (X,-)$, is exact.*
2. *Every α-filtered colimit of exact sequences in $\mathcal{A}$ is again exact.*
3. *$F \in \mathcal{A}$ is an exact functor if and only if $\operatorname{Ext}^1((X,-),F) = 0$ for all $X \in \mathcal{C}$.*
4. *If α-products in $\mathcal{C}$ are exact, then coproducts in $\mathcal{A}$ are exact.*

Proof. (1), (2) The category $\mathcal{A}$ is complete and cocomplete. In fact, limits in $\mathcal{A}$ can be computed in $(\mathcal{C}, \mathrm{Ab})$, and colimits are first computed in $(\mathcal{C}, \mathrm{Ab})$, and then one applies the left adjoint of the inclusion $\mathcal{A} \to (\mathcal{C}, \mathrm{Ab})$. Given an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}$, the induced sequence $0 \to (Z,-) \to (Y,-) \to (X,-)$ is automatically exact. The last map is an epimorphism in $\mathcal{A}$ since $((X,-),F) \to ((Y,-),F)$ is a monomorphism for every left exact $F\colon \mathcal{C} \to \mathrm{Ab}$ by Yoneda's lemma. Therefore the Yoneda embedding $\mathcal{C} \to \mathcal{A}$ is exact. Next observe that an α-filtered colimit of left exact sequences is again left exact by Lemma 1.1. To show that $\mathcal{A}$ is abelian, one needs to check that for every map $\phi\colon X \to Y$ in $\mathcal{A}$ the canonical map

$$\operatorname{Coker}(\operatorname{Ker}\phi \to X) \longrightarrow \operatorname{Ker}(Y \to \operatorname{Coker}\phi)$$

is an isomorphism. This is clear if X and Y are representable functors since $\mathcal{C}$ is abelian. Now write ϕ as α-filtered colimit of maps between representable functors, using Lemma 1.1. The claim follows since taking kernels and cokernels commutes with taking α-filtered colimits.

(3) Let $F \in \mathcal{A}$ and suppose first $\operatorname{Ext}^1((X,-),F) = 0$ for all $X \in \mathcal{C}$. Let $0 \to X \to Y \to Z \to 0$ be exact in $\mathcal{C}$ so that $0 \to (Z,-) \to (Y,-) \to (X,-) \to 0$ is exact in $\mathcal{A}$. Applying $(-,F)$ to this sequence shows that F is an exact functor. Now suppose that F is exact and let $\varepsilon\colon 0 \to F \to E \to (X,-) \to 0$ be an exact sequence in $\mathcal{A}$. Writing E as α-filtered colimit of representable functors $(Y_i,-)$, one finds some i such that the composite $(Y_i,-) \to E \to (X,-)$ is an epimorphism. Taking a cokernel $Y_i \to Z$ of the map $X \to Y_i$, we get the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & (Z,-) \arrow[r] \arrow[d] & (Y_i,-) \arrow[r] \arrow[d] & (X,-) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r] & E \arrow[r] & (X,-) \arrow[r] & 0
\end{tikzcd}$$

Applying again Yoneda's lemma, the exactness of F implies that the sequence ε splits.

(4) Let $(\varepsilon_i)_{i\in I}$ be a family of exact sequences $\varepsilon_i\colon 0 \to X_i \to Y_i \to Z_i \to 0$ in $\mathcal{A}$. We write each ε_i as α-filtered colimit $\varepsilon_i = \operatorname{colim}_{a_i\in\mathcal{J}_i} \varepsilon_{a_i}$ of exact sequences of representable

functors. This is possible since each map in $\mathcal{A}$ is an α-filtered colimit of maps between representable functors and α-filtered colimits are exact. Now let $\mathcal{I}$ be the category of families $(a_i)_{i\in K}$ of objects $a_i \in \mathcal{J}_i$ where $K \subseteq I$ runs through all subsets of less than α elements. For $a = (a_i)_{i\in K}$ and $b = (b_i)_{i\in L}$ let $(a, b) = \prod_{i\in K}(a_i, b_i)$ if $K \subseteq L$ and $(a, b) = \emptyset$ otherwise. The category $\mathcal{I}$ is again α-filtered and we have

$$\varepsilon = \coprod_{i\in I} \varepsilon_i \cong \operatorname{colim}_{(a_i)_{i\in K}}(\coprod_{i\in K} \varepsilon_{a_i}).$$

The assumption on $\mathcal{C}$ implies that $\coprod_{i\in K} \varepsilon_{a_i}$ is exact for all K, and we conclude that ε is exact. □

The next lemma requires some notation. The cardinality of a set X is denoted by $|X|$. Choose a representative set $\mathcal{C}_0$ of objects in $\mathcal{C}$ and let $|\mathcal{C}| = |\coprod_{C,D\in\mathcal{C}_0}(C, D)|$. For a functor $F : \mathcal{C} \to \mathrm{Ab}$ define $|F| = |\coprod_{C\in\mathcal{C}_0} FC|$. Given a cardinal κ, we define for every ordinal ν with $|\nu| < \alpha$ a new cardinal κ_ν by transfinite induction. First define

$$\kappa' = \inf\{|LF| \mid F \in (\mathcal{C}, \mathrm{Ab}), |F| \le \kappa\},$$

where L denotes the left adjoint of the inclusion $\mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab}) \to (\mathcal{C}, \mathrm{Ab})$. Now let $\kappa_0 = \kappa$ and $\kappa_{\nu+1} = (\kappa_\nu + |\mathcal{C}|)'$. For a limit ordinal λ let $\kappa_\lambda = \sum_{\nu<\lambda} \kappa_\nu$. Finally, let $\kappa_\alpha = \sum_{|\nu|<\alpha} \kappa_\nu$.

Lemma 1.5. *Let $\mathcal{C}$ be a small abelian category with α-products. Let $G : \mathcal{C} \to \mathrm{Ab}$ be α-exact, and $F \subseteq G$ be a subfunctor. Then there exists an α-exact subfunctor $F' \subseteq G$ containing F such that $|F'| \le |F|_\alpha$.*

Proof. We define by transfinite induction a sequence

$$F = F_0 \to E_0 \to G_0 \to F_1 \to \cdots \to F_\nu \to E_\nu \to G_\nu \to \cdots$$

of maps in $(\mathcal{C}, \mathrm{Ab})$ where ν runs through all ordinals with $|\nu| < \alpha$. To define E_ν from F_ν, choose for every exact sequence $U \xrightarrow{\phi} V \xrightarrow{\psi} W$ in $\mathcal{C}$ and every $y \in \operatorname{Ker} F_\nu\psi \setminus \operatorname{Im} F_\nu\phi$ some $x \in GU$ such that $G\phi(x) = y$, and denote by Γ_ν the set of all such elements x. Define $E_\nu = F_\nu + \sum_{x\in\Gamma_\nu} G_x$ where G_x denotes the image of the map $(U, -) \to G$ corresponding to $x \in GU$ under the Yoneda isomorphism. Let $G_\nu = LE_\nu$. The inclusion $E_\nu \to G$ factors through the canonical map $E_\nu \to G_\nu$ via some map $\gamma_\nu : G_\nu \to G$ and we define $F_{\nu+1} = \operatorname{Im}\gamma_\nu$. For a limit ordinal λ let $E_\lambda = F_\lambda = \operatorname{colim}_{\nu<\lambda} F_\nu$. Finally, define $F' = \operatorname{colim}_{|\nu|<\alpha} F_\nu$. This functor is exact by construction. It preserves α-products by Lemma 1.1 since $F' \cong \operatorname{colim}_{|\nu|<\alpha} G_\nu$ is an α-filtered colimit of α-left exact functors. We have $|F'| \le |F|_\alpha$ since $|F_\nu| \le |F|_\nu$ for all ν. □

Functors on locally presentable categories. We collect in two lemmas the basic properties of functors on locally presentable categories.

Lemma 1.6. *Let $\mathcal{C}$ be a small α-cocomplete additive category and fix the Yoneda functor $\mathcal{C} \to \mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Let $\mathcal{A}$ be an additive category with α-filtered colimits. Then every additive functor $f : \mathcal{C} \to \mathcal{A}$ has a (essentially) unique factorization*

$$\mathcal{C} \longrightarrow \mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{f'} \mathcal{A}$$

such that f' preserves α-filtered colimits. Moreover, f' preserves all colimits if and only if f preserves α-colimits.

Proof. It is clear how to define f'. Let $F \in \operatorname{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and write $F = \operatorname{colim}_i(-, X_i)$ as α-filtered colimit of representable functors. Then define $f'F = \operatorname{colim}_i fX_i$. We refer to [8, Satz 5.5] for details. ☐

Lemma 1.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be small α-cocomplete additive categories, and let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor which preserves α-colimits. Then the functor*

$$f_*\colon \operatorname{Lex}_\alpha(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}) \longrightarrow \operatorname{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad F \mapsto F \circ f,$$

has a left adjoint $f^*\colon \operatorname{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \operatorname{Lex}_\alpha(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ which sends $(-, X)$ to $(-, fX)$ for all $X \in \mathcal{C}$.

Proof. It is well-known that the functor

$$(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}) \longrightarrow (\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad F \mapsto F \circ f,$$

has a left adjoint $(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to (\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ which sends $(-, X)$ to $(-, fX)$ for all $X \in \mathcal{C}$. Both functors preserve α-left exactness, and therefore the assertion follows. ☐

2. A filtration of the category $D(\mathcal{A})$

The universal property of $D(\mathcal{A})$. Let $\mathcal{A}$ be a cocomplete additive category and denote by $D(\mathcal{A}) = \widehat{\mathcal{A}}^{\mathrm{op}}$ the opposite of the category of coherent functors $\mathcal{A} \to \mathrm{Ab}$. We may think of $D(\mathcal{A})$ as an abelian completion. To make this precise we need the following lemma.

Lemma 2.1. *Let $\mathcal{C}$ be an additive category with cokernels and fix the Yoneda functor $\mathcal{C} \to \widehat{\mathcal{C}}^{\mathrm{op}}$. Then any additive functor $f\colon \mathcal{C} \to \mathcal{D}$ into an abelian category $\mathcal{D}$ has a (essentially) unique factorization*

$$\mathcal{C} \longrightarrow \widehat{\mathcal{C}}^{\mathrm{op}} \xrightarrow{f'} \mathcal{D}$$

such that f' is left exact. Moreover, f' is exact if and only if f is right exact.

Proof. The category $\widehat{\mathcal{C}}$ is abelian since $\mathcal{C}$ has cokernels. Now let $f\colon \mathcal{C} \to \mathcal{D}$ be an additive functor into an abelian category. We obtain a left exact functor $f'\colon \widehat{\mathcal{C}}^{\mathrm{op}} \to \mathcal{D}$ if we send $G \in \widehat{\mathcal{C}}$ with presentation

$$(Y, -) \xrightarrow{(\phi, -)} (X, -) \longrightarrow G \longrightarrow 0$$

to the kernel of $f\phi$. Using the snake lemma, one shows that f' is right exact if and only if f is right exact. ☐

The universal property of $D(\mathcal{A})$ is a direct consequence.

Proposition 2.2. *Let $\mathcal{A}$ be a cocomplete additive category.*

(1) *The category $D(\mathcal{A})$ is abelian, cocomplete, and coproducts are exact. Moreover, the Yoneda functor $\mathcal{A} \to D(\mathcal{A})$ preserves colimits.*
(2) *Suppose that $\mathcal{A}'$ is abelian, cocomplete, and coproducts are exact. Then any functor $f\colon \mathcal{A} \to \mathcal{A}'$ which preserves colimits has a (essentially) unique factorization*

$$\mathcal{A} \longrightarrow D(\mathcal{A}) \xrightarrow{f'} \mathcal{A}'$$

such that f' is exact and preserves coproducts.

Proof. (1) The category $\widehat{\mathcal{A}}$ is abelian since $\mathcal{A}$ has cokernels. If $(X_i)_{i\in I}$ is a family of objects in $\mathcal{A}$, then
$$(\coprod_i X_i,-)\cong\prod_i (X_i,-).$$
Therefore $\widehat{\mathcal{A}}$ has products and they are exact since the products are computed in Ab. We conclude that $D(\mathcal{A})$ has exact coproducts.

(2) Suppose $f\colon \mathcal{A}\to\mathcal{A}'$ preserves colimits. We obtain an exact functor $f'\colon D(\mathcal{A})\to\mathcal{A}'$ from Lemma 2.1, and f' preserves coproducts since coproducts in $\mathcal{A}'$ are left exact. □

The category $D_\alpha(\mathcal{A})$. Let $\mathcal{A}$ be a locally α-presentable category and let $\mathcal{C}=\mathcal{A}_\alpha$. We show how $\mathcal{A}$ can be identified with the category of α-exact functors $\mathcal{C}\to$ Ab. This construction is classical for $\alpha=\aleph_0$; it was introduced by Gruson and Jensen [9] for module categories and later extended by Simson [11] and Crawley-Boevey [4]. Consider the category $D_\alpha(\mathcal{A})=\operatorname{Lex}_\alpha(\widehat{\mathcal{C}},\mathrm{Ab})$. The Yoneda functor
$$h\colon \mathcal{C}\longrightarrow\widehat{\mathcal{C}}^{\mathrm{op}},\quad X\mapsto (X,-),$$
extends to a functor
$$h^*\colon \operatorname{Lex}_\alpha(\mathcal{C}^{\mathrm{op}},\mathrm{Ab})\longrightarrow \operatorname{Lex}_\alpha(\widehat{\mathcal{C}},\mathrm{Ab})$$
by Lemma 1.7, and we consider the composite
$$d\colon \mathcal{A}\xrightarrow{\sim}\operatorname{Lex}_\alpha(\mathcal{C}^{\mathrm{op}},\mathrm{Ab})\xrightarrow{h^*} D_\alpha(\mathcal{A}).$$

Proposition 2.3. *Let $\mathcal{A}$ be a locally α-presentable category. Then $D_\alpha(\mathcal{A})$ is a locally presentable abelian category with exact coproducts and exact α-filtered colimits.*

Proof. This follows from Lemma 1.2 and 1.4. □

Proposition 2.4. *Let $\mathcal{A}$ be a locally α-presentable category. Then $d\colon\mathcal{A}\to D_\alpha(\mathcal{A})$ has the following properties:*

(1) *d preserves colimits and products.*
(2) *d induces an equivalence $\mathcal{A}\to\operatorname{Ex}_\alpha(\widehat{\mathcal{C}},\mathrm{Ab})$.*
(3) *d sends a sequence $0\to X\to Y\to Z\to 0$ in $\mathcal{A}$ to an exact sequence if and only if $0\to(C,X)\to(C,Y)\to(C,Z)\to 0$ is exact for all $C\in\mathcal{C}$.*

Proof. (1) Since h^* is a left adjoint, it preserves colimits. A product of α-exact functors is again α-exact, and therefore d preserves products by the following part (2).

(2) Since h is fully faithful, it follows that h^* is fully faithful. Therefore d is fully faithful. The Yoneda functor h identifies $\mathcal{C}$ with the full subcategory of injectives in $\widehat{\mathcal{C}}^{\mathrm{op}}$. It follows that dX is an exact functor for every $X\in\mathcal{C}$. An arbitrary object in $\mathcal{A}$ is an α-filtered colimit of objects in $\mathcal{C}$. But d preserves α-filtered colimits and an α-filtered colimit of exact functors is exact. Thus dX is exact for all $X\in\mathcal{A}$. Now suppose that $F\in D_\alpha(\mathcal{A})$ is exact. Then $h_*F\colon\mathcal{C}^{\mathrm{op}}\to$ Ab is α-left exact, hence isomorphic to $(-,X)|_{\mathcal{C}}$ for some $X\in\mathcal{A}$. Now dX and F are both α-exact and agree on the representable functors. Thus $F\cong dX$.

(3) Fix a sequence $\varepsilon\colon 0\to X\to Y\to Z\to 0$ in $\mathcal{A}$. Suppose first that $0\to(C,X)\to(C,Y)\to(C,Z)\to 0$ is exact for all $C\in\mathcal{C}$. Write $Z=\operatorname{colim} Z_i$ as α-filtered colimit of objects in $\mathcal{C}$. It follows that ε is the α-filtered colimit of split exact sequences $0\to X\to Y_i\to Z_i\to 0$. Therefore $d\varepsilon$ is the filtered colimit of exact sequences since d preserves α-filtered colimits by (1), and we conclude from Lemma 1.4 that $d\varepsilon$ is exact.

Now suppose that the sequence $0\to dX\to dY\to dZ\to 0$ is exact in $D_\alpha(\mathcal{A})$ and let $C\in\mathcal{C}$. It follows from Lemma 1.4 that $\operatorname{Ext}^1(dC,dZ)=0$ since dZ is exact by (2).

Therefore the sequence $0 \to (dC, dX) \to (dC, dY) \to (dC, dZ) \to 0$ is exact. The functor d is fully faithful, again by (2), and therefore $0 \to (C, X) \to (C, Y) \to (C, Z) \to 0$ is eaxct. $\square$

Definition 2.5. Let $\mathcal{A}$ be a locally presentable additive category and α be a regular cardinal.

(1) A sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$ is called *α-pure exact* if the induced sequence $0 \to (C, X) \to (C, Y) \to (C, Z) \to 0$ is exact for all α-presentable objects C.
(2) A map $X \to Y$ in $\mathcal{A}$ is called *α-pure monomorphism* if there exists an α-pure exact sequence $0 \to X \to Y \to Z \to 0$.

Note that an α-pure exact sequence in $\mathcal{A}$ is a kernel-cokernel sequence provided that $\mathcal{A}$ is locally α-presentable.

Lemma 2.6. *Let $\mathcal{A}$ be a locally α-presentable additive category and let $\phi\colon X \to Y$ be a map in $\mathcal{A}$. Then $d\colon \mathcal{A} \to D_\alpha(\mathcal{A})$ sends ϕ to a monomorphism if and only if ϕ is an α-pure monomorphism.*

Proof. We apply Proposition 2.4. It follows from part (3) that $d\phi$ is a monomorphism if ϕ is an α-pure monomorphism. Now suppose that $d\phi$ is a monomorphism and let F be the cokernel of $d\phi$. The functors dX and dY are exact by part (2), and an application of the snake lemma shows that F is exact. Thus $F \cong dZ$ for some $Z \in \mathcal{A}$ and we get a sequence $0 \to X \to Y \to Z \to 0$ which is α-pure-exact, again using (3). It follows that ϕ is an α-pure monomorphism. $\square$

The inclusion $i\colon \mathcal{A}_\alpha \to \mathcal{A}$ extends to an exact functor $i^*\colon \widehat{\mathcal{A}_\alpha} \to \widehat{\mathcal{A}}$ sending $(X, -)$ to $(iX, -)$ and we write $\bar{F} = i^* F$ for each $F \in \widehat{\mathcal{A}_\alpha}$.

Lemma 2.7. *Let $\mathcal{A}$ be a locally α-presentable additive category. Then*

$$\bar{F}X \cong ((F, -), dX) \cong dX(F)$$

for all $F \in \widehat{\mathcal{A}_\alpha}$ and $X \in \mathcal{A}$.

Proof. The second isomorphisms follows from Yoneda's lemma. For $X \in \mathcal{A}_\alpha$ we have $dX = ((X, -), -)$ and therefore

$$\begin{aligned} \bar{F}X = FX &\cong ((X, -), F) \\ &\cong ((F, -), ((X, -), -)) \\ &\cong ((F, -), dX), \end{aligned}$$

again by Yoneda's lemma. The general case follows since every object in $\mathcal{A}$ is an α-filtered colimit of α-presentable objects and both functors, $\bar{F}$ and d, preserve α-filtered colimits. $\square$

The universal property of $D_\alpha(\mathcal{A})$. The category $D_\alpha(\mathcal{A})$ has a universal property which is a variation of the universal property of $D(\mathcal{A})$, involving the cardinal α.

Proposition 2.8. *Let $\mathcal{A}$ be a locally α-presentable additive category.*

(1) *The category $D_\alpha(\mathcal{A})$ is abelian, cocomplete, and α-filtered colimits are exact. Moreover, the functor $d_\alpha\colon \mathcal{A} \to D_\alpha(\mathcal{A})$ preserves colimits.*

(2) *Suppose that $\mathcal{A}'$ is abelian, cocomplete, and α-filtered colimits are exact. Then any functor $f\colon \mathcal{A} \to \mathcal{A}'$ which preserves colimits has a (essentially) unique factorization*

$$\mathcal{A} \longrightarrow D_\alpha(\mathcal{A}) \xrightarrow{f'} \mathcal{A}'$$

such that f' is exact and preserves coproducts.

Proof. (1) follows from Proposition 2.3 and 2.4.

(2) Suppose that $f\colon \mathcal{A} \to \mathcal{A}'$ preserves colimits. The restriction $\mathcal{A}_\alpha \to \mathcal{A}'$ induces an exact functor $\widehat{\mathcal{A}_\alpha}^{\mathrm{op}} \to \mathcal{A}'$ which preserves α-colimits. Now apply Lemma 1.6 to get a colimit preserving functor $f'\colon D_\alpha(\mathcal{A}) \to \mathcal{A}'$ which extends f. The exactness of f' follows from the fact that each exact sequence in $D_\alpha(\mathcal{A})$ can be written as α-filtered colmit of exact sequences of representable functors. □

Remark. The proof of Proposition 2.8 shows that $f'\colon D_\alpha(\mathcal{A}) \to \mathcal{A}'$ is already determined by the fact that f' is left exact, preserves α-filtered colimits, and $f = f' \circ d_\alpha$.

The filtration of $D(\mathcal{A})$. Fix a locally presentable additive category $\mathcal{A}$. We have a filtration $\mathcal{A} = \bigcup_{\alpha<\infty} \mathcal{A}_\alpha$ and we denote for regular cardinals $\beta \geq \alpha$ or $\beta = \infty$ by $i_{\beta\alpha}\colon \mathcal{A}_\alpha \to \mathcal{A}_\beta$ the inclusion functor where $\mathcal{A}_\infty = \mathcal{A}$. To simplify our notation we write $C_\alpha(\mathcal{A}) = \widehat{\mathcal{A}_\alpha}^{\mathrm{op}}$ and put

$$C_\infty(\mathcal{A}) = C(\mathcal{A}) = D(\mathcal{A}) = D_\infty(\mathcal{A}).$$

We denote by $c_\alpha\colon \mathcal{A}_\alpha \to C_\alpha(\mathcal{A})$ the Yoneda functor which sends $X \in \mathcal{A}_\alpha$ to $(X, -)$; it makes the following diagram commutative

$$\begin{array}{ccc} \mathcal{A}_\alpha & \xrightarrow{c_\alpha} & C_\alpha(\mathcal{A}) \\ \big\downarrow{\scriptstyle i_\alpha} & & \big\downarrow{\scriptstyle j_\alpha} \\ \mathcal{A} & \xrightarrow{d_\alpha} & D_\alpha(\mathcal{A}) \end{array}$$

where $j_\alpha X = (-, X)$. Each inclusion $i_{\beta\alpha}$ induces a fully faithful exact and α-colimit preserving functor $c_{\beta\alpha}\colon C_\alpha(\mathcal{A}) \to C_\beta(\mathcal{A})$ with $c_{\beta\alpha}(X,-) = (i_{\beta\alpha}X, -)$ for all $X \in \mathcal{A}_\alpha$. Thus $c_\beta = c_{\beta\alpha} \circ c_\alpha$, and if $\gamma \geq \beta$, then we obtain $c_{\gamma\alpha} = c_{\gamma\beta} \circ c_{\beta\alpha}$. The next lemma shows that we have a functor $d_{\beta\alpha}\colon D_\alpha(\mathcal{A}) \to D_\beta(\mathcal{A})$ which completes the following diagram.

$$\begin{array}{ccccc} \mathcal{A}_\alpha & \xrightarrow{c_\alpha} & C_\alpha(\mathcal{A}) & \xrightarrow{j_\alpha} & D_\alpha(\mathcal{A}) \\ \big\downarrow{\scriptstyle i_{\beta\alpha}} & & \big\downarrow{\scriptstyle c_{\beta\alpha}} & & \big\downarrow{\scriptstyle d_{\beta\alpha}} \\ \mathcal{A}_\beta & \xrightarrow{c_\beta} & C_\beta(\mathcal{A}) & \xrightarrow{j_\beta} & D_\beta(\mathcal{A}) \end{array}$$

Lemma 2.9. *Let $\mathcal{A}$ be a locally α-presentable additive category, and let $\beta \geq \alpha$ be a regular cardinal or $\beta = \infty$. Then there exists a (essentially) unique colimit preserving functor $d_{\beta\alpha}\colon D_\alpha(\mathcal{A}) \to D_\beta(\mathcal{A})$ such that $d_\beta = d_{\beta\alpha} \circ d_\alpha$. If $\gamma \geq \beta$, then $d_{\gamma\alpha} = d_{\gamma\beta} \circ d_{\beta\alpha}$.*

Proof. The functor

$$C_\alpha(\mathcal{A}) \xrightarrow{c_{\beta\alpha}} C_\beta(\mathcal{A}) \xrightarrow{j_\beta} D_\beta(\mathcal{A})$$

preserves α-colimits. We apply Lemma 1.6 and obtain a unique colimit preserving functor $d_{\beta\alpha}\colon D_\alpha(\mathcal{A}) \to D_\beta(\mathcal{A})$ such that $j_\beta \circ c_{\beta\alpha} = d_{\beta\alpha} \circ j_\alpha$. The equality $d_\beta = d_{\beta\alpha} \circ d_\alpha$ follows from $c_\beta = c_{\beta\alpha} \circ c_\alpha$, and $d_{\gamma\alpha} = d_{\gamma\beta} \circ d_{\beta\alpha}$ follows from $c_{\gamma\alpha} = c_{\gamma\beta} \circ c_{\beta\alpha}$. □

We have also functors $D_\beta(\mathcal{A}) \to D_\alpha(\mathcal{A})$ for $\beta \geq \alpha$.

Lemma 2.10. *Let $\mathcal{A}$ be a locally α-presentable additive category, and let $\beta \geq \alpha$ be a regular cardinal or $\beta = \infty$. Then there exists a (essentially) unique exact and coproduct preserving functor $e_{\alpha\beta} \colon D_\beta(\mathcal{A}) \to D_\alpha(\mathcal{A})$ such that $d_\alpha = e_{\alpha\beta} \circ d_\beta$. If $\gamma \geq \beta$, then $e_{\alpha\gamma} = e_{\alpha\beta} \circ e_{\beta\gamma}$.*

Proof. We have exactness of coproducts and α-filtered colimits in $D_\alpha(\mathcal{A})$ by Proposition 2.3. So we can apply the universal property of $d_\beta \colon \mathcal{A} \to D_\beta(\mathcal{A})$, formulated in Proposition 2.2 and 2.8, to obtain $e_{\alpha\beta} \colon D_\beta(\mathcal{A}) \to D_\alpha(\mathcal{A})$. □

Next we exhibit the relation between the functors $e_{\alpha\beta}$ and $d_{\beta\alpha}$. Recall that a *Serre* subcategory $\mathcal{S}$ of an abelian category $\mathcal{C}$ is a full subcategory such that for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ we have $X \in \mathcal{S}$ if and only if $X', X'' \in \mathcal{S}$. There is an abelian quotient category $\mathcal{C}/\mathcal{S}$ and $\mathcal{S}$ is *colocalizing* if the exact quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{S}$ has a left adjoint [7].

Proposition 2.11. *Let $\mathcal{A}$ be a locally α-presentable additive category, and let $\beta \geq \alpha$ be a regular cardinal or $\beta = \infty$.*

(1) *$e_{\alpha\beta} \circ d_{\beta\alpha} = \mathrm{id}_{D_\alpha(\mathcal{A})}$ and $e_{\alpha\beta}$ is a right adjoint for $d_{\beta\alpha}$.*
(2) *$\mathcal{S}_{\alpha\beta} = \{X \in D_\beta(\mathcal{A}) \mid e_{\alpha\beta}X = 0\}$ is a colocalizing subcategory of $D_\beta(\mathcal{A})$ and $e_{\alpha\beta}$ induces an equivalence $D_\beta(\mathcal{A})/\mathcal{S}_{\alpha\beta} \to D_\alpha(\mathcal{A})$.*

Proof. (1) The definition of $e_{\alpha\beta}$ implies the following factorization of $d_\alpha|_{\mathcal{A}_\alpha} = j_\alpha \circ c_\alpha$:

$$\mathcal{A}_\alpha \xrightarrow{c_\alpha} C_\alpha(\mathcal{A}) \xrightarrow{j_\alpha} D_\alpha(\mathcal{A}) \xrightarrow{d_{\beta\alpha}} D_\beta(\mathcal{A}) \xrightarrow{e_{\alpha\beta}} D_\alpha(\mathcal{A}).$$

The composite $f = e_{\alpha\beta} \circ d_{\beta\alpha} \circ j_\alpha$ is exact since $d_{\beta\alpha} \circ j_\alpha = j_\beta \circ h_{\beta\alpha}$, and this implies $f = j_\alpha$ since both coincide on the full subcategory of injectives in $C_\alpha(\mathcal{A})$. We conclude $e_{\alpha\beta} \circ d_{\beta\alpha} = \mathrm{id}_{D_\alpha(\mathcal{A})}$ from Lemma 1.6 since $e_{\alpha\beta} \circ d_{\beta\alpha}$ preserves colimits.

Next, we show that $e_{\alpha\beta}$ is a right adjoint for $d_{\beta\alpha}$. Suppose first that $\beta < \infty$. With the notation of Lemma 1.7, we have $d_{\beta\alpha} = c^*$ for $c = c_{\beta\alpha}$, and this has a right adjoint c_* with $c_* \circ c^* = \mathrm{id}_{D_\alpha(\mathcal{A})}$ since c^* is fully faithful by the first part of this proof. Therefore $d_\alpha = c_* \circ d_\beta$, and Lemma 2.10, in combination with the remark after the proof of Proposition 2.8, implies $e_{\alpha\beta} = c_*$. Thus $e_{\alpha\beta}$ is a right adjoint for $d_{\beta\alpha}$.

Now suppose $\beta = \infty$ and let $F \in D_\alpha(\mathcal{A})$ and $G \in D_\infty(\mathcal{A})$. If the objects in a presentation of G are γ-presentable, then $G = d_{\infty\gamma}H$ for $H = (G|_{\mathcal{A}_\gamma}, -)$ in $D_\gamma(\mathcal{A})$, and we may assume that $\gamma \geq \alpha$. Using that $e_{\alpha\gamma}$ is a right adjoint for $d_{\gamma\alpha}$ we obtain

$$\begin{aligned}(d_{\infty\alpha}F, G) &= (d_{\infty\gamma}(d_{\gamma\alpha}F), d_{\infty\gamma}H) \cong (d_{\gamma\alpha}F, H) \\ &\cong (F, e_{\alpha\gamma}H) \cong (F, e_{\alpha\gamma} \circ (e_{\gamma\infty} \circ d_{\infty\gamma})H) \\ &\cong (F, e_{\alpha\infty}G).\end{aligned}$$

Thus $e_{\alpha\infty}$ is a right adjoint for $d_{\infty\alpha}$.

(2) is a consequence of (1) and the fact that $e_{\alpha\beta}$ is exact; see Proposition III.5 in [7]. □

Corollary 2.12. *Let $\mathcal{A}$ be a locally presentable additive category. If α is a regular cardinal such that $\mathcal{A}$ is locally α-presentable, then $d_{\infty\alpha} \colon D_\alpha(\mathcal{A}) \to D(\mathcal{A})$ identifies $D_\alpha(\mathcal{A})$ with the full subcategory of colimits of functors $F \in D(\mathcal{A})$ having a presentation*

$$(Y, -) \longrightarrow (X, -) \longrightarrow F \longrightarrow 0$$

such that X and Y are α-presentable. Therefore

$$\bigcup_{\alpha < \infty} D_\alpha(\mathcal{A}) = D(\mathcal{A}).$$

3. Coherent functors

In this section we fix a locally presentable additive category $\mathcal{A}$ and characterize the coherent functors $\mathcal{A} \to \mathrm{Ab}$.

Definition 3.1. Let α be a regular cardinal. A functor $F\colon \mathcal{A} \to \mathrm{Ab}$ is called *α-coherent* if there exists an exact sequence

$$(Y,-) \longrightarrow (X,-) \longrightarrow F \longrightarrow 0$$

such that X and Y are α-presentable objects.

Note that every coherent functor $\mathcal{A} \to \mathrm{Ab}$ is α-coherent for some regular cardinal α since $\mathcal{A} = \bigcup_\alpha \mathcal{A}_\alpha$.

Proposition 3.2. *Let $\mathcal{A}$ be a locally α-presentable category. Then a functor $F\colon \mathcal{A} \to$ Ab is α-coherent if and only F preserves products and α-filtered colimits.*

Proof. Suppose first that F is α-coherent. Each representable functor $(X,-)$ preserves products and α-filtered colimits provided that X is α-presentable. Clearly, this property is preserved if we pass to the cokernel of a map $(Y,-) \to (X,-)$ where X and Y are α-presentable.

Now suppose that F preserves products and α-filtered colimits. Let $\mathcal{C}$ be the full subcategory of α-presentable objects in $\mathcal{A}$. Recall that there exists a *tensor product*

$$(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \times (\mathcal{C}, \mathrm{Ab}) \longrightarrow \mathrm{Ab}, \quad (F,G) \to F \otimes_{\mathcal{C}} G,$$

where for any functor $G\colon \mathcal{C} \to \mathrm{Ab}$, the tensor functor $- \otimes_{\mathcal{C}} G$ is determined by the fact that it preserves colimits and $(-,X) \otimes_{\mathcal{C}} G \cong GX$ for all X in $\mathcal{C}$. Now take $G = F|_{\mathcal{C}}$ and observe that

$$(*) \qquad FX \cong (-,X)|_{\mathcal{C}} \otimes_{\mathcal{C}} G$$

for all $X \in \mathcal{A}$ since F preserves α-filtered colimits and every object in $\mathcal{A}$ can be written as α-filtered colimit of α-presentable object.

Next we use the fact that a module M over any ring R is finitely presented if the corresponding tensor functor $- \otimes_R M$ preserves products. In fact, it is sufficient to assume that $- \otimes_R M$ preserves products of finitely generated projective modules (cf. Lemma I.13.2 in [12]). This result generalizes to rings with several objects. The finitely generated projective $\mathcal{C}$-modules are up to isomorphism of the form $(-,X)$ with $X \in \mathcal{C}$ and the assumption on F implies that the natural map

$$\Big(\prod_i (-,X_i)\Big) \otimes_{\mathcal{C}} G \longrightarrow \prod_i ((-,X_i) \otimes_{\mathcal{C}} G)$$

is an isomorphism for every family $(X_i)_{i\in I}$ in $\mathcal{C}$. We conclude that G has a presentation

$$(Y,-)|_{\mathcal{C}} \longrightarrow (X,-)|_{\mathcal{C}} \longrightarrow G \longrightarrow 0$$

with $X, Y \in \mathcal{C}$. Combining this presentation with the isomorphism $(*)$ gives a presentation

$$(Y,-) \longrightarrow (X,-) \longrightarrow F \longrightarrow 0$$

of F. Thus F is α-coherent. $\square$

4. Covariantly finite subcategories

A full subcategory $\mathcal{B}$ of a category $\mathcal{A}$ is *covariantly finite* if for every object $X \in \mathcal{A}$ there exists a map $X \to Y$ such that $Y \in \mathcal{B}$ and for every $Y' \in \mathcal{B}$ the induced map $(Y, Y') \to (X, Y')$ is surjective. We require in addition that every direct factor of an object in $\mathcal{B}$ belongs to $\mathcal{B}$.

Colocalizing subcategories. Fix an additive category $\mathcal{A}$ with cokernels so that the category $\widehat{\mathcal{A}}$ of coherent functors $\mathcal{A} \to \mathrm{Ab}$ is abelian. We describe the covariantly finite subcategories of $\mathcal{A}$ in terms of colocalizing subcategories of $\widehat{\mathcal{A}}$. Note that a colocalizing subcategory of $\widehat{\mathcal{A}}$ is the same as a localizing subcategory of $D(\mathcal{A}) = \widehat{\mathcal{A}}^{\mathrm{op}}$.

Proposition 4.1. *Let $\mathcal{A}$ be an additive category with cokernels. Then there exists a bijection between the class of covariantly finite subcategories of $\mathcal{A}$ and the class of colocalizing subcategories of $\widehat{\mathcal{A}}$. This bijection sends a covariantly finite subcategory $\mathcal{B}$ to $\{F \in \widehat{\mathcal{A}} \mid F|_{\mathcal{B}} = 0\}$ and a colocalizing subcategory $\mathcal{S}$ to $\{X \in \mathcal{A} \mid FX = 0$ for all $F \in \mathcal{S}\}$.*

Proof. Denote by $i \colon \mathcal{B} \to \mathcal{A}$ the inclusion of a full subcategory. Suppose first that $\mathcal{B}$ is covariantly finite and let $\mathcal{S} = \{F \in \widehat{\mathcal{A}} \mid F|_{\mathcal{B}} = 0\}$. We obtain a functor $i_* \colon \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$ sending F to $F|_{\mathcal{B}}$ which induces an equivalence between $\widehat{\mathcal{A}}/\mathcal{S}$ and $\widehat{\mathcal{B}}$. The right exact functor $i^* \colon \widehat{\mathcal{B}} \to \widehat{\mathcal{A}}$ which sends $(X, -)$ to $(iX, -)$ for all $X \in \mathcal{B}$ provides a left adjoint for the quotient functor $\widehat{\mathcal{A}} \to \widehat{\mathcal{A}}/\mathcal{S}$ and therefore $\mathcal{S}$ is colocalizing.

Conversely, let $\mathcal{S} \subseteq \widehat{\mathcal{A}}$ be colocalizing and let $\mathcal{B} = \{X \in \mathcal{A} \mid FX = 0 \text{ for all } F \in \mathcal{S}\}$. Then the quotient category $\widehat{\mathcal{A}}/\mathcal{S}$ has enough projectives which the left adjoint identifies with the representable functors $(X, -)$ such that $((X, -), F) = FX = 0$ for all $F \in \mathcal{S}$. Thus $\widehat{\mathcal{A}}/\mathcal{S}$ and $\widehat{\mathcal{B}}$ are equivalent, and the quotient functor sends $(X, -)$ to $(X, -)|_{\mathcal{B}}$ which is a quotient of $(Y, -)$ in $\widehat{\mathcal{B}}$ for some $Y \in \mathcal{B}$. This gives the desired map $X \to Y$ so that $\mathcal{B}$ is covariantly finite. □

A criterion. Fix a locally presentable additive category $\mathcal{A}$. The following proposition gives a general criterion for a subcategory of $\mathcal{A}$ to be covariantly finite.

Proposition 4.2. *Let $\mathcal{A}$ be a locally presentable additive category and let $\mathcal{B}$ be a full subcategory which is closed under taking products. Suppose that there exists a set $\mathcal{C}$ of objects in $\mathcal{A}$ such that for every $Y \in \mathcal{B}$ a subobject $X \subseteq Y$ belongs to $\mathcal{B}$ provided that the cokernel sequence $X \to Y \to Z \to 0$ induces an epimorphism $(C, Y) \to (C, Z)$ for all $C \in \mathcal{C}$. Then $\mathcal{B}$ is covariantly finite.*

Proof. Suppose that $\mathcal{A}$ is locally α-presentable. Choosing α sufficiently large, we may assume that $\mathcal{C} \subseteq \mathcal{A}_\alpha$. Therefore the condition on $\mathcal{B}$ implies that $\mathcal{B}$ is closed under taking α-pure subobjects. To prove that $\mathcal{B}$ is covariantly finite we use the embedding $d \colon \mathcal{A} \to D_\alpha(\mathcal{A})$ and Lemma 1.5. Extending the notation introduced before Lemma 1.5, we define $|X| = |dX|$ for every $X \in \mathcal{A}$. Now let $\phi \colon X \to Y$ be a map with $Y \in \mathcal{B}$. Let κ be a cardinal such that $|F|_\alpha \leq \kappa$ for every quotient F of dX in $D_\alpha(\mathcal{A})$. We claim that ϕ factors through some $Y' \in \mathcal{B}$ with $|Y'| \leq \kappa$. In fact, it follows from Lemma 1.5 that $d\phi$ factors through some exact subfunctor $G \subseteq dY$ with $|G| \leq \kappa$. Now $G \cong dY'$ for some $Y' \in \mathcal{A}$ by Proposition 2.4, and $Y' \in \mathcal{B}$ since the map $Y' \to Y$ is an α-pure monomorphism by Lemma 2.6. Let

$$\psi \colon X \longrightarrow \prod_{|Z| \leq \kappa} Z^{(X,Z)}$$

where Z runs through a set of isoclasses of objects in $\mathcal{B}$ with $|Z| \leq \kappa$. The product $\prod_{|Z|\leq\kappa} Z^{(X,Z)}$ belongs to $\mathcal{B}$, and every map $X \to Y$ with $Y \in \mathcal{B}$ factors through ψ by construction. Thus $\mathcal{B}$ is covariantly finite. $\square$

Corollary 4.3. *Let $\mathcal{A}$ be a locally presentable additive category and let $\mathcal{B}$ be a full subcategory. Suppose that $\mathcal{B}$ is closed under taking products and α-pure subobjects for some regular cardinal α. Then $\mathcal{B}$ is covariantly finite.*

Every covariantly finite subcategory of a locally presentable category is automatically closed under taking products and direct factors. Therefore it is attempting to ask whether the converse is true. Unfortunately, an answer to this question requires some set-theoretic assumptions. To be precise, consider the following statement.

(CF) *A full subcategory of a locally presentable category is covariantly finite if and only if it is closed under taking products and direct factors.*

This statement cannot be proved in the set theory GBC (Gödel-Bernays plus axiom of choice). However, assuming the existence of certain large cardinals one can find a model for the set theory GBC where (CF) holds true. We refer to [1] for details.

5. Definable subcategories

The concept of a definable subcatory has been introduced by Crawley-Boevey for module categories [5]. For his definition he used coherent functors $F\colon \operatorname{Mod} R \to \mathrm{Ab}$ having a presentation

$$(Y,-) \longrightarrow (X,-) \longrightarrow F \longrightarrow 0$$

such that X and Y are finitely presented R-modules. The following definition is more general.

Definition 5.1. Let $\mathcal{A}$ be an additive category. A full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called *definable* if there exists a set of coherent functors $F_i\colon \mathcal{A} \to \mathrm{Ab}$ such that $X \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if $F_iX = 0$ for all i.

A characterization. Our aim is to characterize the definable subcategories of a locally presentable additive category. The essential part of this project is the following proposition.

Proposition 5.2. *Let $\mathcal{A}$ be a locally α-presentable additive category and let $\mathcal{B} \subseteq \mathcal{A}$.*

1. *$\mathcal{S} = \{F \in \widehat{\mathcal{A}_\alpha} \mid \bar{F}|_{\mathcal{B}} = 0\}$ is a Serre subcategory of $\widehat{\mathcal{A}_\alpha}$ which is closed under α-products and admits a faithful α-exact functor $\widehat{\mathcal{A}_\alpha}/\mathcal{S} \to \mathrm{Ab}$.*
2. *If $\mathcal{B}$ is closed under taking products, α-filtered colimits, and α-pure subobjects, then $\mathcal{B} = \{X \in \mathcal{A} \mid \bar{F}X = 0 \text{ for all } F \in \mathcal{S}\}$.*

Proof. (1) follows from the isomorphism $\bar{F}X \cong dX(F)$ in Lemma 2.7 since dX is α-exact by Proposition 2.4. To obtain a faithful α-exact functor $\widehat{\mathcal{A}_\alpha}/\mathcal{S} \to \mathrm{Ab}$ take for each $F \in \widehat{\mathcal{A}_\alpha} \setminus \mathcal{S}$ some $X_F \in \mathcal{B}$ with $\bar{F}X_F \neq 0$ and let $G = \prod_F dX_F$. Clearly, G induces a faithful α-exact functor $\widehat{\mathcal{A}_\alpha}/\mathcal{S} \to \mathrm{Ab}$.

(2) Now suppose that $\mathcal{B}$ is closed under taking products, α-filtered colimits, and α-pure subobjects. Let $\mathcal{F}$ be the full subcategory of objects in $D_\alpha(\mathcal{A})$ which are subobjects of objects in $d\mathcal{B}$. Note that $dX \in \mathcal{F}$ implies $X \in \mathcal{B}$ for all $X \in \mathcal{A}$ by our assumptions on $\mathcal{B}$ since subobjects in $D_\alpha(\mathcal{A})$ correspond to α-pure subobjects in $\mathcal{B}$ by Lemma 2.6. Clearly, $\mathcal{F}$ is closed under products and subobjects since $\mathcal{B}$ is closed under products. We claim that $\mathcal{F}$ is also closed under α-filtered colimits. To this end let $X = \operatorname{colim}_{i\in\mathcal{I}} X_i$ be

an α-filtered colimit of objects in $\mathcal{F}$. Choose for each $i \in \mathcal{I}$ a monomorphism $X_i \to Y_i$ with $Y_i \in d\mathcal{B}$. We obtain an α-filtered system of monomorphisms $X_i \to \prod_{i\to j} Y_j$, indexed by $i \in \mathcal{I}$, where $i \to j$ runs through all maps in $\mathcal{I}$ which start in i, and $X_i \to Y_j$ is the composite of $X_i \to X_j$ and the monomorphism $X_j \to Y_j$. The colimit of this system produces a monomorphism $X \to Y$ with $Y \in d\mathcal{B}$ since $\mathcal{B}$ is closed under products and α-filtered colimits. Therefore X belongs to $\mathcal{F}$.

The inclusion functor $\mathcal{F} \to D_\alpha(\mathcal{A})$ has a left adjoint $f\colon D_\alpha(\mathcal{A}) \to \mathcal{F}$ which is constructed as follows. For $X \in D_\alpha(\mathcal{A})$ consider the set of all quotient objects Y_i which belong to $\mathcal{F}$. Then define fX to be the image and tX to be the kernel of the canonical map $X \to \prod_i Y_i$. Also define $\mathcal{T}$ to be the full subcategory

$$\begin{aligned}\mathcal{T} &= \{X \in D_\alpha(\mathcal{A}) \mid (X,Y) = 0 \text{ for all } Y \in \mathcal{F}\}\\ &= \{X \in D_\alpha(\mathcal{A}) \mid (X,dZ) = 0 \text{ for all } Z \in \mathcal{B}\}.\end{aligned}$$

We identify each $X \in \mathcal{S}$ with $(X,-)$ in $D_\alpha(\mathcal{A})$ and claim that $\mathcal{T} = \operatorname{colim}_\alpha \mathcal{S}$, that is, the objects in $\mathcal{T}$ are precisely the α-filtered colimits of objects in $\mathcal{S}$. The isomorphism of Lemma 2.7 shows that $\mathcal{S} \subseteq \mathcal{T}$ and therefore $\operatorname{colim}_\alpha \mathcal{S} \subseteq \mathcal{T}$. Next we show that $tX \in \operatorname{colim}_\alpha \mathcal{S}$ for every α-presentable X. We can write $tX = \operatorname{colim} X_i$ as α-filtered colimit of α-presentable subobjects since the image of every map between α-presentable objects is again α-presentable. If $X_i \notin \mathcal{S}$ for some i then there exists a non-zero map $X_i \to Y$ for some $Y \in d\mathcal{B}$ and this extends to a map $\phi\colon X \to Y$ since $\operatorname{Ext}^1(-,Y)$ vanishes on α-presentable objects by Lemma 1.4. However, ϕ factors through the canonical map $X \to fX$ which implies $\phi|_{tX} = 0$. This is a contradiction and therefore $tX \in \operatorname{colim}_\alpha \mathcal{S}$. Now let $X = \operatorname{colim} X_i$ be an arbitrary object in $D_\alpha(\mathcal{A})$ written as α-filtered colimit of α-presentable objects. We obtain an exact sequence

$$0 \longrightarrow \operatorname{colim} tX_i \longrightarrow X \longrightarrow \operatorname{colim} fX_i \longrightarrow 0$$

with $\operatorname{colim} tX_i \in \operatorname{colim}_\alpha \mathcal{S}$ and $\operatorname{colim} fX_i \in \mathcal{F}$ since both, $\operatorname{colim}_\alpha \mathcal{S}$ and $\mathcal{F}$, are closed under α-filtered colimits. We conclude that $tX \in \operatorname{colim}_\alpha \mathcal{S}$. An immediate consequence is

$$\mathcal{F} = \{Y \in D_\alpha(\mathcal{A}) \mid (X,Y) = 0 \text{ for all } X \in \mathcal{T}\},$$

that is, $(\mathcal{T},\mathcal{F})$ forms a torsion pair.

We are now in a position to prove that $\mathcal{B} = \{X \in \mathcal{A} \mid \bar{F}X = 0 \text{ for all } F \in \mathcal{S}\}$. One inclusion is clear. Therefore let $X \in \mathcal{A}$ with $\bar{F}X = 0$ for all $F \in \mathcal{S}$. Thus $(F,dX) = 0$ for all $F \in \mathcal{S}$, and this implies $(G,dX) = 0$ for all G in $\operatorname{colim}_\alpha \mathcal{S} = \mathcal{T}$. We obtain $dX \in \mathcal{F}$ from the first part of this proof, and this implies $X \in \mathcal{B}$. □

We continue with two lemmas which help to characterize the definable subcategories.

Lemma 5.3. *Let $\mathcal{A}$ be a cocomplete additive category. Given a set of sequences $U_i \to V_i \to W_i$ in $\mathcal{A}$, there exists a coherent functor $F\colon \mathcal{A} \to \mathrm{Ab}$ such that for every object $X \in \mathcal{A}$ the induced sequence $(W_i,X) \to (V_i,X) \to (U_i,X)$ is exact for all i if and only if $FX = 0$.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}\coprod_i U_i & \longrightarrow & \coprod_i V_i & \longrightarrow & P & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \\ & & \coprod_i W_i & \longrightarrow & Q & & \end{array}$$

where the rows are exact and the square is a pushout. Now take for $F\colon \mathcal{A} \to \mathrm{Ab}$ the cokernel of $(Q,-) \to (P,-)$. □

Lemma 5.4. *Let $\mathcal{A}$ be a locally presentable category and $\mathcal{S}$ be a class of α-coherent functors $\mathcal{A} \to \mathrm{Ab}$. Then*

$$\mathcal{B} = \{X \in \mathcal{A} \mid FX = 0 \textit{ for all } F \in \mathcal{S}\}$$

is closed under taking products, α-filtered colimits, and α-pure subobjects.

Proof. A representable functor $(X, -)$ with α-presentable X preserves products, α-filtered colimits, and sends α-pure exact sequences to exact sequences. It follows immediately that every α-coherent functor $F\colon \mathcal{A} \to \mathrm{Ab}$ preserves products and α-filtered colimits. An application of the snake lemma shows that F sends α-pure exact sequences to exact sequences. We conclude that $\mathcal{B}$ is closed under taking products, α-filtered colimits, and α-pure subobjects. □

We are now in a position to give the proof of Theorem 2 which characterizes the definable subcategories of a locally presentable category.

Proof of Theorem 2. (2) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (2): See Lemma 5.3.

(2) $\Rightarrow$ (4): Given a set of coherent functors $F_i\colon \mathcal{A} \to \mathrm{Ab}$ there is a regular cardinal α such that all F_i are α-coherent. Now use Lemma 5.4.

(4) $\Rightarrow$ (2): See Proposition 5.2.

(2) $\Rightarrow$ (1): Fix a set of coherent functors F_i such that $X \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if $F_iX = 0$ for all i. The subcategory $\mathcal{B}$ is closed under products and α-pure subobjects for some regular cardinal α by Lemma 5.4. It follows from Corollary 4.3 that $\mathcal{B}$ is covariantly finite. Moreover, the bijective correspondence between covariantly finite subcategories of $\mathcal{A}$ and colocalizing subcategories of $\widehat{\mathcal{A}}$ shows that the corresponding colocalizing subcategory $\{G \in \widehat{\mathcal{A}} \mid G|_{\mathcal{B}} = 0\}$ is the smallest containing $F = \prod_i F_i$.

(1) $\Rightarrow$ (2): Let $\mathcal{B}'$ be the full subcategory of objects in $\mathcal{A}$ satisfying $FX = 0$. We claim that $\mathcal{B}' = \mathcal{B}$. The subcategory $\mathcal{B}'$ is covariantly finite since (2) implies (1). Now apply the bijective correspondence between covariantly finite subcategories of $\mathcal{A}$ and colocalizing subcategories of $\widehat{\mathcal{A}}$. It is clear that the colocalizing subcategory of $\widehat{\mathcal{A}}$ which consists of the coherent functors vanishing on $\mathcal{B}'$ is the smallest containing F. Therefore (1) implies $\mathcal{B}' = \mathcal{B}$ and this shows (2). □

We give some examples of definable subcategories.

Example 5.5. (1) Let $(X_i)_{i\in I}$ be a family of R-modules (R any ring) and $(n_i)_{i\in I}$ be a family of non-negative integers. Then the R-modules Y such that $\mathrm{Ext}^{n_i}_R(X_i, Y) = 0$ for all i form a definable subcategory of the category of all R-modules.

(2) Let $\mathcal{C}$ be a small additive category and let $U_i \to V_i \to W_i$ be a family of sequences in $\mathcal{C}$. Then the functors $F\colon \mathcal{C} \to \mathrm{Ab}$ such that $FU_i \to FV_i \to FW_i$ is exact for all i form a definable subcategory in the category of all additive functors $\mathcal{C} \to \mathrm{Ab}$.

A classification. Fix a locally presentable additive category $\mathcal{A}$. We have seen in Proposition 5.2 that every definable subcategory of $\mathcal{A}$ is determined by some Serre subcategory of $\widehat{\mathcal{A}_\alpha}$ for some appropriate cardinal α. This leads to a classification of all definable subcategories. To this end denote for every regular cardinal α by Σ_α the set of all Serre subcategories $\mathcal{S}$ of $\widehat{\mathcal{A}_\alpha}$ which are closed under α-products and admit a faithful α-exact functor $\widehat{\mathcal{A}_\alpha}/\mathcal{S} \to \mathrm{Ab}$.

Proposition 5.6. *Let $\mathcal{A}$ be a locally α-presentable additive category and let $\mathcal{S} \subseteq \widehat{\mathcal{A}_\alpha}$.*

(1) $\mathcal{B} = \{X \in \mathcal{A} \mid \bar{F}X = 0$ *for all* $F \in \mathcal{S}\}$ *is closed under taking products,* α*-filtered colimits, and* α*-pure subobjects.*

(2) $\overline{\mathcal{S}} = \{F \in \widehat{\mathcal{A}_\alpha} \mid \bar{F}|_{\mathcal{B}} = 0\}$ *is the smallest Serre subcategory in* Σ_α *containing* $\mathcal{S}$.

Proof. (1) follows from Lemma 5.4.

(2) It follows from Proposition 5.2 that $\overline{\mathcal{S}} \in \Sigma_\alpha$. Now let $\mathcal{T} \in \Sigma_\alpha$ containing $\mathcal{S}$. There exists a faitful α-exact functor $G\colon \widehat{\mathcal{A}_\alpha}/\mathcal{T} \to \mathrm{Ab}$ and the composition with the quotient functor $\widehat{\mathcal{A}_\alpha} \to \widehat{\mathcal{A}_\alpha}/\mathcal{T}$ is α-exact and therefore isomorphic to dY for some $Y \in \mathcal{A}$ by Proposition 2.4. The isomorphism $\bar{F}Y = dY(F)$ from Lemma 2.7 implies $\mathcal{T} = \{F \in \widehat{\mathcal{A}_\alpha} \mid \bar{F}Y = 0\}$. We get $Y \in \mathcal{B}$ since $\mathcal{S} \subseteq \mathcal{T}$, and the definition of $\overline{\mathcal{S}}$ implies $\overline{\mathcal{S}} \subseteq \mathcal{T}$. This completes the proof. $\square$

Now consider $\Sigma = \bigcup_\alpha \Sigma_\alpha$. Given regular cardinals $\alpha_2 \geq \alpha_1$, we call $\mathcal{S}_1 \in \Sigma_{\alpha_1}$ and $\mathcal{S}_2 \in \Sigma_{\alpha_2}$ *equivalent* and write $\mathcal{S}_1 \sim \mathcal{S}_2$, if $\mathcal{S}_2$ is the smallest Serre subcategory in Σ_{α_2} which contains $\bar{F}|_{\mathcal{A}_{\alpha_2}}$ for all $F \in \mathcal{S}_1$. The following lemma explains this equivalence relation.

Lemma 5.7. *Let* $\mathcal{A}$ *be a locally* α*-presentable additive category and let* $\alpha_2 \geq \alpha_1 \geq \alpha$ *be regular cardinals. Then* $\mathcal{S}_1 \in \Sigma_{\alpha_1}$ *and* $\mathcal{S}_2 \in \Sigma_{\alpha_2}$ *are equivalent if and only if*

$$\{X \in \mathcal{A} \mid \bar{F}X = 0 \textit{ for all } F \in \mathcal{S}_1\} = \{X \in \mathcal{A} \mid \bar{F}X = 0 \textit{ for all } F \in \mathcal{S}_2\}.$$

Proof. The inclusion $i\colon \mathcal{A}_{\alpha_1} \to \mathcal{A}_{\alpha_2}$ induces an exact functor $i^*\colon \widehat{\mathcal{A}_{\alpha_1}} \to \widehat{\mathcal{A}_{\alpha_2}}$ which sends $(X,-)$ to $(iX,-)$ for all $X \in \mathcal{A}_{\alpha_1}$. Observe that $i^*F = \bar{F}|_{\mathcal{A}_{\alpha_2}}$. Now let $\mathcal{B}_i = \{X \in \mathcal{A} \mid \bar{F}X = 0 \text{ for all } F \in \mathcal{S}_i\}$ for $i = 1,2$. It follows from Proposition 5.6 that $\mathcal{S}_2 = \{F \in \widehat{\mathcal{A}_{\alpha_2}} \mid \bar{F}|_{\mathcal{B}_2} = 0\}$ and $\overline{i^*\mathcal{S}_1} = \{F \in \widehat{\mathcal{A}_{\alpha_2}} \mid \bar{F}|_{\mathcal{B}_1} = 0\}$ is the smallest Serre subcategory in Σ_{α_2} containing $i^*\mathcal{S}_1$. We conclude, again from Proposition 5.6, that $\mathcal{B}_1 = \mathcal{B}_2$ if and only if $\mathcal{S}_1 \sim \mathcal{S}_2$. $\square$

We obtain now the classification of all definable subcategories via Serre subcategories which is Theorem 3.

Corollary 5.8. *Let* $\mathcal{A}$ *be a locally presentable additive category. Then there exists a bijection between* $\Sigma/{\sim}$ *and the class of definable subcategories of* $\mathcal{A}$*. It sends* $\mathcal{S} \in \Sigma$ *to* $\{X \in \mathcal{A} \mid \bar{F}X = 0$ *for all* $F \in \mathcal{S}\}$.

Proof. Every definable subcategory is of the form $\{X \in \mathcal{A} \mid \bar{F}X = 0 \text{ for all } F \in \mathcal{S}\}$ for some $\mathcal{S} \in \Sigma$ by Proposition 5.2. Moreover, the definable subcategories corresponding to $\mathcal{S}_1$ and $\mathcal{S}_2$ in Σ coincide if and only if $\mathcal{S}_1 \sim \mathcal{S}_2$ by Lemma 5.7. $\square$

6. How to make a functor exact

Let $\mathcal{C}$ be a small abelian category with α-products for some regular cardinal α. In this section we construct for every additive functor $F\colon \mathcal{C} \to \mathrm{Ab}$ an α-exact functor F' and a map $\phi\colon F \to F'$ such that for every α-exact G every map $F \to G$ factors through ϕ. We show that this construction is closeley related to the following problem which arises in the classification of definable subcategories via Serre subcategories of coherent functors.

Problem 6.1. *Let* $\mathcal{C}$ *be a small abelian category with* α*-products for some regular cardinal* α*. When does there exists a faithful* α*-exact functor* $\mathcal{C} \to \mathrm{Ab}$*?*

Let us start with a functor $F\colon \mathcal{C} \to \mathrm{Ab}$ which is α-left exact. We define by transfinite induction a sequence

$$F = F_0 \to F_1 \to F_2 \to \cdots \to F_\nu \to F_{\nu+1} \to \cdots$$

of maps in $\mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$ where ν runs through all ordinals with $|\nu| < \alpha$. To this end fix a representative set of epimorphisms $\pi\colon Y \to Z$ in $\mathcal{C}$. For ν with $|\nu| < \alpha$ let

$$\Gamma_\nu = \coprod_{\pi\colon Y \to Z} F_\nu Z \setminus \operatorname{Im} F_\nu \pi.$$

We get a map $\coprod_{i \in \Gamma_\nu} (Z_i, -) \to F_\nu$ where $(Z_i, -) \to F_\nu$ is the map corresponding to $i \in \Gamma_\nu$ under the Yoneda isomorphism. We define the map $F_\nu \to F_{\nu+1}$ by taking the pushout

$$\begin{array}{ccc} \coprod_{i \in \Gamma_\nu} (Z_i, -) & \xrightarrow{\amalg(\pi_i, -)} & \coprod_{i \in \Gamma_\nu} (Y_i, -) \\ \big\downarrow & & \big\downarrow \\ F_\nu & \longrightarrow & F_{\nu+1} \end{array}$$

Note that $F_\nu \to F_{\nu+1}$ is a monomorphism if α-products in $\mathcal{C}$ are exact. For a limit ordinal λ let $F_\lambda = \operatorname{colim}_{\nu < \lambda} F_\nu$. Finally, we define

$$F^* = \operatorname{colim}_{|\nu| < \alpha} F_\nu.$$

Lemma 6.2. *Let $F\colon \mathcal{C} \to \mathrm{Ab}$ be α-left exact. Then F^* is α-exact and every map $F \to G$ such that G is α-exact factors through the canonical map $F \to F^*$.*

Proof. The functor F^* is α-exact by construction. Now let $F \to G$ be a map such that G is α-exact. In each step the map $F_\nu \to G$ factors through $F_\nu \to F_{\nu+1}$. Thus $F \to G$ factors through $F \to F^*$. □

Corollary 6.3. *Let $\mathcal{C}$ be a small abelian category with α-products for some regular cardinal α. Then the α-exact functors form a covariantly finite subcategory in the category of all additive functors $\mathcal{C} \to \mathrm{Ab}$.*

Proof. Combine Lemma 1.1 and Lemma 6.2. □

We continue with the problem of finding a faithful α-exact functor $\mathcal{C} \to \mathrm{Ab}$.

Proposition 6.4. *Let $\mathcal{C}$ be a small abelian category with α-products for some regular cardinal α and let $\mathcal{A} = \mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$. Then the following are equivalent:*

1. *There exists a faithful α-exact functor $\mathcal{C} \to \mathrm{Ab}$.*
2. *For every $F \in \mathcal{A}$ the canonical map $F \to F^*$ is a monomorphism.*
3. *Every $F \in \mathcal{A}$ is subobject of some α-injective object G, that is, $\mathrm{Ext}^1(-, G)$ vanishes on α-presentable objects.*

Proof. (1) ⇒ (2): Let $G\colon \mathcal{C} \to \mathrm{Ab}$ be faithful and α-exact. Suppose first that $F = (X, -)$ and let K be the kernel of $\phi\colon F \to F^*$. A non-zero element in KY gives rise to a non-zero map $\psi\colon (Y, -) \to F$ and we get with $F' = \operatorname{Im} \psi$ an exact sequence $0 \to F' \to F \to F'' \to 0$ of representable functors with $F' \subseteq K$. If $F' = (X', -)$, then a non-zero element in GX' gives rise to a non-zero map $F' \to G$ which extends to a map $F \to G$ by Lemma 1.4 since G is exact. This map factors through $F \to F^*$ by Lemma 6.2 and therefore $K = 0$. Now let $F \in \mathcal{A}$ be arbitrary and write $F = \operatorname{colim}_{i \in \mathcal{I}} F_i$ as α-filtered colimit of representable functors. The monomorphisms $F_i \to F_i^*$ induce an α-filtered system of monomorpisms

$$F_i \longrightarrow \prod_{i \to j} F_j^*$$

where $i \to j$ runs through all maps in $\mathcal{I}$ starting in i. This gives a monomorphism

$$\psi\colon F = \operatorname{colim}_i F_i \longrightarrow \operatorname{colim}_i \prod_{i \to j} F_j^*$$

into an α-exact functor since each F_j^* is exact. The map ψ factors through $\phi\colon F \to F^*$ by Lemma 6.2 and we conclude that ϕ is a monomorphism.

(2) $\Rightarrow$ (3): An object $G \in \mathcal{A}$ is α-injective if and only if it is α-exact by Lemma 1.4. Therefore $F \in \mathcal{A}$ is subobject of some α-injective object G if we take $G = F^*$.

(3) $\Rightarrow$ (1): Let $\mathcal{C}_0$ be a representative set of objects in $\mathcal{C}$ and choose for each $X \in \mathcal{C}_0$ a monomorphism $(X, -) \to G_X$ such that G_X is α-injective. Then $G = \prod_{X \in \mathcal{C}_0} G_X$ is a faithful α-exact functor $\mathcal{C} \to \mathrm{Ab}$. $\square$

A small abelian category $\mathcal{C}$ with α-products admits a faithful α-exact functor $\mathcal{C} \to \mathrm{Ab}$ if $\mathcal{C}$ has enough projectives or if $\alpha = \aleph_0$. In the first case use $\prod_P (P, -)$ where P runs through a representative set of projectives; in the second case take an injective cogenerator in $\mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$. It is clear that α-products in $\mathcal{C}$ are exact if there exists a faithful α-exact functor $\mathcal{C} \to \mathrm{Ab}$. However this condition is not sufficient. In fact, Neeman constructs in [10] an example which leads to a small abelian category with $\aleph_1$-products which are exact but which does not admit an $\aleph_1$-exact embedding into the category of abelian groups.

Note that every small abelian category $\mathcal{C}$ with exact α-products arises as quotient $\widehat{\mathcal{A}_\alpha}/\mathcal{S}$ for some locally α-presentable additive category $\mathcal{A}$ and some Serre subcategory $\mathcal{S}$ of $\widehat{\mathcal{A}_\alpha}$ closed under α-products, by taking $\mathcal{A} = \mathrm{Lex}_\alpha(\mathcal{C}, \mathrm{Ab})$.

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