On linear row-finite systems of stochastic differential equations

T. S. Rybnikova

Chair of Function Theory and Functional Analysis,
Department of Mechanics and Mathematics,
Moscow State University, 119899 Moscow, Russia,
e-mail: tatiana@rybnikov.mccme.ru

Abstract

Solvability of autonomous and non-autonomous stochastic linear differential equations in $\mathbb{R}^\infty$ is studied. The existence of strong continuous (L_p-continuous) solutions of autonomous linear stochastic differential equations in $\mathbb{R}^\infty$ with continuous (L_p-continuous) right hand sides is proved. Uniqueness conditions are obtained. We give examples showing that both deterministic and stochastic linear non-autonomous differential equations with the same operator in $\mathbb{R}^\infty$ may fail to have a solution for some initial values. We also establish existence and uniqueness conditions for such equations.

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1 Introduction

In this paper we consider infinite systems of linear stochastic differential equations of the form

$$dX_n(t) = dW_n(t) + \sum_{k=1}^{m_n} a_{nk}(t) X_k(t)\, dt.$$

Such systems appear in many applications. A natural state space for such a system is $\mathbb{R}^\infty$, the space of all real sequences. This paper is devoted to the solvability of autonomous and non-autonomous stochastic linear differential equations in $\mathbb{R}^\infty$. The problem is non-trivial as the space is not Banach. In Banach spaces, linear ordinary differential equations and linear stochastic differential equations with continuous operators and continuous (respectively, L_p-continuous) right hand sides are uniquely solvable. In non-normable locally convex spaces even in complete metrisable, the situation is different (see [1]). Namely, linear ordinary differential equations and linear stochastic differential equations with continuous operators need not be solvable (see Example 4.6). On the other hand, they may have infinitely many solutions (see Example 3.7).

Fortunately, $\mathbb{R}^\infty$ possesses special properties. For example, autonomous row-finite systems of ordinary linear differential equations are solvable for any initial condition (see [9]). The same is true for stochastic linear equations in $\mathbb{R}^\infty$ with continuous (respectively, L_p-continuous) right hand sides (see [8]). It should be noted, however, that $\mathbb{R}^\infty$ is quite special in this respect: there exist Fréchet spaces in which not every linear ordinary or stochastic autonomous differential equation with a continuous operator has a solution (see [1]). In section 3, for completeness and the reader's convenience we give the details of the proof that were not included in the short note [8].

Other results of this paper are concerned with non-autonomous systems and relations between the ordinary and stochastic equations with the same operator. We give an example showing that a linear non-autonomous stochastic differential equation in $\mathbb{R}^\infty$ may be unsolvable and establish existence and obtain uniqueness conditions for such systems. This generalizes results of G. Herzog [5] obtained in the deterministic case.

2 Notation and terminology

Let $\mathbb{R}^\infty$ be the space of all real valued sequences endowed with the topology of the coordinatewise convergence and let $\mathbb{R}^\infty_0$ be its dual space, i.e., the space of all finite sequences. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathbb{R}_+ = [0; +\infty)$. Given topological spaces E and F, we denote by $C(E, F)$ the space of all continuous mappings from E to F. As usual, $C[a, b]$ stands for the Banach space of all continuous real valued functions on $[a;b]$ with the norm $||f||_C = \max_{t\in[a;b]} |f(t)|$. If E is a topological vector space then E^* denotes its dual space, and $\mathcal{L}(E)$ denotes the space of all linear continuous operators on E. Given two real numbers a and b, let $a \wedge b = \min(a,b)$. A stochastic process $\xi(t,\omega),\ t \in \mathbb{R}_+$, on $(\Omega, \mathcal{F}, P)$ with values in a topological space is said to be continuous if almost all trajectories $\xi(\cdot,\omega)$ are continuous. A mapping $A : \mathbb{R}_+ \to \mathcal{L}(\mathbb{R}^\infty)$ is said to be continuous if $A(t)x$ continuously depends on t for each $x \in \mathbb{R}^\infty$.

Let E be a locally convex space and let H be a separable Hilbert space embedded in E continuously and densely. For any $v \in E^*$ the functional $h \mapsto \langle v, h\rangle$ is continuous on H. Hence there is a vector $j(v) \in H$ such that for all $h \in H$ we have $\langle k, h\rangle = (j(k), h)_H$. Thus we get a natural embedding $j : E^* \to H$.

Definition 2.1 *A continuous stochastic process $W(t),\ t \geq 0$, on a probability space $(\Omega, \mathcal{F}, P)$ with values in E is called a Wiener process associated with H if, for each $v \in E^*$ with $\|j(v)\|_H = 1$, the one-dimensional process $\langle k, W_t\rangle$ is Wiener.*

In our case $E = \mathbb{R}^\infty$. If $H = l^2$, then for each sequence of independent Wiener processes $W_n(t)$ the process $W(t) = \big(W_n(t)\big)_{n=1}^\infty$ is Wiener in the sense of Definition 2.1. For more details about infinite dimensional Wiener processes see [2], [3].

Let $\mathcal{F}_t, t \in \mathbb{R}_+$, be an increasing family of σ–fields. We will assume that all sets of measure zero are included in $\mathcal{F}_0$. Let us recall that a process $\xi(t,\omega) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called $\mathcal{F}_t$-adapted if, for each $t \in \mathbb{R}_+$, the function $\omega \to \xi(t,\omega)$ is $\mathcal{F}_t$–measurable. A stochastic process $\xi(\cdot,\cdot) = \big(\xi_n(\cdot,\cdot)\big)$ on $\mathbb{R}_+ \times \Omega$ taking values in $\mathbb{R}^\infty$ is said to be adapted if all the coordinate processes $\xi_n(\cdot,\cdot)$ are adapted. The σ–field $\mathcal{P}_\infty$ on $[0;+\infty) \times \Omega$ generated by sets of the form

$$[s;t] \times F,\ \text{where } 0 \leq s < t < \infty, F \in \mathcal{F}_s, \tag{2.1}$$

is called the predictable σ–field. The restriction of the σ–field $\mathcal{P}_\infty$ to $[0;T]\times\Omega$ will be denoted by $\mathcal{P}_T$. A stochastic process $\xi(\cdot,\cdot)$ on $[0;+\infty)\times\Omega$ (respectively, on $[0;T]\times\Omega$) with values in $\mathbb{R}^1$ is called predictable if it is $\mathcal{P}_\infty$–measurable (respectively, $\mathcal{P}_T$–measurable). A stochastic process $\xi(\cdot,\cdot) = \big(\xi_n(\cdot,\cdot)\big)$ on $[0;+\infty) \times \Omega$ (or on $[0;T] \times \Omega$) taking values in $\mathbb{R}^\infty$ is said to be predictable if all the coordinate processes $\xi_n(\cdot,\cdot)$ are predictable. A predictable process is necessarily adapted. An arbitrary adapted continuous scalar process is predictable (see [3], section 3.3).

Along with continuous stochastic processes we consider $\mathbb{R}^\infty$-valued L^p-continuous processes $\xi(\cdot) = (\xi_n(\cdot))_{n=1}^\infty$ for $p \geq 1$, i.e., processes satisfying for all $n \in \mathbb{N}$ the condition

$$\lim_{t \to s} \mathbb{E}|\xi_n(t) - \xi_n(s)|^p = 0.$$

In other words, the map $[0;T] \to L^p(\Omega, \mathcal{F}, P)$, $t \mapsto \xi_n(t, \cdot)$, is continuous for all $n \in \mathbb{N}$. The limit in L^p of the random variables $(\xi_{t+h} - \xi_t)/h$ as $h \to 0$ is said to be the derivative at the point t of the stochastic process ξ_t. The limit in L^p of the Riemann sums is called the integral of the stochastic process; the limit exists for every L^p-continuous process (see [11, ch. 2.1.4]). Note that if a stochastic process ζ_t, $t \in [a;b]$, is $\mathcal{F}_t$-adapted and L^p-continuous, then, for almost all ω, we have $\Big(\big(L^p\big)\int_a^b \zeta_t\, dt\Big)(\omega) = \int_a^b \zeta_t(\omega)\, dt$, where the left hand side is an L^p-integral, and the right hand side is an integral defined pathwise (see [11, ch. 2.1.4]). It follows that if $\zeta(\cdot)$ is a $\mathcal{F}_t$-adapted L^p-continuous process then $\big(L^p\big)\int_a^t \zeta_s\, ds$ is $\mathcal{F}_t$-adapted too. Obviously, the right-hand side of the latter equality is $\mathcal{F}_t$-measurable; hence, so is the left hand side. For more details, see [11, ch. 2.1.4].

Suppose $f \in C(\mathbb{R}_+, \mathbb{R}^\infty)$ and let $\xi(t)$, $t \in \mathbb{R}_+$, be an adapted stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^\infty$. Let us remark that an operator A is in $\mathcal{L}(\mathbb{R}^\infty)$ if and only if its matrix A is row-finite. Suppose that we are given a continuous mapping $A : \mathbb{R}_+ \to \mathcal{L}(\mathbb{R}^\infty)$.

Definition 2.2 *A process $X(t) = \big(X(t,\omega),\, \omega \in \Omega,\ t \in \mathbb{R}_+\big)$ with values in $\mathbb{R}^\infty$ is called a strong solution of the equation*

$$dX(t) = A(t)X(t)\, dt + f(t)\, dt + d\xi(t),\ X(0) = \xi(0), \tag{2.2}$$

if it is adapted and, for a.a. ω, satisfies coordinatewise the equation

$$X(t,\omega) = \int_0^t A(s)X(s,\omega)\, ds + \int_0^t f(s)\, ds + \xi(t,\omega),\ \forall t \geq 0, \tag{2.3}$$

i.e., $$X_n(t,\omega) = \int_0^t \sum_{i=1}^\infty a_{ni}(s)X_i(s,\omega)\, ds + \int_0^t f_n(s)\, ds + \xi_n(t,\omega),\ \forall t \geq 0.$$

In particular, the coordinates of the process $AX(s,\omega)$ (i.e., the functions $\sum_{i=1}^\infty a_{ni}(s)X_i(s,\omega)$) must be locally integrable with respect to s for a.a ω. Certainly we could write (2.2) in the following compact form

$$dX(t) = AX(t)\, dt + d\zeta(t)$$

with $\zeta(t) = \int_0^t f(s)\,ds + \xi(t,\omega)$. Nevertheless in many natural cases (for example if $\xi(\cdot,\cdot)$ is a Wiener process) it is convenient to single out the random component $\xi(\cdot,\cdot)$ and to write our equation in form (2.2).

We say that a sequence of real valued continuous stochastic processes $u_n(t,\omega)$ converges in probability to a process $u(t,\omega)$ uniformly with respect to $t \in [0;T]$ if

$$P\{\omega : \|u_n(\cdot,\omega) - u(\cdot,\omega)\|_C > a\} \to 0$$

for each $a > 0$ as $n \to \infty$ (in other words the corresponding $C[0;T]$-valued random elements converge in probability).

3 Autonomous systems

Lemma 3.1 (i) *For each continuous adapted process*

$$\xi(t,\omega) : [0;T] \times \Omega \to \mathbb{R}$$

there exists a sequence $u_n(t,\omega)$ of adapted processes with smooth trajectories $u_n(\cdot,\omega)$ such that $u_n(t,\cdot) \to \xi(t,\cdot)$ in probability uniformly with respect to $t \in [0;T]$ and, in addition, for a.a. ω, $u_n(0,\omega) = \xi(0,\omega)$, $u_n^{(k)}(0,\omega) = 0$ and all $k,n \in \mathbb{N}$.

(ii) *For each L^p-continuous adapted real valued process $\xi(t)$, $t \in [0;T]$, there exists a sequence $u_n(t)$ of L^p-smooth adapted real valued processes such that $u_n(t) \to \xi(t)$ in $L^p(\Omega,\mathcal{F},P)$ uniformly with respect to $t \in [0;T]$ and, in addition, $u_n(0) = \xi(0)$, $u_n^{(k)}(0) = 0$ for each $k \in \mathbb{N}$.*

Proof. Consider a sequence p_n of smooth probability densities on $\mathbb{R}$ such that $\operatorname{supp}(p_n) \subseteq [0;\frac{1}{n}]$. We set

$$\begin{aligned} u_n(t,\omega) = \big(\xi(\cdot,\omega) * p_n\big)(t) &= \int_0^{\frac{1}{n}} \xi(t-s,\omega)p_n(s)\,ds \\ &= \int_{t-\frac{1}{n}}^t \xi(z,\omega)p_n(t-z)\,dz. \end{aligned} \tag{3.1}$$

for all ω such that the integrals exist and we put $u_n(t,\omega) = \xi(0,\omega)$ otherwise. Note that as $\xi(\cdot,\cdot)$ is a continuous process the integrals are well defined for a.a. ω. We set $\xi(t,\omega) \equiv \xi(0,\omega)$ for $t < 0$. Then the trajectory $u_n(\cdot,\omega)$ is smooth, $u_n(0) = \xi(0)$, $u_n^{(k)}(0) = 0$ for each $\omega \in \Omega$ and all $k \in \mathbb{N}$. Since

$$\int_{-\infty}^{+\infty} p_n(t)\,dt = 1 \quad \text{and} \quad \operatorname{supp} p_n = [0;\frac{1}{n}],$$

we see that $||u_n(\cdot,\omega)-\xi(\cdot,\omega)||_C \to 0$ as $n\to\infty$ for a.a. $\omega\in\Omega$. It follows that $P\big\{\|u_n(\cdot,\omega)-\xi(\cdot,\omega)\|_C > a\big\} \to 0$ as $n\to\infty$ for each $a>0$. Finally, (3.1) implies that u_n is $\mathcal{F}_t$-adapted.

In the case of a L^p-continuous process consider the similar maps $u_n(t)$ and ξ_t on $[0;T]$ with values in the Banach space L^p, and conclude that $u_n(t)\to\xi_t$ in L^p uniformly with respect to $t\in[0;T]$. In addition, the function $u_n(t)$ is smooth because of the smoothness of p_n. It remains to note that by (3.1) the process $u_n(t)$ is adapted and $u_n(0)=\xi_n(0),\ u_n^{(k)}(0)=0$ for all $k\in\mathbb{N}$. The lemma is proved.

The following lemma generalizes a result from [9] proved in the deterministic case (i.e., $\xi=0$). In the stochastic case, some additional technicalities arise due to the measurability issues.

Lemma 3.2 *The equation whose coordinate form is*

$$dX_n(t)=X_{n+1}(t)\,dt+d\xi_n(t)+f_n(t)\,dt,\quad X_n(0)=\xi_n(0), \tag{3.2}$$

has a strong solution $X(t,\omega)$ for each $f\in C(\mathbb{R}_+,\mathbb{R}^\infty)$ and for each $\mathcal{F}_t$–adapted continuous (L^p-continuous) $\mathbb{R}^\infty$-valued stochastic process $\xi(t),\ t\in\mathbb{R}_+$. In addition, $X(t,\omega)$ is a continuous (respectively, L^p-continuous) process.

Proof. We construct a solution on the interval $[0;T]$. To simplify calculations we assume that $T=1$.

I. There exist two sequences $\{\varphi_n\}$ and $\{\psi_n\}$ of $\mathcal{F}_t$–adapted processes, satisfying the following conditions for a.a. ω:

$$\varphi_n(\cdot,\omega)\in C^{(n-1)}([0;1])\ \forall n\in\mathbb{N};\ \varphi_1(0)\equiv\xi_1(0),$$
$$\varphi_n^{(k)}(0)\equiv 0,\ 0\le k\le n-1,\ \forall n\ge 2; \tag{3.3}$$
$$P\{\|\varphi_n^{(k)}(\cdot,\omega)\|_C>2^{-n}\}\le 2^{-n},\ \forall n\ge 2,\ 0\le k\le n-1,\ t\in[0;1]; \tag{3.4}$$
$$\psi_n(\cdot,\omega)\in C^\infty([0;1]),\ \psi_n(0)\equiv\xi_{n+1}(0),\ n\in\mathbb{N};\ \psi_0\equiv 0; \tag{3.5}$$
$$\psi_n(t,\omega)=(\varphi_n^{(n-1)}(t,\omega)-\xi_n(t,\omega))'-f_n(t)+\psi'_{n-1}(t,\omega),\ n\in\mathbb{N}. \tag{3.6}$$

Assume for a moment that such sequences are already constructed. Then the solution $X(t,\omega)=\big(X_1(t,\omega),X_2(t,\omega),\dots\big)$ of equation (3.2) can be obtained in the following way: using (3.4) we see that the series $\sum\limits_{n=1}^{\infty}\varphi_n(\cdot,\omega)$ converges in probability uniformly with respect to $t\in[0;1]$.

Set

$$X_1(\cdot,\omega) = \begin{cases} \sum\limits_{n=1}^{\infty} \varphi_n(\cdot,\omega) & \text{if the series converges,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the series converges for a.a. ω. For $n \geq 2$ we define the process X_n by

$$X_n(\cdot,\omega) = \begin{cases} \big(X_{n-1}(t,\omega) - \xi_{n-1}(t,\omega)\big)' - f_{n-1}(t) & \begin{array}{l}\text{if the derivative} \\ \text{is well defined,}\end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Using (3.6), one can check by induction that $\big(X_{n-1}(t,\omega) - \xi_{n-1}(t,\omega)\big)'$ exists for a.a. ω. In addition,

$$X_n(t,\omega) = \psi_{n-1}(t,\omega) + \sum_{j=n}^{\infty} \varphi_j^{(n-1)}(t,\omega), \tag{3.8}$$

under the assumption that the series $\sum_{j=n}^{\infty} \varphi_j^{(n-2)}(t,\omega)$ can be differentiated termwise. Let us prove that this assumption is satisfied. Fix $n \in \mathbb{N}$. The sequence of the partial sums $S_l = \sum_{j=n}^{l} \varphi_j^{(n-1)}$, $l \geq n$, is a Cauchy sequence in probability. Indeed, it follows from (3.4) that for each $a > 0$, $m \in \mathbb{N}$ such that $a \geq 2^{-m}$ and any positive integers $l \geq i \geq \max(m,n)$, we have

$$\begin{aligned} P\big\{\|S_l - S_i\|_C > a\big\} &\leq P\big\{\|S_l - S_i\|_C > 2^{-m}\big\} \\ &= P\Big\{||\sum_{j=i+1}^{l} \varphi_j^{(n-1)}||_C > 2^{-m}\Big\} \leq P\Big\{\sum_{j=i+1}^{l} ||\varphi_j^{(n-1)}||_C > 2^{-m}\Big\} \\ &\leq \sum_{j=i+1}^{l} P\Big\{||\varphi_j^{(n-1)}||_C > 2^{-j}\Big\} \leq \sum_{j=i+1}^{l} 2^{-j} < 2^{-i}. \end{aligned} \tag{3.9}$$

Therefore, the series $\sum_{j=n}^{\infty} \varphi_j^{(n-1)}(\cdot,\omega)$ converges in probability uniformly with respect to $t \in [0;1]$ for all $n \in \mathbb{N}$. Put

$$F(\cdot,\omega) = \sum_{j=n}^{\infty} \varphi_j^{(n-1)}(\cdot,\omega), \; S(\cdot,\omega) = \sum_{j=n}^{\infty} \varphi_j^{(n-2)}(\cdot,\omega).$$

We shall regard the processes $S(\cdot,\omega)$, $F(\cdot,\omega)$, $\varphi_j^{(n-1)}(\cdot,\omega)$, $\varphi_j^{(n-2)}(\cdot,\omega)$ as random elements with values in $C[0;1]$ and denote them by the same symbols without arguments. Since $\sum_{j=n}^{l}\varphi_j^{(n-1)} \to F$ in probability as $l\to\infty$ and $\sum_{j=n}^{k}\varphi_j^{(n-2)} \to S$ in probability as $k\to\infty$, by the Riesz theorem, there exist subsequences of partial sums $F_{l_i} = \sum_{j=n}^{l_i}\varphi_j^{(n-1)}$ and $S_{k_i} = \sum_{j=n}^{k_i}\varphi_j^{(n-2)}$, which converge almost everywhere to F and S, respectively. Putting $m_i = \max l_{k_i}$ we have $F_{m_i}\to F$ and $S_{m_i}\to S$ for a.a. ω, in other words, the trajectories of the processes converge uniformly for a.a. ω. Clearly, $F_{m_j}(t,\omega) = S'_{m_j}(t,\omega)$ for a.a. ω. Differentiating the sequences, we have $\sum_{j=n}^{\infty}\varphi_j^{(n-1)}(\cdot,\omega) = \Big(\sum_{j=n}^{\infty}\varphi_j^{(n-2)}(\cdot,\omega)\Big)'$ for a.a. ω. This proves (3.8).

In order to prove that X_n, $n\in\mathbb{N}$, are adapted let us set

$$F(\cdot,\omega) = \begin{cases} \lim_{i\to\infty} F_{m_i}(\cdot,\omega) & \text{if the limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that by our construction the sequence F_{m_i} converges uniformly with respect to t for almost all ω. Fix $t\in[0;1]$. By the construction, $\varphi_n^{(k)}(t)$ for $k\le n-1$ are $\mathcal{F}_t$-adapted. Therefore, F_{m_i} are $\mathcal{F}_t$–adapted. The set of all ω such that the subsequence $F_{m_i}(t,\omega)$ has a limit is $\mathcal{F}_t$–measurable. The above modification F is an $\mathcal{F}_t$-adapted process, as well as $X_n = \psi_{n-1} + F$. Together with (3.7) this yields that $X = (X_n)_{n=1}^{\infty}$ is a solution of equation (3.2). We obviously have from (2.3) that X is continuous.

II. It remains to prove that $\{\varphi_n\}$ and $\{\psi_n\}$ exist. Set $\varphi_1 = \xi_1(0)$ and $\psi_0\equiv 0$. To construct φ_n for all $n\ge 2$, first represent $f_n(t)$ on $[0;1]$ as a sum $f_n(t) = \widetilde{f}_n(t) + \widehat{f}_n(t)$, with smooth $\widetilde{f}_n(t)$, and $\widehat{f}_n(t)$ such that $|\widehat{f}_n(t)| \le \frac{1}{3\cdot 2^n}$ on $[0;1]$. Then, by using Lemma 1, find u_n with smooth trajectories such that

$$P\Big\{\|u_n(\cdot,\omega) - \xi_n(\cdot,\omega)\|_C > \frac{1}{3}\cdot 2^{-n}\Big\} \le 2^{-n},$$

and pick adapted process $g_n(\cdot,\omega)$ with smooth trajectories on $[0;1]$ satisfying for all ω the following conditions: $g_n(0)\equiv 0$, $g_n'(0,\omega) = \eta(\omega)$ and $\|g_n(\cdot,\omega)\|_C < K(n)$, where

$$\eta(\omega) = \xi_{n+1}(0) + \widetilde{f}_n(0) - \psi'_{n-1}(0) \quad \text{and} \quad K(n) = \frac{1}{3\cdot 2^n}.$$

We can take for g_n the function $g_n(t,\omega) = \frac{K(n)}{2}\arctan\frac{2t\eta(\omega)}{K(n)}$. Then the process φ_n, defined by the equation

$$\varphi_n^{(n-1)}(t,\omega) = \xi_n(t,\omega) - u_n(t,\omega) + \int_0^t \widehat{f}_n(\tau)d\tau + g_n(t,\omega),$$
$$\varphi_n^{(k)}(0) \equiv 0,\ 0 \le k \le n-1, \quad (3.10)$$

satisfies condition (3.4). Indeed,

$$P\{\|\varphi_n^{(n-1)}(\cdot,\omega)\|_C > 2^{-n}\} \le P\Big\{\|\xi_n(\cdot,\omega) - u_n(\cdot,\omega)\|_C > \frac{1}{3}\cdot 2^{-n}\Big\} +$$
$$P\Big\{\Big|\int_0^t \widehat{f}_n(u)du\Big| > \frac{1}{3}\cdot 2^{-n}\Big\} + P\Big\{\|g_n(\cdot,\omega)\| > \frac{1}{3}\cdot 2^{-n}\Big\} =$$
$$P\Big\{\|\xi_n(\cdot,\omega) - u_n(\cdot,\omega)\|_C > \frac{1}{3}\cdot 2^{-n}\Big\} \le 2^{-n},$$

and $\|\varphi_n^{(k-1)}(\cdot,\omega)\|_C = \max_{t\in[0;1]}\Big|\int_0^t \varphi_n^{(k)}(s,\omega)ds\Big| \le \|\varphi_n^{(k)}(\cdot,\omega)\|_C$ for $k = 1,2,\dots,n-1$ for all ω such that the first norm is well defined, i.e., for a.a. ω. Knowing $\psi_{n-1}(\cdot,\omega)$ and $\varphi_n(\cdot,\omega)$ and using (3.6) we obtain $\psi_n(\cdot,\omega)$. Hence

$$\psi_n(t,\omega) = -u_n'(t,\omega) - \widetilde{f}_n(t) + g_n'(t,\omega) + \psi_{n-1}(t,\omega)$$

is a process with a.a. smooth trajectories, in addition, $\psi_n(0) \equiv \xi_{n+1}(0)$, $\varphi_n^{(k)}(0) \equiv 0$ for $0 \le k \le n-1$. Finally, φ_n and ψ_n satisfy conditions (3.3)–(3.6). To complete the proof it remains to note that according to (3.10) the process $\varphi_n^{(k)}$ for $k \le n-1$ is adapted as well as ψ_n, $n \in \mathbb{N}$, according to (3.6).

For the case of L^p-continuous processes the proof is similar and is omitted. The lemma is proved.

The following lemma is well known and proved by standard iteration arguments. For some related results see [1, Theorem 6.15], [10].

Lemma 3.3 *Let E be a separable Banach space. Suppose that $A \in \mathcal{L}(E)$, $f \in C(\mathbb{R}_+, E)$ and that $\xi(t), t \in \mathbb{R}_+$, is continuous (or L^p-continuous), adapted E-valued process. Then the equation*

$$dX(t) = AX(t)\,dt + f(t)\,dt + d\xi(t), \quad X(0) = \xi(0), \quad (3.11)$$

has a unique strong solution X taking values in E. In addition, X is continuous (respectively, L^p-continuous).

We shall employ the following lemma which can be obtained as a by-product result of the reasoning in [9]. However, for application to stochastic equations we need a more specific formulation.

Lemma 3.4 *Let $A \in \mathcal{L}(\mathbb{R}^\infty)$. Then there exists a bijective operator $C \in \mathcal{L}(\mathbb{R}_0^\infty)$ with $C^\top \in \mathcal{L}(\mathbb{R}^\infty)$ such that $C^\top A (C^\top)^{-1}$ has the form*

$$A' = \begin{pmatrix} A_1 & 0 & & & \cdots & 0 \\ B_{21} & A_2 & 0 & & \cdots & 0 \\ B_{31} & B_{32} & A_3 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{3.12}$$

where A_i are either infinite dimensional matrices of the form

$$A_i = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

or finite dimensional matrices and B_{ij} are blocks (finite or infinite).

Proof. Let $\{e_k\}_{k=1}^\infty$, $e_k = (0, \dots, 0, 1, 0, \dots)$, be the standard basis in $\mathbb{R}_0^\infty$. Let us construct a new basis $v_i = Ce_i, i \in \mathbb{N}$, where C is such that $C^\top A (C^\top)^{-1}$ has the form (3.12). The process of changing the basis is similar to the one described in [9]. The process is divided in one or several (may be countably many) steps. Denote by v_i^n the i-th vector chosen at the n-th step. At the same time we construct the matrix A'. At the n-th step we construct the blocks $A_n, B_{n,1}, \dots, B_{n,n-1}$.

Step 1. First put $v_1^1 = e_1$. If $A^\top v_1^1 \in Lin\{v_1^1\}$ then $A_1 = \alpha$, where $\alpha \in \mathbb{R}$ is such that $A^\top v_1^1 = \alpha v_1^1$. Then $M_1 = Lin\{v_1^1\}$ and the step is completed. Otherwise, $A^\top v_1^1 \notin Lin\{v_1^1\}$ and we set $v_2^1 = A^\top v_1^1$. Recall that there are only finitely many nonzero elements in each row of the matrix A. Hence $A^\top$ is column-finite and $v_2^1 \in \mathbb{R}_0^\infty$. In this case the procedure is continuing. Let us describe inductively the continuation. Assume that the step is not completed and that we have already constructed vectors $v_i^1 \in \mathbb{R}_0^\infty$, for $i \le j$. Denote $M = Lin(v_1^1, \dots, v_j^1)$.

If $A^\top v_j^1 \notin M$, then set $v_{j+1}^1 = A^\top v_j^1$. Since $v_j^1 \in \mathbb{R}_0^\infty$ and $A^\top$ is column-finite, $v_{j+1}^1 \in \mathbb{R}_0^\infty$. We obtain one more row in the block A_1 and in the matrix A': this row has number j, its elements vanish except for the $(j+1)$-th element equal to 1. We will now return to our original starting point as explained above. If

$$A^\top v_1^j \in M,$$

then the j-th row of the block A_1 and matrix A' is the expansion of the foregoing sum with respect to the basis $(v_1^1, \dots, v_j^1)$, i.e. $(\alpha_1, \dots, \alpha_j)$ such that $A^\top v_j^1 = \sum_{i=1}^{j} \alpha_i v_i^1$. In this case the step is completed.

If $Lin\{v_i^1\} = \mathbb{R}_0^\infty$ then the process of changing basis in $\mathbb{R}_0^\infty$ is completed. Otherwise we proceed to the next step.

Let us describe the step number n. By the induction hypothesis, we are given subspaces $M_j = Lin\{v_i^j\}$ for $1 \leq j \leq n-1$, where vectors v_i^j are linearly independent and their number (possibly infinite) depends on j. Denote $L_{n-1} = Lin\{M_1 \dots M_{n-1}\}$. By the inductive hypothesis, we have that $e_1, ..., e_{n-1} \in L_{n-1}$ and that the blocks A_j and B_{ji} are already constructed for $j \leq n-1, i < j$.

Step n**.** Set $v_1^n = e_m$, where $m = \min\{k : \ e_k \notin L_{n-1}\}$. Now we construct v_j^n by induction on j. Fix $j \in \mathbb{N}$. Assume that the vectors $v_i^n \in \mathbb{R}^\infty, i \leq j$, are already defined and $M_n = Lin\{v_i^n,\}$, $i \leq j$. If $A^\top v_j^n \notin M = Lin(L_{n-1} \bigcup v_1^n, \dots, v_j^n)$, then put

$$v_{j+1}^n = A^\top v_j^n.$$

By the same reason as above, $v_{j+1}^n \in \mathbb{R}_0^\infty$.

We obtain one more row in the block A_n. This row has number j and its elements are zeros except for the $(j+1)$-th element which is equal to 1. This row with other zero elements will be a new row of the matrix A'. All the elements of the j-th rows of the blocks $B_{n,1}, \dots B_{n,n-1}$ are zeros. Now we return to the situation of the beginning of step n. If $A^\top v_j^n \in M$, then the next row of the matrix A' is the vector

$$(\beta_{11}, \dots \beta_{ij}, \dots, \alpha_1, \dots, \alpha_j)$$

such that $A^\top v_j^n = \sum_{l=1}^{n-1} \sum_i \beta_{li} v_i^l + \sum_{i=1}^{j} \alpha_i v_i^n$. Note that the first sum is the projection of the vector $A^\top v_j^n$ onto L_{n-1}, the second one is its projection onto $M_n = Lin(v_1^n, \dots, v_j^n)$. The j-th row of A_n is $(\alpha_1, \dots, \alpha_j)$, the j-th row of $B_{n,k}$, $1 \leq k \leq n-1$, is $(\beta_{k1}, \dots, \beta_{kl}, \dots)$, $l \in \{i, \ e_i \in M_k\}$. Now the step is completed.

If $Lin\{M_1, \dots, M_n\} = \mathbb{R}_0^\infty$ then the process of changing basis in $\mathbb{R}_0^\infty$ is completed. Otherwise we proceed to the next step.

The process consists of finitely or countably many steps. As a result we have $H = \{v_k^n\}$ that is an algebraic basis in $\mathbb{R}_\infty$. Indeed, the set

H is linearly independent and its span comprises all $e_m,\ m \in \mathbb{N}$. The matrix C has the desired form as follows by direct calculations. The lemma is proved.

Theorem 3.5 *The equation*

$$dX(t) = AX(t)dt + f(t)dt + d\xi(t), \quad X(0) = X_0, \tag{3.13}$$

has a strong solution for any $A \in \mathcal{L}(\mathbb{R}^\infty), f \in C(\mathbb{R}_+, \mathbb{R}^\infty)$ and for any $\mathcal{F}_t$-adapted continuous (L^p-continuous) stochastic process ξ with values in $\mathbb{R}^\infty$. In addition, $X(t,\omega)$ is a continuous (respectively, L^p-continuous) process.

Proof. Consider the equation

$$dY(t) = A'Y(t)dt + g(t)dt + d\zeta(t), \tag{3.14}$$

obtained from (3.13) by the linear transformation $C^\top$ constructed in Lemma 3.4. In (3.14) A' has the form (3.12). Since $\xi(\cdot)$ is $\mathcal{F}_t$-adapted and continuous (respectively, L^p-continuous), $\zeta(\cdot)$ has the same properties.

Why is system (3.14) solvable? Let A_i act on the subspace M_i. First, the restriction of the solution of (3.14) to M_1 satisfies the system

$$dY^1(t) = A_1Y^1(t)dt + g^1(t)dt + d\zeta^1(t), \tag{3.15}$$

where $g^1(t)$ (respectively, $\zeta^1(t)$) is the restriction of $g(t)$ (respectively, $\zeta(t)$) to M_1. Note that either system (3.15) is finite or satisfies the conditions of Lemma 3.2. In both cases it has a strong solution $Y^1(\cdot)$. Since we already know the coordinates Y_i^1 of the solution in the subspace M_1, we can substitute them in equation (3.14). Now, in equation (3.14) all the elements of the blocks B_{i1} located below A_1 are multiplied by the known functions Y_i^1. Thus we are able to solve the equation on the subspace M_2. In this case we obtain the system

$$dY^2(t) = A_2Y^2(t)dt + g^2(t)dt + d\zeta^2(t) + B_{21}Y^1(t)dt, \tag{3.16}$$

where $g^2(t)$ and $\zeta^2(t)$ are the restrictions of $g(t)$ and $\zeta(t)$, respectively, to the subspace M_2. System (3.16) is solvable for the same reasons as system (3.15). Now, arguing inductively, let us suppose that we know the restrictions of the solution to the subspaces $\{M_1, \dots, M_{n-1}\}$. Substitute the known coordinates in (3.14). As before we find the solution Y_i^n on M_n for all n such that A_n are defined. Finally, we find a solution of (3.13) by the reverse transformation putting $x = (C^\top)^{-1}y$. The theorem is proved.

Remark 3.6 The above proof uses an idea from [9] (where the deterministic case $\xi = 0$ was considered) and from [7] (where ξ_n are independent Wiener processes). The main difficulty in the stochastic case is to obtain an adapted solution. It is easily seen that, in general, (3.13) may have non-adapted solutions and that an adapted solution of (3.13) may be non-unique.

Example 3.7 Consider the equation

$$x_n'(t) = x_{n+1}(t), x_n(0) = z_n.$$

It is easily seen that for each z this equation has infintely many solutions.

Indeed, by Borel's theorem there exists a C^∞-function ϕ on the line with $\phi^{(n)}(0) = z_n$ for all n. It yields a solution $x_n(t) = \phi^{(n)}(t)$ with $x_n(0) = z_n$. Evidently, such a function ϕ is not unique, and therefore the above equation has infinitely many solutions.

Consider the equation (3.2). Adding any solution of the homogeneous problem with zero initial condition to the solution of (3.2) constructed in Lemma 3.2 we get one more solution of (3.2). Since the homogeneous problem has infinitely many solutions, the same is true for (3.2).

Further, let $W(\cdot,\cdot)$ be a Wiener process on the line. Set

$$x_n(t,\omega) = W(1,\omega) \cdot \phi^{(n)}(t)$$

and add $x(t,\omega) = \big(x_n(t,\omega)\big)_{n=1}^{\infty}$ to the solution of (3.2) constructed in Lemma 3.2. Then we get a non-adapted solution of (3.2).

Remark 3.8 Under the conditions of the theorem, assume that $\xi(t)$ is continuous. Then the solution $X(t)$ of (2.2) is predictable because $X(t)$ is $\mathcal{F}_t$-adapted and continuous. For the same reason $\xi(t)$ is predictable.

Remark 3.9 The following assertions are equivalent:

(a). Equation (2.2) is uniquely solvable.

(b). The equation $dX(t) = AX(t)\,dt$, $X(0) = 0$, is uniquely solvable.

(c). $\exp(tA) = \sum\limits_{k=0}^{\infty} \frac{t^k}{k!} A^k$ is convergent in $\mathcal{L}(\mathbb{R}^\infty)$ for all $t \in \mathbb{R}$.

(d). The spectrum $\sigma(A)$ is at most countable.

(a) is equivalent to (b). Indeed, for all ω the difference between any two solutions of (2.2) with the same initial value is a solution of the equation $dX(t) = AX(t)\,dt$ with zero initial condition. (b) and (c) are equivalent according to [1], example 4.24. Finally, the equivalence of (c) and (d) was proved in [4].

4 Non-autonomous systems

In this section we discuss non-autonomous row-finite systems of linear differential stochastic equations. We give an example showing that such systems may fail to have a solution, while autonomous systems are solvable for any initial condition. We establish existence and uniqueness conditions for such systems.

Lemma 4.1 *Let $A(\cdot) = \big(a_{ij}(\cdot)\big) : [a;b] \to \mathcal{L}(\mathbb{R}^\infty)$ be continuous. Then for each i there exists $N(i)$ such that $a_{ij}(t) \equiv 0$ on $[a;b]$ for each $j \geq N(i)$.*

Proof. Assume the converse. Fix $k \in \mathbb{N}$. Suppose that there exists an infinite sequence of indices i_n, $n \in \mathbb{N}$, such that the elements

$$a_{ki_1}(\cdot), \dots, a_{ki_n}(\cdot), \dots$$

of the k-th row of $A(\cdot)$ are not identically zero. Consider the family of functionals $a_k(t) = (a_{k1}(t), \dots, a_{kn}(t), \dots)$, $t \in [a;b]$, on $\mathbb{R}^\infty$. Since $A(\cdot)$ is continuous on $[a;b]$, (i.e., $\big(A(\cdot)x\big)_k$ is continuous on $t \in [a;b]$ for each $x \in \mathbb{R}^\infty$) the family $a_k(t)$ is uniformly bounded on each $x \in \mathbb{R}^\infty$. Therefore, by the Banach–Steinhaus theorem, the family $a_k(t)$ is uniformly bounded on each compact set in $\mathbb{R}^\infty$. But this family is not bounded on the set

$$x_{i_1} \in \left[\frac{-1}{\|a_{ki_1}\|_C}; \frac{1}{\|a_{ki_1}\|_C}\right], \dots, x_{i_n} \in \left[\frac{-n}{\|a_{ki_n}\|_C}; \frac{n}{\|a_{ki_n}\|_C}\right], \dots,$$

with other zero coordinates x_j, for $j \notin \{i_1, i_2, \dots, i_n \dots\}$. This contradiction completes the proof.

Collorary 4.2 *A function $A(t) = \big(a_{ij}(t)\big) : [a;b] \to \mathcal{L}(\mathbb{R}^\infty)$ is continuous if and only if all functions $a_{ij}(\cdot)$ are continuous and there is only a finite number of elements $a_{ij}(\cdot)$ in each row that are not identically zero on $[a;b]$.*

For a matrix function $Z(t) : [0;T] \to \mathcal{L}(\mathbb{R}_0^\infty)$ and for $y \in \mathbb{R}_0^\infty$, let

$$U(Z,y) = lin\Big(\{Z(t_1)\cdot \ldots \cdot Z(t_k)y : t_1,\ldots,t_k \in [0;T], k \in \mathbb{N}\} \cup \{y\}\Big),$$

be the smallest subspace containing y and invariant for all $Z(t), t \in [0;T]$. We say that Z is uniformly locally algebraic if

$$\dim U(Z,y) < \infty, \forall y \in \mathbb{R}_0^\infty.$$

According to a result of Herzog [5] we have

Theorem 4.3 *Let $A(t) : [0;T] \to \mathcal{L}(\mathbb{R}^\infty)$ be continuous with $A^\top(t) : [0;T] \to \mathcal{L}(\mathbb{R}_0^\infty)$ uniformly locally algebraic. Then the problem*

$$y'(t) = -A^\top(t)y(t), \quad y(t_0) = y_0, \tag{4.1}$$

is uniquely solvable on $[0;T]$ for all $(t_0,y_0) \in [0;T] \times \mathbb{R}_0^\infty$ and for each initial condition there is a differentiable matrix function $M(t) : [0;T] \to \mathcal{L}(\mathbb{R}_0^\infty)$ such that M is uniformly locally algebraic, $M(t)$ is invertible in $\mathcal{L}(\mathbb{R}_0^\infty)$ for all $t \in [0;T]$ and $y(t) = M(t)y_0, t \in [0;T]$, is a solution of (4.1)*. Furthermore, the problem*

$$x'(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0, \tag{4.2}$$

is uniquely solvable on $[0;T]$ for each initial condition $(t_0,y_0) \in [0;T] \times \mathbb{R}^\infty$ for all $f \in C([0;T],\mathbb{R}^\infty)$, and for each initial condition the matrix function $\Lambda(t) : [0;T] \to \mathcal{L}(\mathbb{R}^\infty), \Lambda(t) = (M^{-1})^\top(t)$, is differentiable and

$$x(t) = \Lambda(t)x_0 + \int_{t_0}^{t} \Lambda(t)\Lambda^{(-1)}(s)f(s)\,ds, \quad t \in [0;T],$$

is a solution of (4.2)*.*

Remark 4.4 The mapping $x(t) = \Lambda(t)x_0$ is a solution of the homogeneous problem

$$x'(t) = A(t)x(t), \quad x(t_0) = x_0,$$

corresponding to (4.2).

We now extend this result to stochastic differential equations.

Theorem 4.5 *Let $A(t) : [0;T] \to \mathcal{L}(\mathbb{R}^\infty)$ be continuous with $A^\top(t) : [0;T] \to \mathcal{L}(\mathbb{R}_0^\infty)$ uniformly locally algebraic. Then the problem*

$$dX(t) = A(t)X(t)\,dt + d\xi(t), \quad X(0) = \xi(0), \tag{4.3}$$

is uniquely solvable on $[0;T]$ for each $\mathcal{F}_t$–adapted continuous (respectively, L^p-continuous) $\mathbb{R}^\infty$-valued stochastic process $\xi(t),\ t \in [0;T]$, and the solution is given by

$$X(t,\omega) = \xi(t,\omega) + \Lambda(t) \int_0^t \Lambda^{(-1)}(s)A(s)\xi(s,\omega)\,ds, \quad t \in [0;T]. \tag{4.4}$$

Proof. The uniqueness for problem (4.3) follows from the uniqueness for problem (4.2). Indeed, let $X_1(t,\omega)$ and $X_2(t,\omega)$ be solutions of problem (4.3) with the same initial value. Then $X_2(t,\omega) - X_1(t,\omega)$ satisfies the homogeneous equation (4.2) with zero initial value. According to Theorem 4.3 we have $X_2(t,\omega) - X_1(t,\omega) = 0,\ t \in [0;T]$.

Formula (4.4) is a standart expression for the solution of (4.3), but we have to show that the right-hand side is well defined and is indeed a solution. Here we again need our assomptions on A.

By Theorem 4.3, the matrix functions $\Lambda(\cdot)$ and $\Lambda^{(-1)}(\cdot)$ and $A(\cdot)$ are continuous, hence, by Lemma 4.1, the integral in (4.4) is well defined both in the case of continuous and L^p-continuous process. It follows from (4.4) that $X(t,\omega)$ is $\mathcal{F}_t$–adapted.

To prove that (4.4) is a solution of (4.3), we have to check the equality

$$\xi(t,\omega) + \Lambda(t) \int_0^t \Lambda^{(-1)}(s)A(s)\xi(s,\omega)\,ds = \xi(t,\omega) + \int_0^t A(s)X(s)\,ds. \tag{4.5}$$

This equality is true for $t = 0$. Subtracting $\xi(t,\omega)$ from both sides of (4.5) and differentiating (which is justified by Lemma 4.1), we get

$$\Lambda'(t) \int_0^t \Lambda^{(-1)}(s)A(s)\xi(s,\omega)\,ds + \Lambda(t)\Lambda^{(-1)}(t)A(t)\xi(t,\omega) = A(t)X(t) \tag{4.6}$$

Since $\Lambda'(t) = A(t)\Lambda(t)$ by the construction, we see that (4.6) is equivalent to

$$A(t)\left(\Lambda(t) \int_0^t \Lambda^{(-1)}(s)A(s)\xi(s,\omega)\,ds + \xi(t,\omega)\right) = A(t)X(t).$$

The above equation is true according to (4.4). Hence (4.4) is indeed a solution of (4.3). This completes the proof.

The following example generalizes the example constructed in [5].

Example 4.6 Let $(W_n)_{n=1}^{\infty}$ be independent standard Brownian motions on the line and let $(f_n)_{n=1}^{\infty}$ be a sequence in $C\big([0;T],\mathbb{R}\big)$. Consider the deterministic homogeneous linear equation

$$x_n'(t) = f_n(t)x_{n+1}(t),\ x_n(0) = x_{0n}, \quad n \in \mathbb{N}, \tag{4.7}$$

and the stochastic linear equation

$$\xi_n(t) = \xi_{0n} + \int_0^T f_n(t)\xi_{n+1}(t)dt + n^{-1}W_n(t), \quad n \in \mathbb{N}. \tag{4.8}$$

Then there exists a sequence $(f_n)_{n=1}^{\infty}$ such that both (4.7) and (4.8) are unsolvable for some initial values.

Proof. First, let $(f_n)_{n=1}^{\infty}$ be a sequence in $C\big([0;T],(0;+\infty)\big)$ such that

$$\max \operatorname{supp} f_n \le \min \operatorname{supp} f_{n+1}, \quad n \in \mathbb{N},$$

and $|f_n(t)| < C,\ t \in [0;T], n \in \mathbb{N}$. Then equation (4.7) is uniquely solvable on $[0;T]$ and the solution is given by

$$x_n(t) = x_{0n} + \int_0^t f_n(s)\ ds \cdot x_{0n+1}, \quad t \in [0;T], \quad n \in \mathbb{N}.$$

Setting $\alpha_n = \int_0^T f_n(s)ds, \quad k, n \in \mathbb{N}$, we have

$$x(T) = Bx_0, \quad \text{where} \quad B = \begin{pmatrix} 1 & \alpha_1 & 0 & 0 & \dots \\ 0 & 1 & \alpha_2 & 0 & \dots \\ 0 & 0 & 1 & \alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \in L(\mathbb{R}^\infty).$$

Now let $(t_k)_{k=1}^{\infty}$ be a sequence in $[0;T]$ with $t_1 = 0$ and $t_k < t_{k+1}, k \in \mathbb{N}$, and $t_\infty = \lim_{k\to\infty} t_k$. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $C\big([0;T],(0;+\infty)\big)$ such that

$$\max\big([t_k,t_{k+1}] \cap \operatorname{supp} f_n\big) \le \min\big([t_k,t_{k+1}] \cap \operatorname{supp} f_{n+1}\big), \quad n, k \in \mathbb{N}.$$

We set

$$\alpha_{kn} = \int_{t_k}^{t_{k+1}} f_n(s)ds \; k, n \in \mathbb{N}$$

and

$$B_k = \begin{pmatrix} 1 & \alpha_{k1} & 0 & 0 & \dots \\ 0 & 1 & \alpha_{k2} & 0 & \dots \\ 0 & 0 & 1 & \alpha_{k3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \in L(\mathbb{R}^\infty), \quad k \in \mathbb{N}.$$

Now, problem (4.7) is uniquely solvable on $[0; t_\infty)$ and

$$x(t_{k+1}) = B_k \cdot B_{k-1} \cdots B_1 x_0, \quad k \in \mathbb{N}.$$

Let us prove that there exist initial values $x_0 \in \mathbb{R}^\infty$ such that $x_{k1} = x_1(t_k) \to \infty$ as $k \to \infty$. Since the sequence f_n, $n \in \mathbb{N}$, is uniformly bounded, we see that $\alpha_{kn} \to 0$ as $k \to \infty$ uniformly with respect to $n \in \mathbb{N}$. Let

$$g(n) = \frac{1}{\alpha_{1,n-1} \cdot \alpha_{2,n-2} \cdot \ldots \cdot \alpha_{n-1,1}}.$$

Then $g(n) \to \infty$ as $n \to \infty$. Assume that all the coordinates $x_{0,n}$ of the initial value are nonnegative and there exist infinitely many numbers $i \in \mathbb{N}$ such that $x_{0,n_i} > g(n_i)$. Then

$$\begin{aligned} x_{n_i-1,1} &= x_{0,1} + \alpha_{1,1}x_{0,2} + \alpha_{2,1}(x_{0,2} + \alpha_{1,2}x_{0,3}) + \ldots \\ &\quad + \alpha_{n_i-1,1}(x_{0,2} + \ldots + \alpha_{n_i-2,2} \cdot \alpha_{n_i-3,3} \cdot \ldots \cdot \alpha_{1,n_i-1}x_{0,n_i}) \\ &= x_{n_i-2,1} + \alpha_{n_i-1,1}(x_{0,2} + \ldots + \alpha_{n_i-2,2} \cdot \alpha_{n_i-3,3} \cdot \ldots \cdot \alpha_{1,n_i-1}x_{0,n_i}) \\ &\qquad > x_{n_i-2,1} + 1, \end{aligned}$$

and $x_1(t_{n_i}) \to \infty$. Hence for such an initial value there is no solution of (4.7) on the interval $[0; t_\infty + \varepsilon], \varepsilon > 0$.

We shall use the following estimate which enables us to compare solutions of deterministic and stochastic equations. Let w_t be a Wiener process in $\mathbb{R}^n$ and let ξ_t, η_t satisfy the equations

$$\xi_t = \xi_0 + \int_0^t f(\xi_s, \omega)ds + \alpha w_t,$$

$$\eta_t = \xi_0 + \int_0^t f(\eta_s, \omega)ds + \beta w_t,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a map satisfying the Lipschitz condition with constant L and ξ_0 is a random variable. Then according to [6, Ch. 2.5, Corollary 5] we have

$$\mathbb{E}|\xi_t - \eta_t|^2 \le C(\alpha - \beta)^2 \exp((4L+1)t),$$

where C does not depend on n.

In equations (4.7) and (4.8) the mapping f has the form

$$\begin{pmatrix} 0 & f_1 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & f_2 & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & f_3 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \tag{4.9}$$

and acts as

$$f(t) : (\xi_1, \xi_2, \dots, \xi_j, \dots) \mapsto (f_1(t)\xi_2, f_2(t)\xi_3, \dots, f_k(t)\xi_{k+1}, \dots).$$

The map f satisfies the Lipschitz condition with constant $L = \sup\limits_{j \in \mathbb{N}} f_j(t)$ on $[0;T]$. By assumption, we have $L < \infty$. Let $\xi_n(\cdot)$ be a solution of (4.8) on $[0;T]$. From the above estimate we have

$$\sum_{n=1}^{\infty} \mathbb{E}(\xi_n(t) - x_n(t))^2 \le C \sum_{n=1}^{\infty} n^{-2} \exp((4L+1)t).$$

Hence $\sum_{n=1}^{\infty} \big(\xi_n(t) - x_n(t)\big)^2$ converges P-a.s. Therefore, $\big(\xi_n(t) - x_n(t)\big)$ is bounded P-a.s for any $n \in \mathbb{N}$. and almost all paths of the process $\xi_1(\cdot,\cdot)$ tend to infinity as $t \to t_\infty$. Thus (4.8) has no solution on $[0; t_\infty + \varepsilon]$.

Remark 4.7 The example shows that both deterministic and stochastic linear differential equations with the same operator may fail to have a solution. It would be interesting to construct an example where one of the two equations is solvable while the other is unsolvable.

Remark 4.8 It should be noted that all the results above are valid for the space $\mathbb{R}^S$ with countable set S in place of $S = \mathbb{N}$. This is obvious, since we can write $s = \{s_n\}_{n=1}^{\infty}$ and make use of the isomorphism $(y_n) = (x_{s_n})$ between $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^S$.

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